1 Nyquist Plots and Stability Analysis in Frequency Domain

Bode plots are very useful tools in determining the system behavior at different sinusoidal excitation frequencies. Using Gain Margin and Phase margin arguments some limited stability insight can be obtained from the Bode plots. A more complete frequency domain technique to determine stability is using *Nyquist Stability Criteria*. As we will see Bode Plots and Nyquist plots are closely related. Nyquist Plots are also referred to as polar plots.

Mathematically Nyquist plots are based on Principle of Argument, in these notes we will not prove that the Nyquist plots work. However, we will briefly describe how it works. Consider the pole zero plot of a system with transfer function $G(s)$ in Figure 1-a(top left). If we evaluate the points of a closed contour C_1 in the right half plane, we will obtain an other contour C_2. Here C_1 is in the same complex plane as the poles and zeros of the system. On the other hand, C_2 is the plot of real and imaginary parts of $G(\lambda)$ where λ is on C_1. The fact that we will use in Nyquist plots is that such a C_2 circles the origin if C_1 encloses a pole of $G(s)$ in it. We call the plane where C_1 lives the s-plane and the plane where C_2 lives the function plane. Compare the different transfer functions and curves C_1 in upper and lower rows of Figure 1.

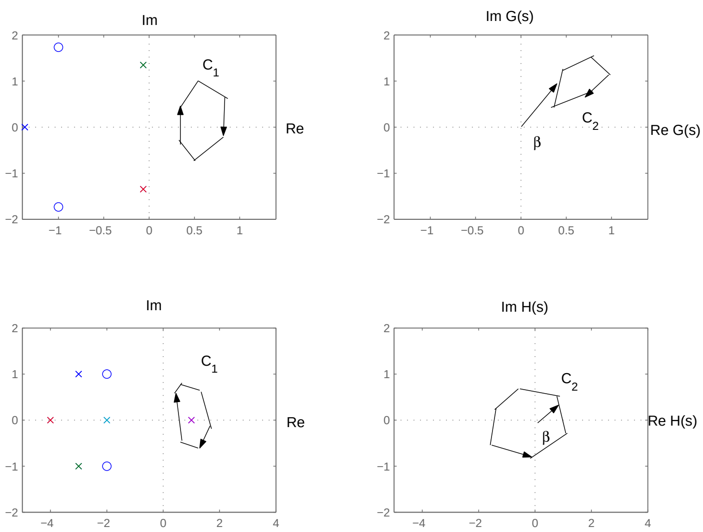

Figure 1: Evaluating contours C_1 for two different systems(two figure on the left) and the resulting polar plots(two plots on the right) con1

Consider a unity feedback system, we have a transfer function of the form,

$$T(s) = \frac{KG(s)}{1 + KG(s)}.$$

where $T(s)$ and $G(s)$ are the close loop and plant transfer functions. Modifying the fact above for $1 + KG(s)$, if a closed contour C_1 contains a pole of $1 + KG(s)$, the corresponding contour in the function plane circles -1. In some cases it is more advantages to plot the Nyquist plot of $G(s)$, in this case one analysis the encirclement of $-1/K$. The advantage of doing so is that one wouldn't have to plot the Nyquist plot over and over again for different K but only move the $-1/K$ point on the Nyquist plot of $G(s)$ as K changes.

Following example illustrates how to plot Nyquist plots.

nyex1 **Example 1.1** *Consider the plant transfer function*

$$G(s) = \frac{1}{(s+3)(s+4)}.$$

Determine the stability of the closed loop system given in Figure 2 for $K = 5$ using the Nyquist plot of $G(s)$.

Let us choose C_1 to be a semi-circle on the right had plane so that if there are any poles of the closed loop transfer function on the RHP the mapped contour C_2 will circle -1. The radius of the semi circle is infinite so that it covers the whole RHP. The part of C_1 from a to b is on the positive imaginary axis. On the function plane this maps to part on the negative half plane(the lower part). In this section of the plot $s = i\omega$ as in a Bode Plot, that is,

$$G(i\omega) = \frac{1}{(i\omega+3)(i\omega+4)}.$$

One can use Table 1 as a guide to plot this lower part of the Nyquist plot. At point b the radius is infinite, since $G(s)$ is a proper transfer function (the order of the denominator is less than that of the numerator), at point b, the magnitude of the transfer function goes to zero. Similarly the points c and d are infinitely far from the origin the magnitude of G at these points is zero. Hence all these points, b, c, d and all the points in between map to 0 in the function plane (on C_2).

Table 1: $|G(i\omega)|$ and $\measuredangle G(i\omega)$ at certain ω for $G(i\omega) = \frac{1}{(i\omega+3)(i\omega+4)}$

ω	$\lvert G(i\omega)\rvert$	$\measuredangle G(i\omega)$
0	0.4167	0
1.0000	0.3835	-32.4712
5.0000	0.1339	-110.3764
10.0000	0.0445	-141.4993
20.0000	0.0121	-160.1593
30.0000	0.0055	-166.6948
90.0000	0.0006	-175.5460

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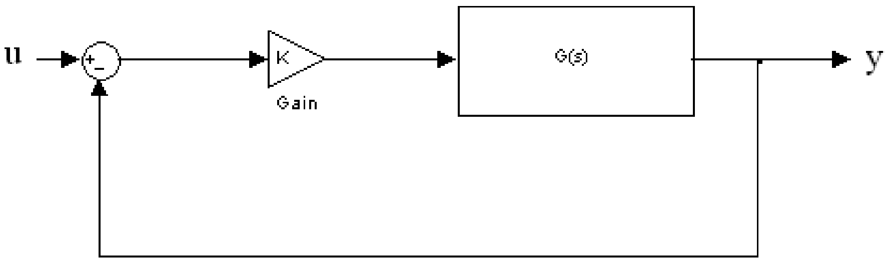

Figure 2: Closed loop system with G(s) as the plant transfer function ex1sys

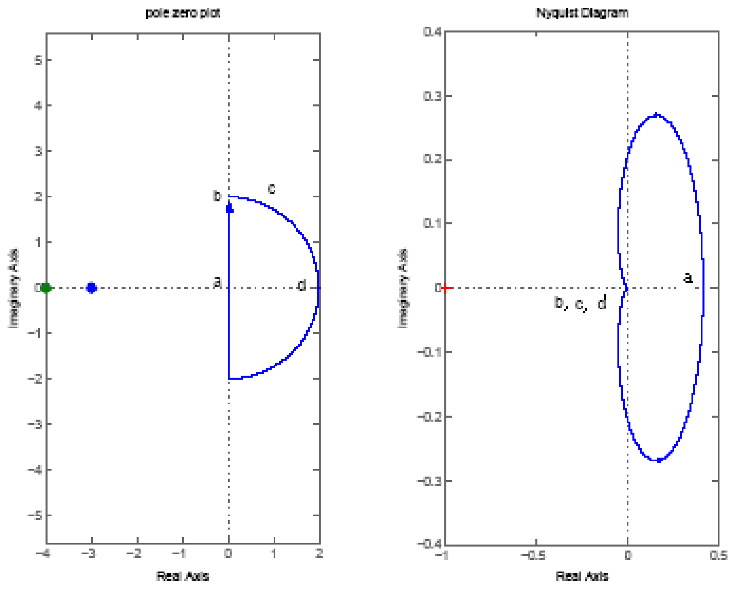

Figure 3: Pole Zero plots and Nyquist plot for $G(s) = 5/((s+3)(s+4))$ ex1ny

2 Nyquist Stability Criteria

We use Nyquist plots to determine the stability of a **closed loop** system given the **open loop** system transfer function. As before we will use the closed loop characteristic equation to determine stability.

Recall that for a unity feedback system the closed loop transfer function is given by,

$$T(s) = \frac{KG(s)}{1 + KG(s)} .$$

In this case the closed loop characteristic equation is

$$1 + KG(s) = 0 .$$

From Cauchy's Principle we know that if $1 + KG(s)$ has poles on the right hand side the mapping of a contour around the right hand plane will encircle the origin.

Consider part of a Nyquist plot in Figure 4. Assume that this part of a Nyquist plot where we we are using $KG(s)$ as the mapping function. Then the encirclements of -1 are significant to us. The point -1 is circled twice in clockwise(CW) and once by a counterclockwise(CCW) direction. In this case effectively we have one CW encirclement of -1.

According to Nyquist the number of right hand side poles of a transfer function

$$H(s) = \frac{KG(s)}{1 + KG(s)} ,$$

is given by

number of RHP poles of H(s)	= number if CW encirclements -1 on the KG plane	+ number of RHP of KG(s)

(2.1) nyc

The first term on right side of this relation is the number of encirclements read on the Nyquist plot. The second term is the number of open loop right side poles. Note that above procedure can be applied to a Nyquist plot of $G(s)$ in $1 + KG(s) = 0$, In this case one will look for the net encirclements of $-1/K$. That is in relation 2.1 replace -1 by $-1/K$.

Let us revisit Example 1.1 and check the closed loop stability. In Figure 3 we see that the Nyquist plot does not encircle -1. It is clear that if we increase K the contour on the $KG(s)$ plane won't encircle -1; see Figure 5. Since the open loop transfer function $G(s) = K/((s + 3)(s + 4))$ does not have any right hand plane poles using the Nyquist criteria 2.1, we see that the closed loop system does not have any right hand plane poles. The Gain margin is infinite for this system. Similarly PM margin is never zero.

We can determine the gain and phase margins from Nyquist plots. Recall that gain margin, GM is the allowable increase in gain K before instability. In terms of Nyquist plots we can phrase this as allowable increase in gain K before there is any CW encirclements of -1 (or -1/K); see Figure 6. Similarly Phase margin, PM is the difference of the phase of the system to $-180°$, that is $PM = 180 + \phi(i\omega_g)$, here ω_g is the gain crossover frequency. In a

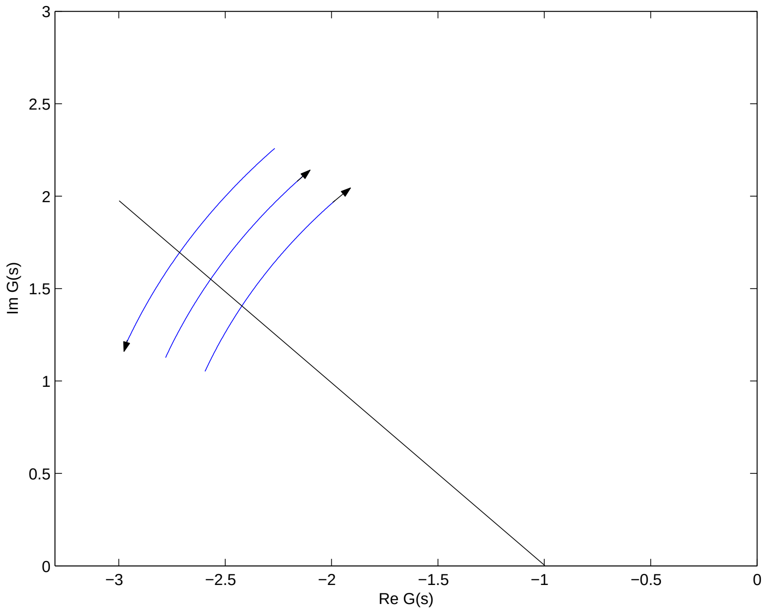

Figure 4: Encirclements of -1, twice in CW and once in CCW direction, effectively there is one CW encirclement. np1

Nyquist plot phase margin is the angle between the negative real axis and point where the magnitude is 1; see Figure 6. Note that here the gain margin does not have a unit when we defined gain margin in Bode plots section it was in dB, both are the same constant. Let the system be unstable for at K_u then

$$GM = \frac{K_u}{K} = \frac{1}{|G(i\omega_p)|}, \quad \text{and} \quad GM(dB) = 20\log\frac{K_u}{K}\ ,.$$

In above equations ω_p is the phase crossover frequency. When the margin is less than 1 the system is stable, otherwise it is unstable. The Phase margin is positive when the system is stable. Note that positive phase margin is defined in CCW direction from negative real axis.

Example 2.1 *Sketch the Nyquist plot and check closed loop stability for the plant transfer function*

$$G(s) = \frac{1}{s(s+5)}\ ,$$

where $G(s)$ is a unity feedback loop shown in Figure 2 with K=3. See Figure 7.

Example 2.2 *Sketch the Nyquist plot for $KG(s)$ and check closed loop stability, where the plant transfer function*

$$G(s) = \frac{1}{s^2(s+2)}\ ,$$

and $K = 4$, consider Figure 2. For the Nyquist plot see Figure 8.

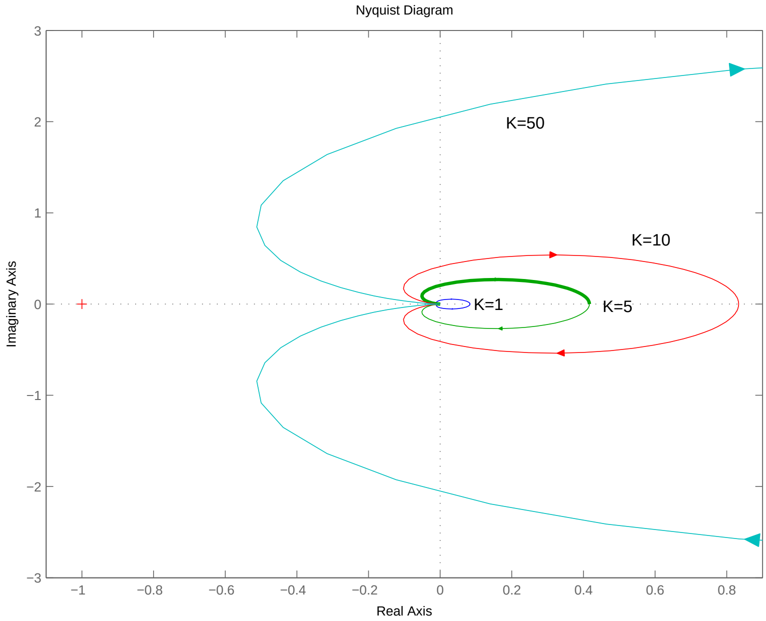

Figure 5: Pole Zero plots and Nyquist plot for $G(s) = K/((s+3)(s+4))$, for K=1,5,10,50. `np`

Example 2.3 *Sketch the Nyquist plot for $KG(s)$ and check closed loop stability, where the plant transfer function*

$$G(s) = \frac{1}{s(s+4)(s+2)},$$

and $K = 4$, consider Figure 2. For the Nyquist plot see Figure 9.

Example 2.4 *Sketch the Nyquist plot for $KG(s)$ and check closed loop stability, where the plant transfer function*

$$G(s) = \frac{s+2}{s(s-8)},$$

and $K = 1$, consider Figure 2. Notice that the open loop system is unstable. See Figure 10 for the Nyquist plot.

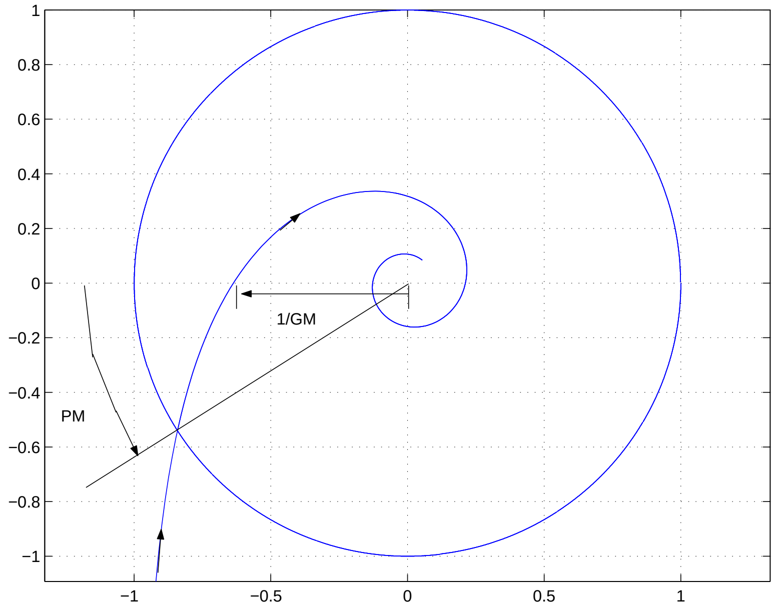

Figure 6: Gain Margin, Phase Margin Defined on a Nyquist plot . nygp

Table 2: $|G(i\omega)|$ and $\measuredangle G(i\omega)$ at certain ω for $G(s) = \frac{1}{s(s+5)}$. tab12

ω	$3\|G(i\omega)\|$	$\measuredangle G(i\omega)$
0	Inf	-90.0000
0.1000	5.9988	-91.1458
0.2000	2.9976	-92.2906
1.0000	0.5883	-101.3099
1.5000	0.3831	-106.6992
2.0000	0.2785	-111.8014
5.0000	0.0849	-135.0000
10.0000	0.0268	–153.4349
50.0000	0.0012	-174.2894
100.0000	0.0003	-177.1376

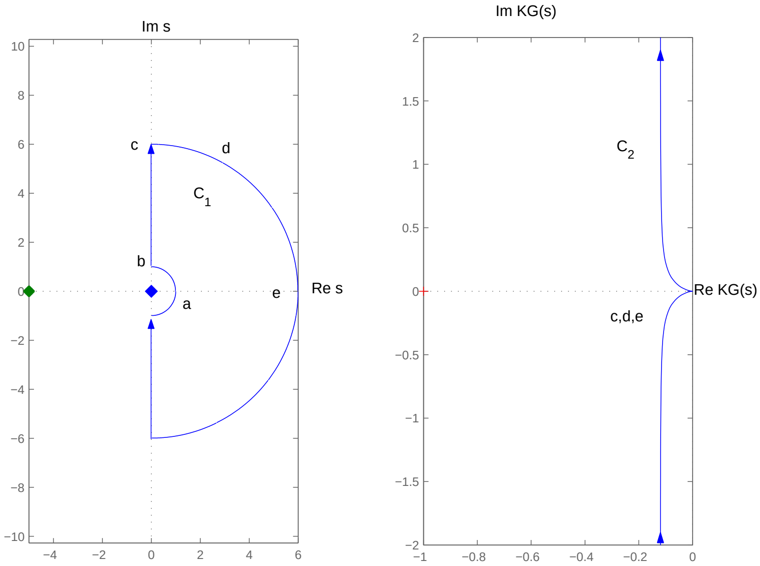

Figure 7: Pole Zero plots and Nyquist plot for $G(s) = 3/(s(s+5))$ ex2ny

Table 3: $|G(i\omega)|$ and $\measuredangle G(i\omega)$ at certain ω for $G(s) = \frac{1}{s^2(s+2)}$ tab22

ω	$\lvert G(i\omega)\rvert$	$\measuredangle G(i\omega)$
0	Inf	-180.0000
0.1000	49.9376	-182.8624
0.2000	12.4380	-185.7106
1.0000	0.4472	-206.5651
1.5000	0.1778	-216.8699
2.0000	0.0884	-225.0000
5.0000	0.0074	-248.1986
10.0000	0.0010	–258.6901
50.0000	0.0000	-267.7094

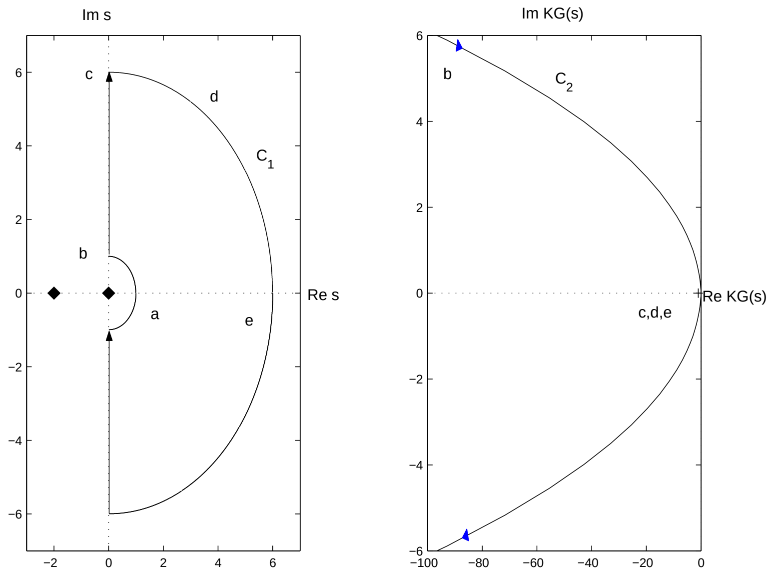

Figure 8: Pole Zero plots and Nyquist plot for $G(s) = 4/(s^2(s+2))$ `ex3ny`

Table 4: $|G(i\omega)|$ and $\measuredangle G(i\omega)$ at certain ω for $G(s) = \frac{1}{s(s+4)(s+2)}$ `tab122`

ω	$\|G(i\omega)\|$	$\measuredangle G(i\omega)$
0	Inf	-90.0000
0.1000	1.2481	-94.2945
0.2000	0.6211	-98.5730
1.0000	0.1085	-130.6013
1.5000	0.0624	-147.4259
2.0000	0.0395	-161.5651
5.0000	0.0058	-209.5388
10.0000	0.0009	-236.8887
50.0000	0.0000	-263.1355
100.0000	0.0000	-266.5636

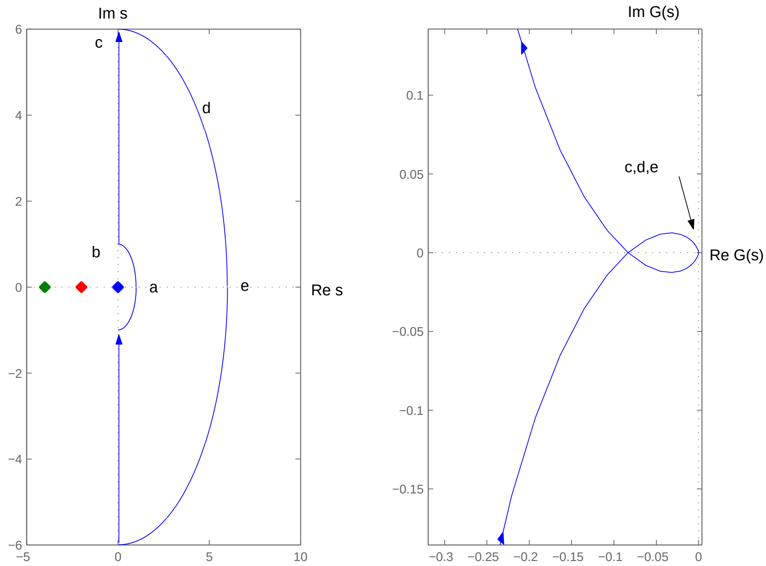

Figure 9: Pole Zero plots and Nyquist plot for $G(s) = 4/(s(s+4)(s+2))$ ex4ny

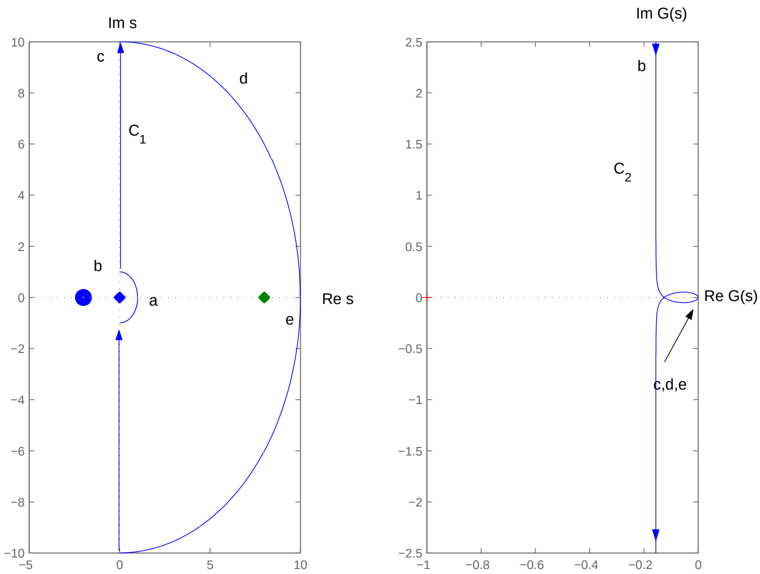

Figure 10: Pole Zero plots and Nyquist plot for $G(s) = (s+2)/(s(s-8))$ ex5ny

Table 5: $|G(i\omega)|$ and $\measuredangle G(i\omega)$ at certain ω for $G(s) = \frac{s+2}{s(s-8)}$ tab2

ω	$\lvert G(i\omega)\rvert$	$\measuredangle G(i\omega)$
0.1000	2.5029	93.5786
0.2000	1.2558	97.1427
0.3000	0.8421	100.6784
0.5000	0.5144	107.6126
1.0000	0.2774	123.6901
1.5000	0.2048	137.4896
2.0000	0.1715	149.0362
5.0000	0.1142	-169.7960
100.0000	0.0100	-95.7197