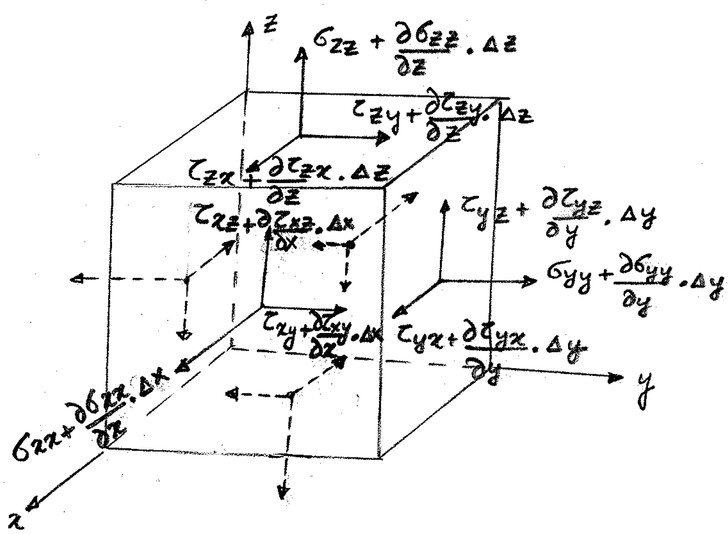
# SUMMARY OF CHAPTER 1

- Stress Tensor

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$




Stress block at point Q

- Equations of Equilibrium:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \bar{F}_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \bar{F}_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \bar{F}_z = 0$$

3 at each point in the body.

- Strain - displacement relations:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}.$$

$\varepsilon \longrightarrow$ normal strains
$\gamma \longrightarrow$ shear strains

$u(x,y,z)$
$v(x,y,z)$
$w(x,y,z)$ } displacement

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \times \frac{E}{(1-2\nu)(1+\nu)}$$

$$\{\sigma\}_{6\times1} = [C]_{6\times6}\{\varepsilon\}_{6\times1}$$

STRESS-STRAIN RELATIONS

- PLANE STRESS:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

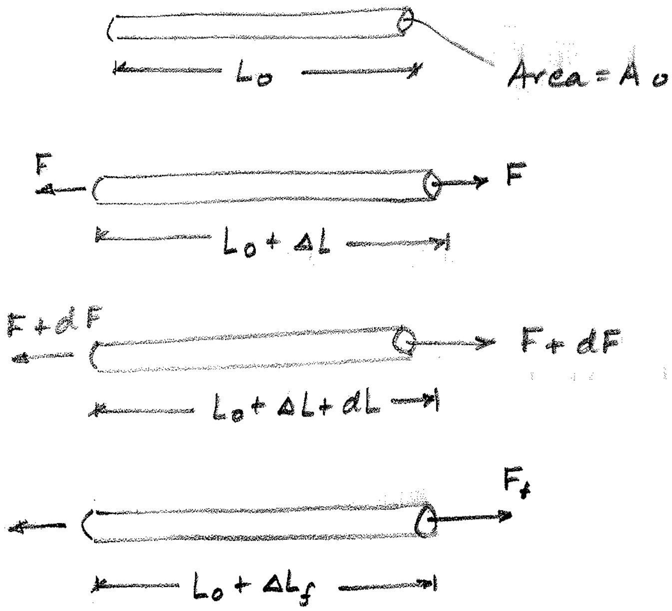
$$\varepsilon_z = -\frac{\nu}{1-\nu}(\varepsilon_x + \varepsilon_y) \quad \text{and} \quad \sigma_z = 0$$

- PLANE STRAIN:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1-\nu^2}{E} \begin{bmatrix} 1 & -\frac{\nu}{1-\nu} & 0 \\ -\frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{2}{1-\nu} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}$$

$$\varepsilon_z = 0 \quad \text{and} \quad \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

## Strain Energy:




$\varepsilon = \dfrac{\Delta L}{L_o}$ ; $\sigma = \dfrac{F}{A_o}$ ; $W_1 = F \cdot \Delta$

$\varepsilon = \dfrac{\Delta L + dL}{L_o}$ ; $\sigma = \dfrac{F + dF}{A_o}$ ; $dW = (F + dF) d$

$\varepsilon_f = \dfrac{\Delta L_f}{L_o}$ ; $\sigma = \dfrac{F_f}{A_o}$ ; $W_f = F_f \cdot \Delta$

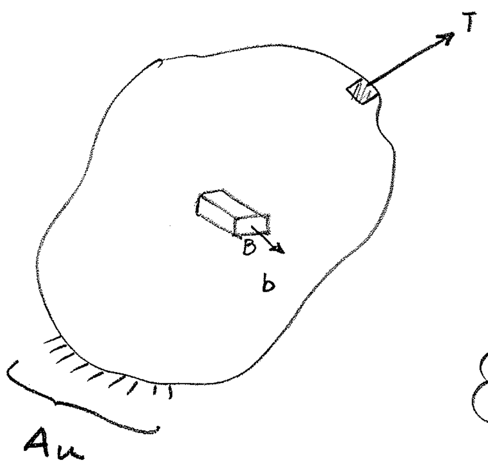
$$dW = (F + dF) \cdot dL = F\,dL$$

$$\therefore W = \int_0^{\varepsilon_f} F\,dL = \int_0^{\varepsilon_f} \sigma \cdot A_o \cdot d\varepsilon \cdot L_o = (A_o L_o) \int_0^{\varepsilon_f} \sigma \cdot d\varepsilon$$

$$\bar{u} = u / V_o = \int_{\varepsilon_o}^{\varepsilon_f} \sigma_x \, d\varepsilon_x$$

$$u = \int_{vol} \bar{u} \, dV$$

$$\bar{u} = \int_{\varepsilon_o}^{\varepsilon_f} \sigma_x \, d\varepsilon_x + \int_{\varepsilon_o}^{\varepsilon_f} \sigma_y \, d\varepsilon_y + \int_{\varepsilon_o}^{\varepsilon_f} \tau_{xy} \, d\gamma_{xy} + \int_{\varepsilon_o}^{\varepsilon_f} \tau_{yz} \, d\gamma_{yz} + \int_{\varepsilon_o}^{\varepsilon_f} \tau_{xz} \, d\gamma_{xz}$$




Body force $[b]$

Surface traction $\{T\}$

virtual displacement $\{\delta u\}$

$$\boxed{\delta W_e = \delta W_i} \quad \text{Equilibrium equation}$$

$$\delta W_e = \int_{Vol.} [b]\{\delta u\}\, dV + \int_{Surf} \{T\}\{\delta u\}\, dA$$

$$\delta W_e = \int_{Vol} [b]\{\delta u\}\, dV + \int_{Surf} [\sigma]\{\hat{n}\}\{\delta u\}\, dA \qquad \text{———} \quad (1)$$

virtual disp.

$$\delta W_i = \delta U = \int_{Vol} \bar{U}\, dV$$

$$= \int_{Vol} \left(\sigma_x \cdot \delta\varepsilon_x + \sigma_y \cdot \delta\varepsilon_y + \sigma_z \cdot \delta\varepsilon_z + \tau_{xy} \cdot \delta\gamma_{xy} + \tau_{yz} \cdot \delta\gamma_{yz} + \tau_{xz}\, \delta\gamma_{xz}\right) dV$$

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$$\int_{Vol.} f(x,y,z) \cdot \frac{\partial}{\partial x} g(x,y,z)\, dV = \int_{Surf} f(x,y,z) \cdot g(x,y,z)\, n_x\, dA - \int_{Vol} g(x,y,z) \cdot \frac{\partial f}{\partial x}(x,y,z)\, dA$$

Consider: $\displaystyle\int_{Vol} \sigma_x \cdot \delta\varepsilon_x$ $\qquad \delta\varepsilon_x = \dfrac{\partial(\delta u)}{\partial x}$

$$= \int (\sigma_x \cdot \delta u)\, n_x\, dA - \int_{Vol} \delta u \cdot \left(\frac{\partial \sigma_x}{\partial x}\right) \cdot dV$$

Consider: $\displaystyle\int_{Vol} (\tau_{xy} \cdot \delta\gamma_{xy})\, dV$ $\qquad \delta\gamma_{xy} = \dfrac{\partial(\delta v)}{\partial x} + \dfrac{\partial(\delta u)}{\partial y}$

$$= \int_{Vol} \tau_{xy} \cdot \frac{\partial}{\partial x}(\delta v)\, dV + \int_{Vol} \tau_{xy} \cdot \frac{\partial}{\partial y}(\delta u)\, dV$$

$$= \int_{Surf} \tau_{xy} \cdot \delta v \cdot n_x\, dV - \int_{Vol} \delta v \cdot \frac{\partial \tau_{xy}}{\partial x} \cdot dV + \int_{Surf} (\tau_{xy} \cdot \delta u \cdot n_y)\, dV - \int_{Vol} \delta u \cdot \frac{\partial \tau_{xy}}{\partial y} \cdot dV$$

Collecting terms.

$$\delta W_i = \int_{surf.} [\sigma]\{\hat{n}\}\{\delta u\}\, dA - \int_{vol.} \begin{bmatrix} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} \\ - \quad - \quad - \\ - \quad - \quad - \end{bmatrix} \cdot \{\delta u\}\, dV$$

$\delta W_i = \delta W_e.$

$$\Rightarrow \int_{surf} [\sigma]\{\hat{n}\} \cdot \{\delta u\}\, dA - \int_{vol.} [A]\{\delta u\}\, dV = \int_{vol.} [b] \cdot \{\delta u\}\, dV + \int_{surf} [\sigma]\{\hat{n}\} \cdot \{\delta u\}\, dA$$

$$\therefore \int_{vol} \{[A] + [b]\}\{\delta u\}\, dV = 0$$

For arbitrary $\{\delta u\}$

$$[A] + [b] = 0 \longrightarrow$$ equilibrium equations.

Weak form of static equilibrium

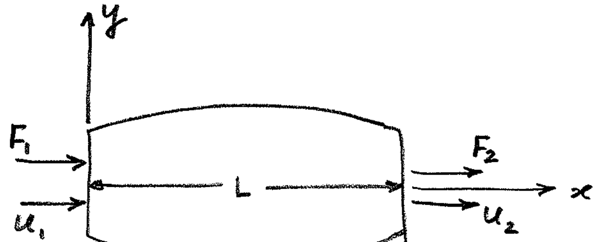
**Example:** Rayleigh-Ritz Method: a specific way of implementing the Principal of Virtual Displacements

Total Potential Energy $= \Pi = W_i - W_e$

Principle of virtual displacements $\quad \delta\Pi = \delta W_i - \delta W_e = 0$

Approximate the displacements as a sum of known functions with unknown coefficients

Use the virtual displacement as the same as real displacements.




Given: Axial bar of length 'L', constant modulus of elasticity $E$, $\quad A(x) = A_0 \times \left(1 + \beta \sin \frac{\pi x}{L}\right)$ $\quad \beta$ is known

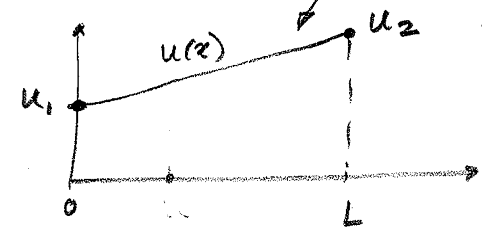
$F_1$ and $F_2$ act @ $x = 0$ & $x = L$

corresponding deflections are $u_1$ & $u_2$

Solve using Rayleigh-Ritz

Assume $\quad u(x) = u_1\left(1 - \frac{x}{L}\right) + u_2\left(\frac{x}{L}\right) \quad$ —— (1)

linear variation assumed

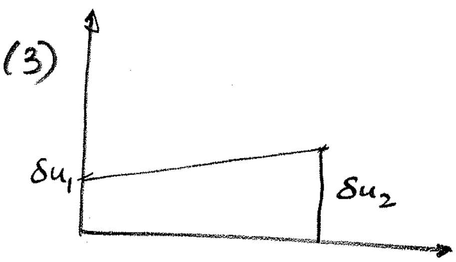


$$\therefore \quad \varepsilon_x = \frac{\partial u}{\partial x} = u_1\left(-\frac{1}{L}\right) + \frac{u_2}{L} = \frac{u_2 - u_1}{L}$$

$$\therefore \quad \sigma_x = E \times \left(\frac{u_2 - u_1}{L}\right) \quad \text{———} \quad (2)$$

To this system in equilibrium, apply a virtual displacement

$$\delta u(x) = \delta u_1 \left(1 - \frac{x}{L}\right) + \delta u_2 \left(\frac{x}{L}\right) \quad \text{—} (3)$$




$$\therefore \quad \delta \varepsilon_x = \frac{\partial(\delta u)}{\partial x} = -\frac{\delta u_1}{L} + \frac{\delta u_2}{L}$$

$$\delta \varepsilon_x = \frac{\delta u_2 - \delta u_1}{L} \quad \text{———} \quad (4)$$

← virtual strain due to virtual disps.

$$\delta W_e = F_1 \times \delta u_1 + F_2\, \delta u_2 \quad \text{———} \quad (5)$$

$$\delta W_i = \int \sigma_x \cdot \delta \varepsilon_x \, dV = \int_{x=0}^{L} E \times \left(\frac{u_2 - u_1}{L}\right) \times \left(\frac{\delta u_2 - \delta u_1}{L}\right) \times A(x)\, dx$$

$$= E \left(\frac{u_2 - u_1}{L}\right)\left(\frac{\delta u_2 - \delta u_1}{L}\right) \times \int_0^L A_0 \left(1 + \beta \sin\frac{\pi x}{L}\right) \cdot dx$$

$$= A_0 E \left(\frac{u_2 - u_1}{L}\right)\left(\frac{\delta u_2 - \delta u_1}{L}\right) \times \left[x + \beta \times \frac{\left(-\cos \pi x/L\right)}{\pi/L}\right]_0^L$$

$$= A_0 E \left(\frac{u_2 - u_1}{L}\right)\left(\frac{\delta u_2 - \delta u_1}{L}\right) \times \left[L + \beta \frac{1}{\pi/L} - \left(0 + \beta \times \frac{-1}{\pi/L}\right)\right]$$

$$= A_0 E \left(\frac{u_2 - u_1}{L}\right)\left(\frac{\delta u_2 - \delta u_1}{L}\right) \times \left(L + \frac{2\beta L}{\pi}\right)$$

$$\therefore \delta W_i = A_0 E \frac{(u_2-u_1)}{\cancel{L}} \times \cancel{L}\left(1+\frac{2\beta}{\pi}\right) \frac{(\delta u_2 - \delta u_1)}{L}.$$

$$\delta W_i = -A_0 E \frac{(u_2-u_1)}{L}\left(1+\frac{2\beta}{\pi}\right) \cdot \delta u_1 + A_0 E \frac{(u_2-u_1)}{L}\left(1+\frac{2\beta}{\pi}\right)\delta u_2$$

$$\delta W_e = \delta W_i$$

$$\therefore F_1 \, \delta u_1 + F_2 \, \delta u_2 = \_\_\_\_\_ \, \delta u_1 + \_\_\_\_\_ \, \delta u_2$$

$$\left.\begin{aligned} \therefore F_1 &= -A_0 E \left(\frac{u_2-u_1}{L}\right)\left(1+\frac{2\beta}{\pi}\right) \\ F_2 &= E A_0 \left(\frac{u_2-u_1}{L}\right)\left(1+\frac{2\beta}{\pi}\right) \end{aligned}\right\}$$

Since $\delta u_1$ & $\delta u_2$ are arbitrary

$$\boxed{\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{E A_0}{L}\left(1+\frac{2\beta}{\pi}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}$$

Solution.

## SUMMARY OF CHAPTER 2

### VARIATIONAL METHODS

(i) FUNCTIONAL $$A = \int_{x_1}^{x_2} F(x, u, u', u'')\, dx \qquad (1)$$

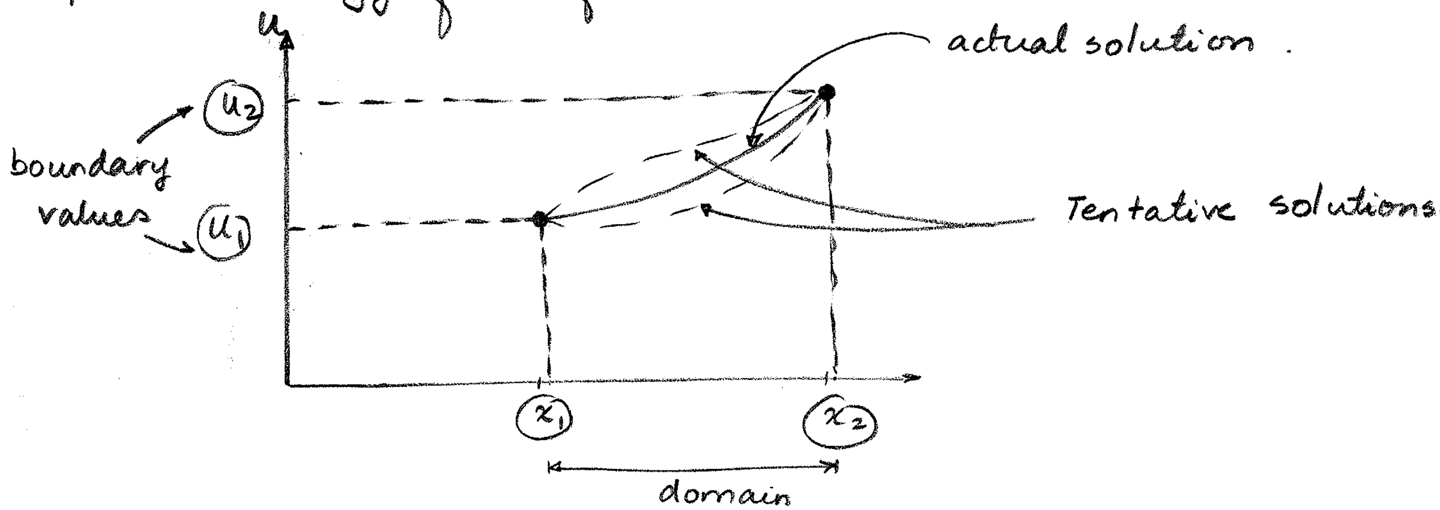
function of other functions

$u \rightarrow u(x)$

$u' \rightarrow \frac{d}{dx} u(x)$

$u'' \rightarrow \frac{d}{dx^2} u(x)$

In mechanics, the functional has a clear meaning, e.g., the potential energy of a deformable solid.




The integral (1) is defined in the region or domain $[x_1, x_2]$

$u(x_1) = u_1$ & $u(x_2) = u_2$ ⟵ boundary values

- The actual solution for $u(x)$ is not known!
- A tentative solution is tried for the problem, and the functional is expressed in terms of tentative solution.
- From all such possible solutions that satisfy the boundary conditions, the solution which satisfies the variational principle governing the behavior will be the one that makes the functional A stationary.

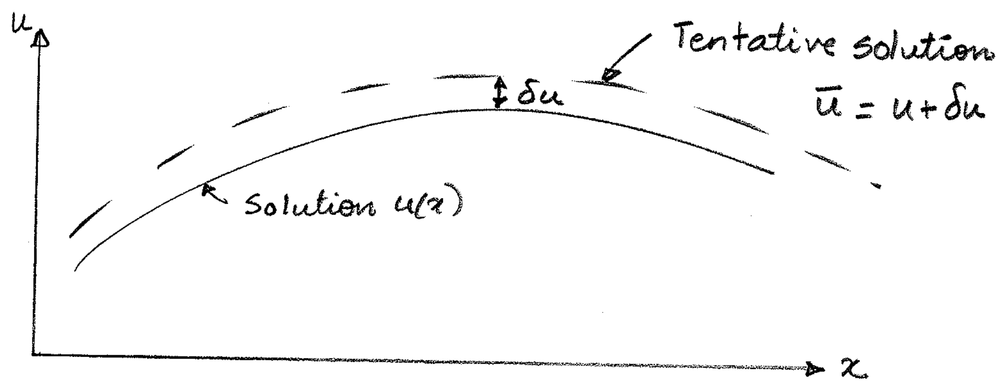
- The mathematical procedure used to select the correct solution from a number of tentative solutions is called the calculus of variations.

- Any tentative solution $\bar{u}$ in the neighborhood of the exact solution, may be represented as the sum of the the exact solution $u$ and a variation of $u$, $\delta u$

$$\bar{u} = u + \delta u$$

- The variation in $u = u(x)$ is defined as an infinitesimal change in $u$ for a fixed value of $x$, i.e., $\delta x = 0$

- Notation $\delta u \longrightarrow$ variational notation.




- The $\delta$ can be treated as an operator, similar to the differential operator, $d$

- The variation operator is commutative with both differentiation and integration

$$\delta\left(\int F\, dx\right) = \int (\delta F)\, dx$$

variation of integral = integral of variation

$$\delta\left(du/dx\right) = d\left(\delta u/dx\right)$$

variation of derivative = derivative of variation

- Variation of a function of several variables or functional

$$\delta F = \frac{\partial F}{\partial u} \cdot \delta u + \frac{\partial F}{\partial u'} \cdot \delta u' + \frac{\partial F}{\partial u''} \cdot \delta u''$$

- Consider the functional $A = \int_{x_1}^{x_2} F\, dx$

Small variations in this functional A correspond to small variations in the solution.

By analogy to the max or min of simple functions in ordinary calculus, the vanishing of the first variation of the functional A will give the condition for A to be stationary.

$$\delta A = \int_{x_1}^{x_2} \delta F\, dx = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial u} \cdot \delta u + \frac{\partial F}{\partial u'} \cdot \delta u' + \frac{\partial F}{\partial u''} \cdot \delta u''\right) dx = 0$$

Integration by parts for the respective terms:

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial u} \cdot \delta u \, dx \qquad \text{———} \qquad (a)$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial u'} \cdot \delta u' \, dx = \left[ \frac{\partial F}{\partial u'} \cdot \delta u \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left( \frac{\partial F}{\partial u'} \right) \delta u \, \delta x$$

and

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial u''} \cdot \delta u'' \, dx = \left[ \frac{\partial F}{\partial u''} \cdot \delta u' \right]_{x_1}^{x_2} - \left[ \frac{d}{dx}\left( \frac{\partial F}{\partial u''} \right) \delta u \right]_{x_1}^{x_2}$$

$$+ \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left( \frac{\partial F}{\partial u''} \right) \delta u \, dx$$

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$$\delta A = \int_{x_1}^{x_2} \left[ \frac{\partial F}{\partial u} - \frac{d}{dx}\left( \frac{\partial F}{\partial u'} \right) + \frac{d^2}{dx^2}\left( \frac{\partial F}{\partial u''} \right) \right] \delta u \, dx$$

$$+ \left[ \frac{\partial F}{\partial u'} - \frac{d}{dx}\left( \frac{\partial F}{\partial u''} \right) \right] \delta u \Bigg|_{x_1}^{x_2}$$

$$+ \left[ \left( \frac{\partial F}{\partial u''} \right) \delta u' \right]_{x_1}^{x_2} = 0$$

The variation $\delta u$ is arbitrary;

$$\therefore \frac{\partial F}{\partial u} - \frac{d}{dx}\left( \frac{\partial F}{\partial u'} \right) + \frac{d^2}{dx^2}\left( \frac{\partial F}{\partial u''} \right) = 0 \qquad \text{———} \qquad (1)$$

$$\left[ \frac{\partial F}{\partial u'} - \frac{d}{dx}\left( \frac{\partial F}{\partial u''} \right) \right] \delta u \Bigg|_{x_1}^{x_2} = 0 \qquad \text{———} \qquad (2)$$

$$\left[ \left( \frac{\partial F}{\partial u''} \right) \delta u' \right]_{x_1}^{x_2} = 0 \qquad \text{———} \qquad (3)$$

Equation (1) is the governing differential equation and is called the EULER equation or the Euler-Lagrange equation.

$$\left[\frac{\partial F}{\partial u'} - \frac{d}{dx}\left(\frac{\partial F}{\partial u''}\right)\right]_{x_1}^{x_2} = 0 \quad \text{and} \quad \left.\frac{\partial F}{\partial u''}\right|_{x_1}^{x_2} = 0$$

or

$$\left.\delta u\right|_{x_1}^{x_2} = 0 \quad \text{or} \quad \left.\delta u'\right|_{x_1}^{x_2} = 0$$

forced boundary conditions

natural boundary conditions.

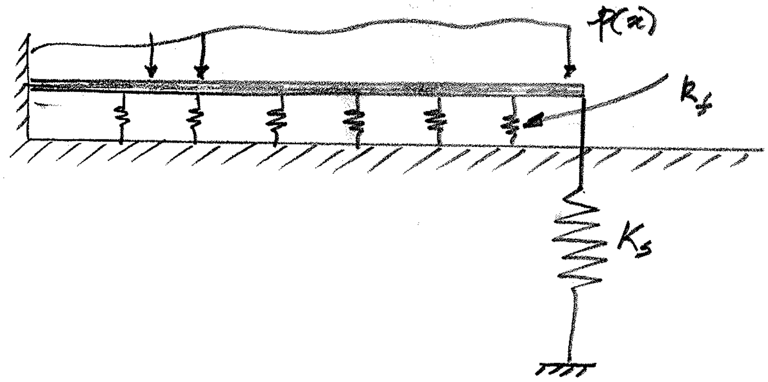
Either the natural or the free boundary conditions need to be satisfied at each end.

- CALCULUS OF VARIATIONS has produced one governing differential equation and four boundary conditions.

- When applied to the integral of a functional, the COV approach provides a complete description of the problem including the governing differential equation and inherent boundary conditions.

## EULER-LAGRANGE FORMULATION:

Example: Beam on Elastic Foundation




Beam with one end rigidly supported and the other end propped on an elastic spring

Geometric or forced boundary condition $\begin{cases} u(x=0)=0 \\ u'(x=0)=0 \end{cases}$

$$\Pi = U - W_e$$

$$= \int_0^L \frac{1}{2} EI(u'')^2 dx + \int_0^L \frac{k_f u^2}{2} dx + \frac{K_s}{2} u^2(L) + \int_0^L p(x)\, u \, dx$$

where $k_f$ is the subgrade reaction spring constant per unit length L and $K_s$ is the spring constant for flexible end support.

$\therefore$ First variation of $\pi$ $\quad \delta\pi = \int_0^L \delta F \, dx$

$$F = \frac{1}{2} EI(u'')^2 + \frac{k_f u^2}{2} - p u + \frac{K_s}{2} u(L)^2$$

$$\therefore \delta F = \frac{\partial F}{\partial u} \cdot \delta u + \frac{\partial F}{\partial u'} \cdot \delta u' + \frac{\partial F}{\partial u''} \cdot \delta u''$$

$$= \frac{k_f}{\cancel{2}} \times \cancel{2} u \times \delta u + \frac{K_s}{\cancel{2}} \cancel{2} u(L) \times \delta u(L) - p \times \delta u + \frac{1}{\cancel{2}} EI \times \cancel{2} (u'') \times \delta u''$$

$$\therefore \int_0^L \delta F \, dx = \int_0^L (k_f u - p) \delta u + \int_0^L EI u'' \delta u'' + K_s u(L) \times \delta u(L)$$

Integrate by parts (the second term):

$$\int_0^L EI u'' \times \delta u'' = EI u'' \times \delta u' \Big|_0^L - \int_0^L EI u''' \times \delta u' \, dx$$

$$= EI u'' \times \delta u' \Big|_0^L - EI u''' \cdot \delta u \Big|_0^L + \int_0^L EI u^{IV} \times \delta u \, dx.$$

$$\therefore \delta\pi = \int_0^L \delta F \, dx = \int_0^L \left(EI u^{IV} + k_f u - P\right) \delta u + EI u'' \times \delta u' \Big|_0^L - EI u''' \cdot \delta u \Big|_0^L + K_s u(L) \times \delta u(L)$$

But $\quad u, u' = 0 \quad @ \; x = 0$

$\therefore \; \delta u, \delta u' = 0 \quad @ \; x = 0$

$$\therefore \int_0^L \left(EI\, u^{IV} + k_f u - p\right)\delta u + EI u'' \times \delta u' \Big|_{x=L} - EI u''' \cdot \delta u \Big|_{x=L} + K_s u \cdot \delta u \Big|_{x=L} = 0$$

$\therefore \quad EI\, u^{IV} + K_f u - p = 0 \longrightarrow$ *governing differential equation.*

$$\left.\begin{aligned} EI u'' \times \delta u' \Big|_{x=L} &= 0 \\ (K_s u - EI\, u''')\,\delta u \Big|_{x=L} &= 0 \end{aligned}\right\} \text{boundary conditions}$$

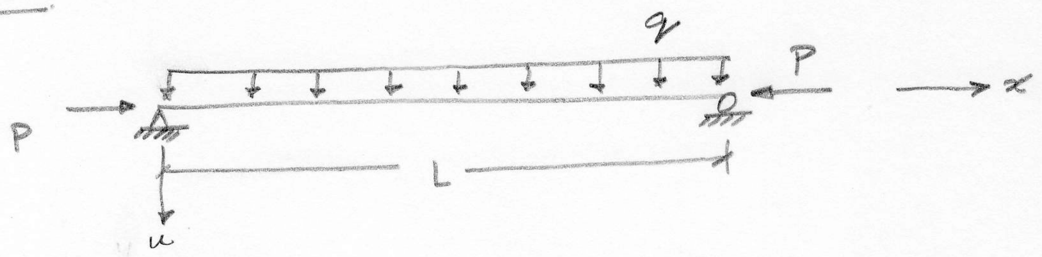
$\therefore$ At $\; x = L$

either $\quad \boxed{EI u'' = 0}$ $\quad$ or $\quad \boxed{\delta u' = 0}$

**and**, either $\quad \boxed{K_s u - EI u''' = 0}$ $\quad$ or $\quad \boxed{\delta u = 0}$

natural $\qquad$ forced

Example:




$$\pi = U - We$$

$$= \frac{1}{2} EI \int_0^L (u'')^2 dx - \frac{1}{2} P \int_0^L (u')^2 dx - q \int_0^L u \, dx$$

$$= \int_0^L \left( \frac{1}{2} EI (u'')^2 - \frac{1}{2} P (u')^2 - qu \right) dx$$

$$\therefore \delta\pi = \int_0^L \delta F \, dx$$

$$\delta F = \frac{\partial F}{\partial u} \cdot \delta u + \frac{\partial F}{\partial u'} \cdot \delta u' + \frac{\partial F}{\partial u''} \cdot \delta u''$$

$$= -q \cdot \delta u - P u' \cdot \delta u' + EI u'' \times \delta u''$$

$$\therefore \int_0^L \delta F \, dx = \int_0^L -q \cdot \delta u - \underline{\int_0^L P u' \cdot \delta u'} + \underline{\int_0^L EI u'' \cdot \delta u''}$$

$$- \int_0^L P u' \cdot \delta u' = -P u' \times \delta u \Big|_0^L + \int_0^L P u'' \cdot \delta u \, dx$$

$$\int_0^L EI u'' \cdot \delta u'' = EI u'' \cdot \delta u' \Big|_0^L - EI u''' \cdot \delta u \Big|_0^L + \int_0^L EI u^{IV} \cdot \delta u \, dx$$

$$\therefore \delta\pi = \int_0^L \delta F \, dx = \int_0^L \left(EI u^{iv} + Pu'' - q\right) \delta u \, dx$$

$$+ \left(-Pu' - EI u'''\right) \delta u \Big|_0^L + EI u'' \cdot \delta u' \Big|_0^L$$

If $\delta\pi = 0 \rightarrow$ Equilibrium.

$$\therefore EI u^{iv} + Pu'' - q = 0$$

$\leftarrow$ governing differential equation

$$\boxed{Pu' + EI u''' = 0 \quad \text{or} \quad \delta u = 0 \quad @ \; x = 0 \; \& \; x = L.}$$

Boundary condition 1

$$\boxed{EI u'' = 0 \quad \text{or} \quad \delta u' = 0 \quad @ \; x = 0 \; \& \; x = L.}$$

Boundary condition 2