1 Stability

We have seen examples of modelling linear systems and obtaining their responses to different inputs. All inputs we used in our examples were bounded in time. That is the input was always less than some positive number M. Similarly the output of the system was always bounded including the time approaching to infinity. We call such systems **Bounded Input Bounded Output**(BIBO). In other words a system with input $U(s)$, output $Y(s)$ is BIBO stable if $|Y(s)| < K$ for some $|U(s)| < M$.
Remark Let $G(s) = Y(s)/U(s) = N(s)/D(s)$ be the transfer function of a system then $G(s)$ is BIBO stable if all poles of $G(s)$ are in the Left Hand s Plane (LHP), that is, the zeros of $D(s)$ has all negative real parts. Otherwise $G(s)$ is unstable.

Example 1.1

$$\begin{aligned} G(s) &= \frac{s^2+4s+2}{(s+2)(s+4)(s^2+4s+5)} \quad \textit{stable} \\ H(s) &= \frac{6s+1}{(s-2)(s+8)(s^2+2s+2)} \quad \textit{unstable} \end{aligned}$$

When all the poles of $G(s)$ has negative real parts the response of the system to impulse decays in time and the system is called stable. In other words we have stability for

$$\mathrm{Re}(p_k) < 0\,,$$

where p_k is a pole of $G(s)$. Such a system has bounded output to any bounded input. We will call BIBO stability shortly *stability.* The poles of $G(s)$ in the LHP will be referred to *stable poles* and the poles in the RHP will be referred to *unstable poles.* Use MATLAB commands *roots, tf2zpk* to find the poles of a transfer function.

If $G(s)$ has even only one unstable pole than the system is unstable.

Let us see why the system output is bounded when the poles of the system are in the LHP with examples.

stb1 **Example 1.2** *Find the unit impulse response of the system,*

$$G(s) = \frac{s+2}{(s+3)(s^2+4s+5)}\,.$$

unst **Example 1.3** *Find the unit step response of the system,*

$$G(s) = \frac{s+4}{(s-4)(s+2)(s+6)}\,.$$

Notice that the poles on the imaginary axis are unstable. This complies with the fact that the stable poles have to have real parts strictly less than zero.

Consider the following example with complex conjugate poles on the imaginary axis.

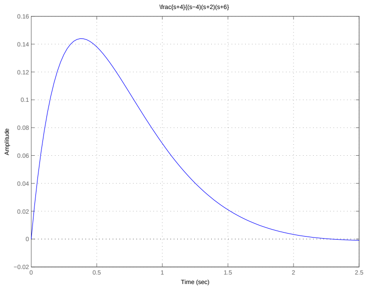

Figure 1: Response of system in Example 1.2, stable response `stb1f`

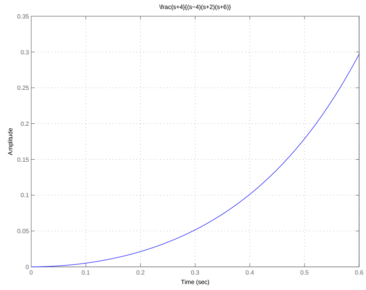

Figure 2: Response of system in Example 1.3, unstable response `unstf`

`imex` **Example 1.4** *Let $G(s) = Y(s)/U(s)$. Find response, $y(t)$ to step input where*

$$G(s) = \frac{s+1}{(s+2)(s^2+4)}.$$

The response of the system is given in Figure 3. It is clear that the response of the system to the bounded step input is a bounded output. This seem to contradict to what we claimed before.

Now assume the input $u(t) = sin(2t)$ for $t > 0$. In this case the output $Y(s)$ is given by,

$$Y(s) = \frac{s+1}{(s+2)(s^2+4)} \frac{4}{s^2+4}.$$

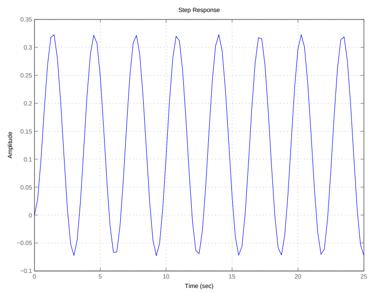

Figure 3: Response of system in Example 1.4, Response to unit step imexf

The response of the system to $\sin(2t)$ *is given in Figure 4.*

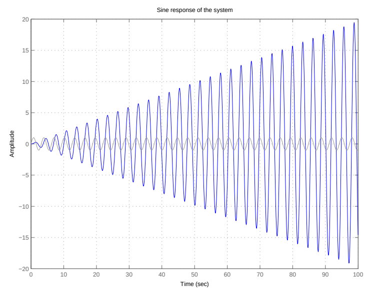

Figure 4: Response of system in Example 1.4, Response to sin(2t) imexf2

Now let us see why a system with poles in the LHP give bounded output to bounded inputs. Let $G(s)$ be a stable transfer function, that is, all poles of $G(s)$ are in the LHP,

$$G(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\Pi_{k=1}^{r}(s-p_k)^{n_k}} \quad \mathrm{Re}(p_k) < 0 \quad (1 \leq k \leq r)\,.$$

Recall that the partial fraction expansion of $G(s)$ is given by,

$$\begin{aligned} G(s) = \frac{N(s)}{\Pi_{k=1}^{r}(s-\lambda_k)^{n_k}} \quad &= \quad \frac{a_{1,1}}{s-\lambda_1} + \frac{a_{1,2}}{(s-\lambda_1)^2} + \cdots + \frac{a_{1,n_1}}{(s-\lambda_1)^{n_1}} + \\ &+ \frac{a_{2,1}}{s-\lambda_2} + \frac{a_{2,2}}{(s-\lambda_2)^2} + \cdots + \frac{a_{2,n_2}}{(s-\lambda_2)^{n_2}} \\ &\vdots \end{aligned}$$

$$+\frac{a_{r,1}}{s-\lambda_r}+\frac{a_{r,2}}{(s-\lambda_r)^2}+\cdots+\frac{a_{r,n_r}}{(s-\lambda_r)^{n_r}}$$

The impulse response of the system $g(t)=\mathcal{L}[G(s)]^{-1}$ is obtained as,

$$\begin{aligned} g(t) &= a_{1,1}e^{-t\lambda_1}+a_{1,2}te^{-t\lambda_1}+\cdots+\frac{a_{1,n_1}}{(n_1-1)!}t^{n_1-1}e^{-t\lambda_1} \\ &\vdots \\ & a_{r,1}e^{-t\lambda_r}+a_{r,2}te^{-t\lambda_r}+\cdots+\frac{a_{r,n_r}}{(n_r-1)!}t^{n_r-1}e^{-t\lambda_r} \end{aligned} \quad (1.1) \quad \boxed{\texttt{gt}}$$

The input-output relation in frequency domain is a multiplication, the inverse Laplace transform of this relation is a convolution in time domain, that is,

$$Y(s)=G(s)U(s) \quad \text{and} \quad y(t)=\int_0^t u(t-\tau)g(\tau)d\tau\,.$$

When the input is bounded by some positive constant M, that is $|u(t)|<M$ for all $t\geq 0$, we have

$$|y(t)|=\left|\int_0^t u(t-\tau)g(\tau)d\tau\right|\leq\int_0^t |u(t-\tau)||g(\tau)|d\tau\leq M\int_0^t |g(\tau)|d\tau\,. \quad (1.2) \quad \boxed{\texttt{yt}}$$

In other words $|y(t)|\leq M\int_0^t|g(\tau)|d\tau$. By consulting (1.1) we see that $\int_0^t|g(\tau)|d\tau$ has only terms containing $t^k e^{pt}$. Here $p=a+bi$ is a complex number. For $k=0$ we have,

$$\int_0^t|e^{p\tau}|d\tau=\int_0^t|e^{(a+bi)\tau}|d\tau=\int_0^t|e^{a\tau}|d\tau=\frac{1}{a}\left[e^{at}-1\right]\,.$$

Hence if $a<0$ then $0<e^{at}<1$ and we have

$$\int_0^t|e^{p\tau}|d\tau\leq\frac{1}{a}\qquad(t\geq 0)\,.$$

Similarly using integration by parts we obtain a bound when $k>0$, for example for $k=1$ we obtain

$$\begin{aligned}\int_0^t|te^{p\tau}|d\tau &= \int_0^t te^{a\tau}d\tau \\ &= \left[\frac{\tau}{a}e^{a\tau}\right]_0^t-\frac{1}{a}\int_0^t|e^{a\tau}|d\tau \\ &= \frac{t}{a}e^{at}-\frac{1}{a^2}\left[e^{at}-1\right]\end{aligned}$$

Since as time approaches infinity e^{at} and te^{at} approaches zero and the integral

$$\int_0^t|te^{p\tau}|d\tau<\frac{1}{a^2}\,,$$

is also bounded. Repeating integration by parts we can find integrals for $t^k e^{pt}$ for $k > 1$. Note that $t^k e^{pt}$ goes to zero for any finite k and negative a. From (1.1) and (1.2) we see that all the integrals involved are bounded, therefore the final integral is bounded. Hence if the system is stable it will have a bounded output for all bounded inputs. In other words if all the system poles are on the left hand s plane, the response of the system will be bounded to bounded input. Assume $g(t)$ has r distinct roots, then we have,

$$g(t) = \alpha_1 e^{p_1 t} + \alpha_2 e^{p_2 t} + \cdots + \alpha_r e^{p_r t},$$

where $p_k = a_k + b_k i$ and the response $y(t)$ can be express as

$$\begin{aligned} |y(t)| &\leq M \int_0^t |g(\tau)| d\tau \\ &\leq M \int_0^t |\alpha_1 e^{p_1 \tau} + \alpha_2 e^{p_2 \tau} + \cdots + \alpha_r e^{p_r \tau}| d\tau \\ &\leq M \left[\left| \frac{\alpha_1}{a_1} \right| + \left| \frac{\alpha_2}{a_2} \right| + \cdots + \left| \frac{\alpha_r}{a_r} \right| \right] \\ &\leq N \qquad \text{for finite } N \geq M.. \end{aligned}$$

Hence the location of the poles of the system dictates the stability of the system.

2 Characteristic equation and system pole locations

In the previous section we have seen that the system poles determine stability of the system. Since the transfer function is a rational function, its denominator is a polynomial. For a stable system we want all the system poles to be on the left hand s plane. In this section we will introduce methods to determine the location of the poles of a system using the coefficients of the denominator of the transfer function, that is the characteristic equation.

First consider the simpler systems we have seen in Lecture 2, the first and second order systems. It is clear that the coefficients of a first order system should be all positive to have stability, that is, $s + a = 0$ where $a > 0$. Similarly the second order system is stable if and only if all its coefficients are positive. However, for higher order systems positivity of all coefficients does not indicate stability.

Example 2.1 *Consider the polynomial*

$$s^3 + s^2 + 2s + 2\,.$$

Reverse of this statement is not true, that is, if some coefficients of the characteristic polynomial are negative then the system is unstable. Moreover, if certain coefficients are zero this indicates poles on the imaginary axis or in the right half s plane. Examples of these cases could be, $s^3 + 4s + 11s - 6$ and $s^2 + 4$.

2.1 Routh-Hurwitz Stability Test

Routh-Hurwitz test is a method to determine the system stability of higher order polynomials using the coefficients of the polynomials. This test is useful when it is tedious or hard to calculate the system poles, like when you don't have MATLAB running around.

To use R-H test the following conditions should be satisfied,

- All the coefficients of the polynomial should be positive, otherwise we know that the system is unstable.
- If there is a pole at zero, it should be removed. Recall that we could have at most one pole at zero for stability, so we use R-H test to find where rest of the poles are.

Consider an n^{th} order polynomial given by,

$$s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_n \, .$$

Then we form the following array of coefficients

$$\begin{array}{c|ccccc} s^n & 1 & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\ s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\ s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array} \tag{2.3}$$

Here the coefficients b_1, b_2 are given by

$$\begin{aligned} b_1 &= \frac{-\begin{vmatrix} 1 & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{a_{n-1}} = \frac{a_{n-1}a_{n-2} - 1a_{n-3}}{a_{n-1}} \\ b_2 &= \frac{-\begin{vmatrix} 1 & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{a_{n-1}} = \frac{a_{n-1}a_{n-4} - 1a_{n-5}}{a_{n-1}} \, . \end{aligned} \tag{2.4}$$

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Similarly for $k > 2$ we obtain b_k by omitting the coefficients right above them. The coefficients on the fourth row are given in terms of the coefficients on the first three rows. We have

$$\begin{aligned} c_1 &= \frac{-\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}}{b_1} = \frac{b_1 a_{n-3} - b_2 a_{n-1}}{b_1} \\ c_2 &= \frac{-\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix}}{b_1} = \frac{b_1 a_{n-5} - b_3 a_{n-1}}{b_1} \, . \end{aligned} \tag{2.5}$$

ck

Similarly rest of the coefficients in the array can be found.

Example 2.2 *Form the Routh array for the following polynomial,*

$$p(s) = 3s^4 + 5s^3 + 4s^2 + 7s + 8\,.$$

First two rows are readily given by,

$$\begin{array}{c|ccc} s^4 & 3 & 4 & 8 \\ s^3 & 5 & 7 & \\ s^2 & b_1 & b_2 & \\ s^1 & c_1 & c_2 & \\ s^0 & d_1 & d_2 & \end{array} \tag{2.6}$$

Using the above relations given in (2.4) we obtain,

$$b_1 = \frac{5(4) - 3(7)}{5} = -0.2 \quad b_2 = \frac{5(8) - 3(0)}{5} = 8\,.$$

Similarly using (2.5) we have,

$$c_1 = \frac{-0.2(7) - 8(5)}{-0.2} = 207 \quad c_2 = \frac{-0.2(0) - 3(0)}{5} = 0\,.$$

Finally, the last non-zero coefficient in the table is,

$$d_1 = \frac{207(8) - 0(-0.2)}{207} = 8\,.$$

The complete Routh array is given by,

$$\begin{array}{c|ccc} s^4 & 3 & 4 & 8 \\ s^3 & 5 & 7 & \\ s^2 & -0.2 & 8 & \\ s^1 & 207 & 0 & \\ s^0 & 8 & & \end{array} \tag{2.7}$$

We use the first column of the Routh array to check the location of zeros of polynomials.

- If there is no sign change in the first column then the polynomial has no right hand plane zeros.
- The number of sign changes in the first column indicates the number of left hand plane zeros.

In above example there are two sign changes first 5 to -0.2 and second -0.2 to 207.

Special cases When there is a zero on the first column or the Routh array or if one of the rows is all zeros we can not directly apply the Routh test. We will consider these cases by examples.

If a first term is zero in a column

Example 2.3 *Consider the polynomial,*

$$p(s) = s^3 + s^2 + 9\,s + 9\,.$$

Determine the location of the zeros $p(s)$.

Example 2.4 *Find the whereabouts of the zeros of the following polynomial.*

$$q(s) = 4\,s^4 + 7\,s^3 + 3\,s^2 + 3\,s + 6\,.$$

Note that if,

- If the sign of coefficients above ϵ is same as the one below, then the polynomial has a pair of *imaginary roots.*
- If the sign of coefficients above and below ϵ have opposite signs, this is regarded as one sign change in applying the Routh test.

If a row is all zeros

Example 2.5 *Consider the following polynomial*

$$r(s) = s^5 + 2\,s^4 + 8\,s^3 + 11\,s^2 + 16\,s + 12\,.$$

Determine the location of the zeros $r(s)$.

- If a row of Routh array is all zeros then the polynomial has zeros of the form $s_1 = -s_2 = a$, where a real or purely imaginary.
- Roots of the Auxiliary polynomial are roots of the original polynomial of the form $s_1 = -s_2 = a$, where a real or purely imaginary.

3 Block Diagrams

A Block Diagram is a representation of a system where components of the system are shown pictorially. It gives a sense of how signals and parts of the system are related in a graphical way. We will use block diagrams to analyze input output behavior of complicated systems.

Recall that we have defined the transfer function $G(s) = Y(s)/U(s)$, in a block diagram we denote this relation as depicted in Figure 5. Two transfer functions $G_1(s)$ and $G_2(s)$ could

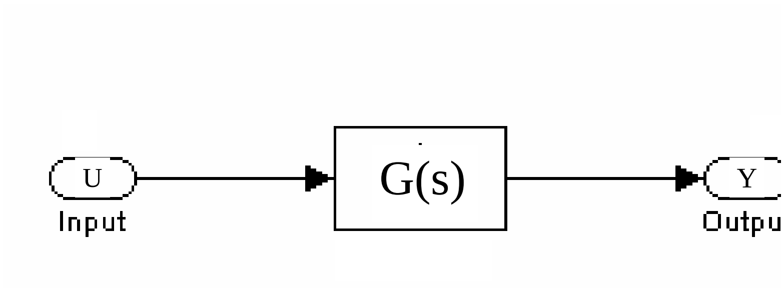

Figure 5: Transfer Function

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be in series or in parallel. If the transfer functions are in series we have $G(s) = G_1(s)G_2(s)$; See Figure 6. If they are in parallel then we add the two to get the equivalent transfer function $G(s)$, that is, $G(s) = G_1(s) + G_2(s)$; see Figure 7.

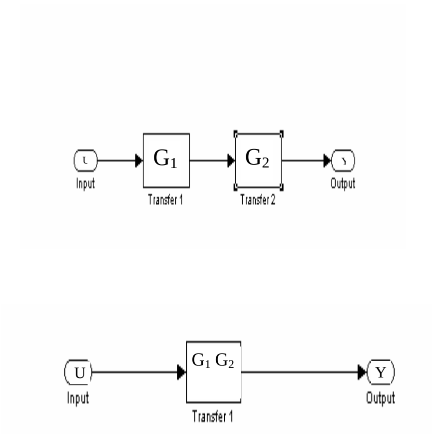

Figure 6: Transfer functions in Series

g1g2

Recall that Open Loop systems are such that the input is not a function of the output. The systems in Figures 5,6,7 are examples of Open Loop systems. On the other hand if the input of the system is a function of the output then we call such systems as Closed Loop. In Figure 8 we illustrate a closed loop system. Here notice the input E to the plant G is a function of U and the output Y. Here U is the input to the overall system, E input to G, Y the output, B the output of H, notice that we have,

$$\begin{aligned} E &= U + B = U + HY \\ Y &= EG = (U + HY)G \end{aligned}$$

Using the above relations we can obtain a relation between the overall input U and the output Y. Rearranging the last equation above we have,

$$\begin{aligned} Y(1 - GH) &= GU \\ \frac{Y}{U} &= \frac{G}{1 - GH}. \end{aligned}$$

In other words the overall transfer function of the system Q is given by,

$$Q = \frac{Y}{U} = \frac{G}{1 - GH}.$$

If the summation point in the block diagram takes the difference instead of the sum than that is if we have $E = U - B$, then the overall transfer function would be,

$$Q = \frac{Y}{U} = \frac{G}{1 + GH}.$$

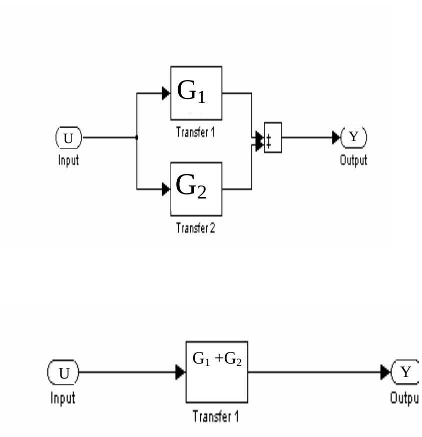

Figure 7: Transfer functions in Series g1g2p

Notice the sign chance in the denominator. Other fundamental block diagram operations are summarized in the Figures 9 and 10.

A complicated block diagram with summation points, feedbacks and other type of connections can be reduced to a single equivalent open loop transfer function. Let us see this on an example

Example 3.1 *Reduce the block diagram given in Figure 11 to obtain an equivalent transfer function.*

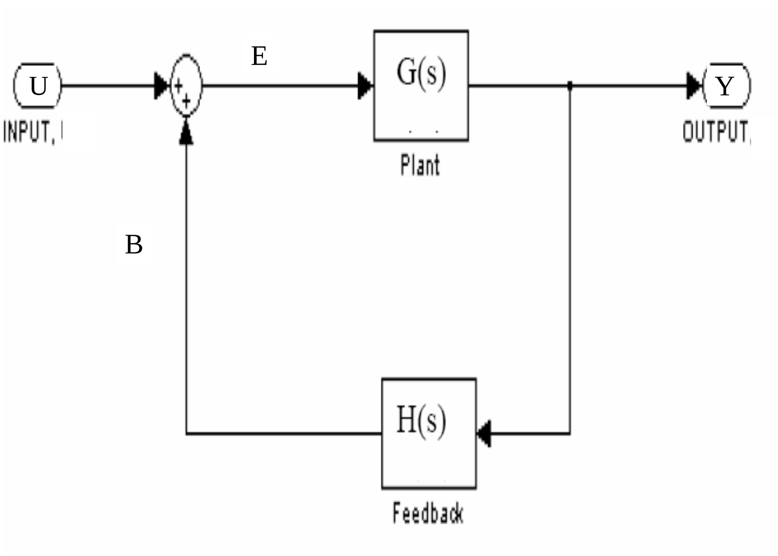

Figure 8: Closed Loop Block Diagram gh

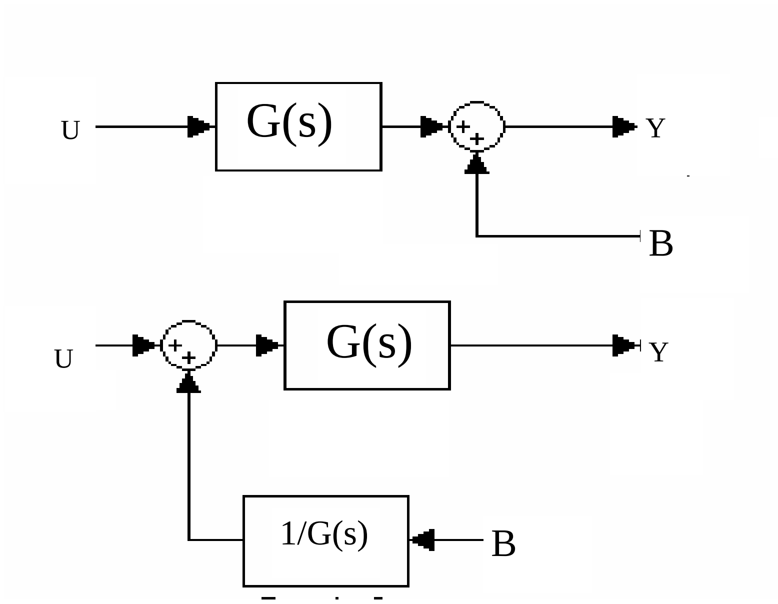

Figure 9: Moving a summation point Ahead of a Block mspab

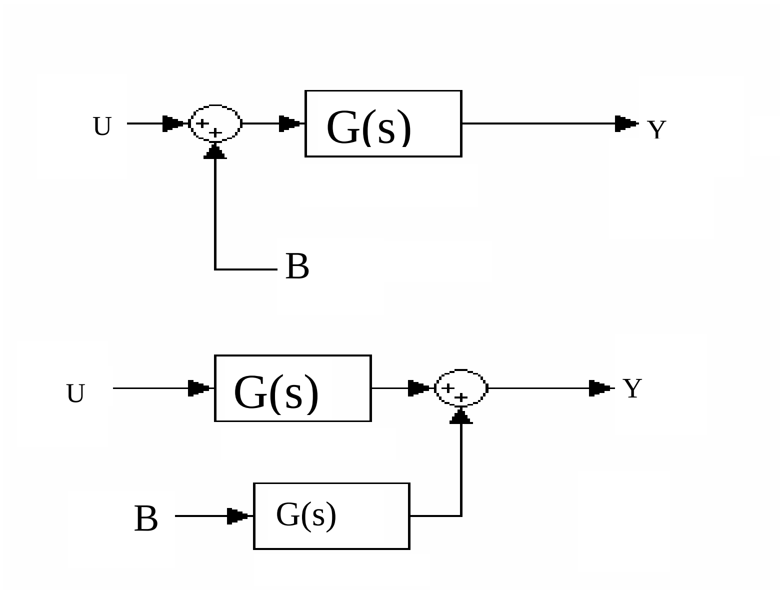

Figure 10: Moving a summation point Beyond of a Block mspbb

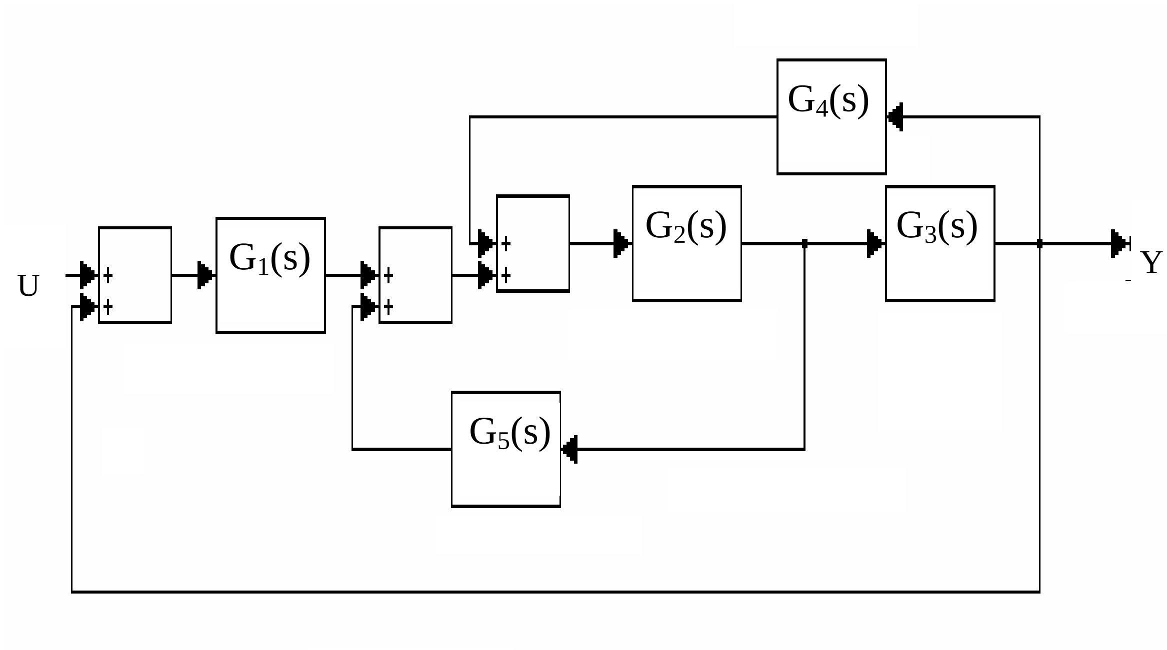

Figure 11: Example Block Diagram br