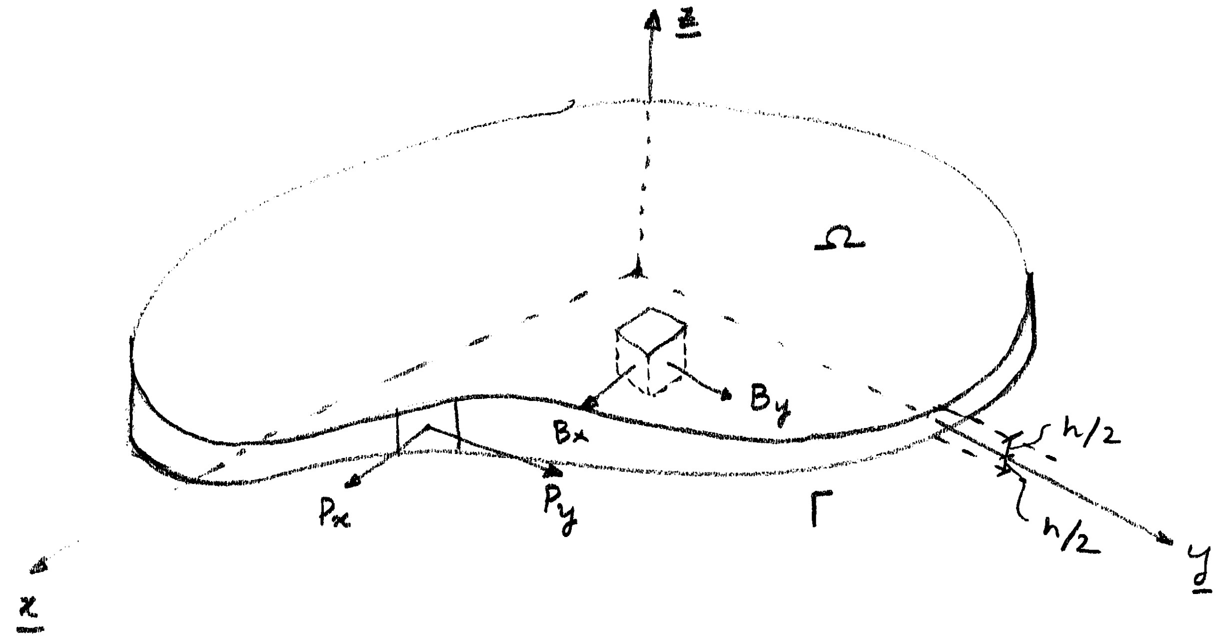
EXAMPLE: CONSIDER A PLANAR (2D) PROBLEM:



$\Omega \rightarrow$ Region

$\Gamma \rightarrow$ Boundary

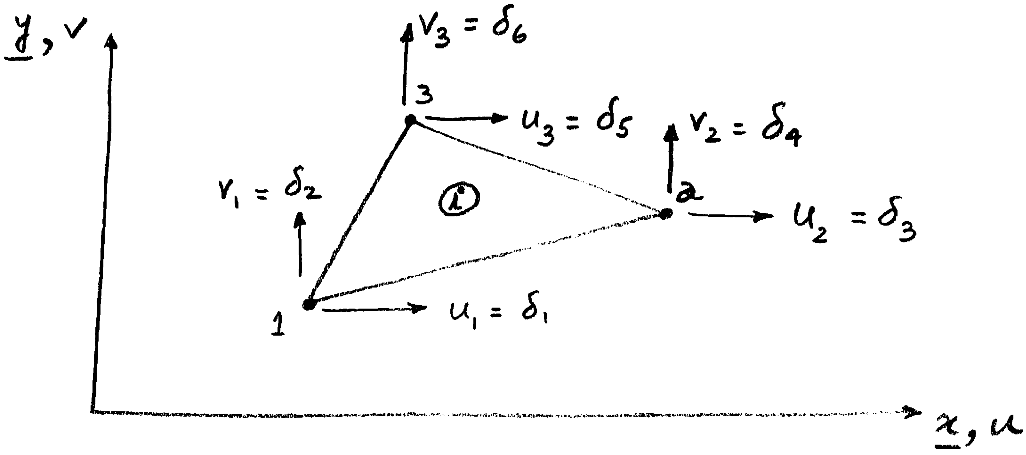
B_x and $B_y \rightarrow$ Body Force components in x and y directions

P_x and $P_y \rightarrow$ Boundary force components in x and y directions

$h \rightarrow$ thickness

Q.> What is the simplest discretization of this problem?

A.> Triangles?



- LOCAL & GLOBAL COORDINATE SYSTEMS ARE PARALLEL
- Element nodes are numbered counter-clockwise
- Each element has three nodes
- Each node has two nodal displacements

Element nodal displacement vector $= \{\delta\}_i = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix}_i = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}_i$

- Element has six degrees-of-freedom
- Assume trial functions:

$$\boxed{\begin{aligned} u_i^*(x,y) &= a_1 + a_2 x + a_3 y \\ v_i^*(x,y) &= a_4 + a_5 x + a_6 y \end{aligned}}$$

where super script * has been added that this trial function is approximate

$a_i \rightarrow$ generalized coordinates

$$\therefore \left\{ \phi^*(x,y) \right\}_i = \begin{Bmatrix} u_i^*(x,y) \\ v_i^*(x,y) \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

$$= [\bar{N}]_i \{a\}_i$$

where, $$[\bar{N}]_i = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix}$$

is called the element "shape function matrix" with respect to the element generalized coordinates vector $\{a\}_i$

$\left\{ \phi^*(x,y) \right\}_i \longrightarrow$ "displacement field" for the element

STRAIN - DISPLACEMENT RELATIONS.

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad ; \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad ; \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Strain in x-dir^n — Strain in y-dir^n — Shear strain in x-y dir^n

$$\therefore \quad \varepsilon_x = a_2 \quad ; \quad \varepsilon_y = a_6 \quad ; \quad \gamma_{xy} = a_3 + a_5$$

since,

$$u_i^*(x,y) = a_1 + a_2 x + a_3 y$$
$$v_i^*(x,y) = a_4 + a_5 x + a_6 y$$

- NOTE THAT THE STRAIN COMPONENTS ARE CONSTANT.

If the coordinates of

$1 \longrightarrow (x_1, y_1)$

$2 \longrightarrow (x_2, y_2)$

$3 \longrightarrow (x_3, y_3)$

$$\therefore \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix}_i \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}_i$$

- If we take the local coordinate system ORIGIN at node 1 and specify the coordinates of element nodes 2 & 3 w.r.t. node 1, then $\boxed{x_1 = 0,\ y_1 = 0}$

$$\therefore \quad \{\delta\}_i = [A]_i \{a\}_i$$

where,
$$[A]_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix}_i$$

- Inverting this relationship gives

$$\{a\}_i = [A]_i^{-1} \{\delta\}_i$$

where,
$$[A]_i^{-1} = \frac{1}{\Delta} \begin{bmatrix} \Delta & 0 & 0 & 0 & 0 & 0 \\ y_2 - y_3 & 0 & y_3 & 0 & -y_2 & 0 \\ x_3 - x_2 & 0 & -x_3 & 0 & x_2 & 0 \\ 0 & \Delta & 0 & 0 & 0 & 0 \\ 0 & y_2 - y_3 & 0 & y_3 & 0 & -y_2 \\ 0 & x_3 - x_2 & 0 & -x_3 & 0 & x_2 \end{bmatrix}_i$$

in which $\Delta = x_2 y_3 - x_3 y_2$

$= 2$ (area of elemental triangle)

- Substituting this into the element displacement field

$$\left\{\phi^*(x,y)\right\}_i = \begin{Bmatrix} u_i^*(x,y) \\ v_i^*(x,y) \end{Bmatrix} = [\bar{N}]_i \{a\}_i$$

$$= \underbrace{[\bar{N}]_i\ [A]_i^{-1}}\ \{\delta\}_i$$

$$= [N]_i \{\delta\}_i$$

where $[N]_i = [\bar{N}]_i \{A\}_i^{-1}$ $\longrightarrow$ Shape functions w.r.t. nodal displacements

- <u>ELEMENT STRAIN VECTOR</u>

$$\{\varepsilon\}_i = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_i = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \left\{\phi^*(x,y)\right\}_i$$

$$= [\bar{B}]_i\ [A]_i^{-1} \{\delta\}_i$$

$$= [B]_i \{\delta\}_i$$

where $$[\bar{B}]_i = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix}$$

$$\therefore \quad [\bar{B}]_i = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Strain - displacement transformation matrix $[B]_i$

$$[B]_i = [\bar{B}]_i \, [A]_i^{-1}$$

- STRESS - STRAIN RELATIONS:

$$\{\sigma\}_i = [D]_i \{\varepsilon_i\} = [D]_i \, [\bar{B}]_i \, [A]_i^{-1} \{\delta\}_i$$

$$= [D]\,[B]\,\{\delta\}_i$$

$$= [DB]_i \, \{\delta\}_i$$

Stress - displacement transformation matrix $[DB]_i$

$$[DB]_i = [D]_i \, [\bar{B}]_i \, [A]_i^{-1}$$

where $[D]_i$ is the elasticity matrix of the material

For a 2D isotropic material $[D_i] = \mu \begin{bmatrix} 1 & D_{12} & 0 \\ D_{12} & 1 & 0 \\ 0 & 0 & D_{33} \end{bmatrix}$

$$[D]_i = \mu \begin{bmatrix} 1 & D_{12} & 0 \\ D_{12} & 1 & 0 \\ 0 & 0 & D_{33} \end{bmatrix}$$

where, μ, D_{12}, and D_{33} depend on whether we have a plane strain or plane stress

COEFFICIENTS OF ELASTICITY MATRIX

	Plane Strain	Plane Stress
μ	$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$	$\frac{E}{1-\nu^2}$
D_{12}	$\frac{\nu}{1-\nu}$	ν
D_{13}	$\frac{1-2\nu}{2(1-\nu)}$	$\frac{1-\nu}{2}$

- Substituting the elements of matrices $[D]_i$ and $[\bar{B}]_i$

$$[\sigma]_i = \begin{bmatrix} 0 & \mu & 0 & 0 & 0 & \mu D_{12} \\ 0 & \mu D_{12} & 0 & 0 & 0 & \mu \\ 0 & 0 & \mu D_{33} & 0 & \mu D_{33} & 0 \end{bmatrix} [A]_i^{-1} \{\delta\}_i$$

- As explained in the next section, we can formulate the total potential energy expression for the element, and then minimize it with respect to $\{\delta\}_i$. This will give the generalized element stiffness matrix $[\bar{K}]_i$ as

$$[\bar{K}]_i = \iiint_{V_i} [\bar{B}]_i^T [D]_i [\bar{B}]_i \, dV_i$$

For the planar element

$$[\bar{K}]_i = h \iint_{A_i} [\bar{B}]_i^T \, [D]_i \, [\bar{B}]_i \, dA_i$$

After substituting and solving:

$$[\bar{K}]_i = \frac{h\,\Delta\mu}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & D_{12} \\ 0 & 0 & D_{33} & 0 & D_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_{33} & 0 & D_{33} & 0 \\ 0 & D_{12} & 0 & 0 & 0 & 1 \end{bmatrix}$$

- Finally, the element stiffness matrix $[K]_i$

$$[K]_i = \left([A]_i^{-1}\right)^T \, [\bar{K}]_i \, [A]_i^{-1}$$

CE 595 : FINITE ELEMENT METHOD

DERIVATION OF THE ELEMENT EQUILIBRIUM EQUATION:

- The element stiffness equilibrium equation can be derived by using the method of virtual work, or the minimum potential energy principle (Rayleigh-Ritz)

- (i) Using Method of Virtual Work:

  - Introduce a set of arbitrary nodal displacements $\{\delta u\}$

    The real external loads acting on the element $\{F\}$ are forced to move through this displacement.

  - Then by the principle of virtual work for a typical element i

$$\delta W_e = \delta U_i \qquad \text{or} \qquad \delta W_e - \delta U_i = 0$$

where δW_e = external virtual work done by real loads

δU_i = internal virtual work or internal virtual strain energy.

- Note that

$$\{\delta \varepsilon\} = [\bar{B}][A]^{-1}\{\delta u\}$$

$$\{\sigma\} = [D][\bar{B}][A]^{-1}\{u\}$$

$$\therefore \delta U_i = \int_V \{\delta u\}^T \left([A]^{-1}\right)^T [\bar{B}]^T [D] \left([\bar{B}][A]^{-1}\{u\} - \{\varepsilon_0\}\right) dV$$

$$\delta W_e = \{\delta u\}^T \{F\}$$

$$\therefore \int_V \{\delta u\}^T \left([A]^{-1}\right)^T [\bar{B}]^T [D] \left([\bar{B}][A]^{-1}\{u\}\right) dV$$

$$= \{\delta u\}^T \{F\}$$

Since $\{\delta u\}^T \longrightarrow$ arbitrary virtual displacement

$$\underline{\int_V \left([A]^{-1}\right)^T [\bar{B}]^T [D] [\bar{B}] [A]^{-1} dV \{u\} = \{F\}}$$

Remember that $[A]^{-1}$ is full of coordinates

$\therefore$ constant

$$\therefore \left([A]^{-1}\right)^T \underbrace{\int_V [\bar{B}]^T [D] [\bar{B}] dV}_{} [A]^{-1}$$

$$\downarrow$$

$[\bar{K}]_i \longrightarrow$ generalized element stiffness matrix

& $[K]_i = \left([A]^{-1}\right)^T [\bar{K}]_i [A]^{-1} \longrightarrow$ element stiffness matrix

(ii) Using Principle of <u>Minimum Potential Energy</u>.

- The total potential energy π_i of the element can be written as

$$\pi_i = U_i - W_e = \frac{1}{2}\int_{V_i} \{\varepsilon\}_i^T \{\sigma\}_i \, dV_i - \{u\}_i^T \{F\}_i$$

Where, U_i = internal work done or internal strain energy

W_e = external work done by nodal loads.

$$\therefore \{\varepsilon\}_i = [\bar{B}]_i [A]_i^{-1} \{u\}_i$$

$$\{\sigma\}_i = [D]_i \left([\bar{B}]_i [A]_i^{-1} \{u\}_i\right)$$

$$\therefore \pi_i = \frac{1}{2}\left\{ \{u\}_i^T \left([A]_i^{-1}\right)^T \left(\int_{V_i} [\bar{B}]_i^T [D]_i [\bar{B}]_i \, dV_i\right) [A]_i^{-1} \{u\}_i\right.$$

$$- \{u_i\}^T \{F\}_i \; .$$

$$\frac{\partial \pi_i}{\partial \{u\}_i^T} = 0$$

$$\therefore \underbrace{\left([A]_i^{-1}\right)^T \left(\int_{V_i} [\bar{B}]_i^T [D]_i [\bar{B}]_i \, dV\right) [A]_i^{-1}}_{[\bar{K}]_i \;\rightarrow\; \text{same as before}} \{u\}_i = \{F\}_i$$

FORMULATION OF ELEMENT LOAD VECTOR

A typical element would in general be subjected to:

(1) Distributed body Forces

&

(2) Distributed boundary Forces

(1) DISTRIBUTED BODY FORCES : such as those due to self-weight. These are defined as loads acting on a unit volume of the material within the element with directions corresponding to those of displacements at the point.

- Let, $\{X\}_i = \begin{Bmatrix} B_x(x,y) \\ B_y(x,y) \end{Bmatrix}_i$ denote the forces per unit volume of the material.

- Assume that the body force components B_x and B_y are constant within the element, since the stress and strain are also constant within.

$$\therefore \quad \{X\}_i = \begin{Bmatrix} B_x \\ B_y \end{Bmatrix}_i$$

- When we establish equilibrium for the finite element, the right hand side:

W_e = work done by external forces

δW_e = virtual work done by external forces

We = external work done

$$\text{displacement field} = \left\{ \phi(x,y) \right\}_i = \left\{ \begin{matrix} u(x,y) \\ v(x,y) \end{matrix} \right\}_i = \underbrace{[\bar{N}]_i}_{\text{shape function}} [A]_i^{-1} \underbrace{\{u\}_i}_{\text{nodal displacements}}$$

$$\therefore \text{ External work done} = We = \int_V \underbrace{\{\phi\}_i^T}_{\substack{\text{displacement field} \\ \text{transposed}}} \underbrace{\{X\}_i}_{\text{body forces}} \, dV$$

$$\therefore We = \int_V \{u\}_i^T \left([A]_i^{-1}\right)^T [\bar{N}]_i^T \left\{ \begin{matrix} B_x \\ B_y \end{matrix} \right\}_i dV$$

$$We = \{u\}_i^T \left([A]_i^{-1}\right)^T \int_V [\bar{N}]_i^T \{X\}_i \, dV$$

$$\pi = U - We$$

$$\therefore \frac{\partial \pi}{\partial \{u\}_i} = 0 \longrightarrow \text{Equilibrium}$$

$$\therefore \text{ Right hand side} = \frac{\partial We}{\partial \{u\}_i} = \left([A]_i^{-1}\right)^T \int_V [\bar{N}]_i^T \{X\}_i \, dV$$

$$\therefore \{F\} = \left([A]_i^{-1}\right)^T \int_V [\bar{N}]_i^T \{X\}_i \, dV$$

- <u>Define</u> the generalized element body force vector $\{\bar{Q}\}_i$

$$\{\bar{Q}\}_i = \iiint_{V_i} [\bar{N}]_i^T \{x\}_i \, dV_i = h \iint_{A_i} [\bar{N}]_i^T \{x\}_i \, dA_i$$

Where, A_i represents the area of the i^{th} element.

On carrying the multiplication and integration over A_i, we arrive at the vector $\{\bar{Q}\}_i$ in the form:

$$\{\bar{Q}\}_i = \frac{h\Delta}{2} \begin{Bmatrix} B_x \\ B_x I_2 \\ B_x I_3 \\ B_y \\ B_y I_2 \\ B_y I_3 \end{Bmatrix} \longrightarrow \left(\begin{matrix} \text{Remember } B_x \\ \text{\& } B_y \text{ are constants} \\ \text{assumed} \end{matrix} \right)$$

in which,

$$I_1 = \iint_{A_i} dA_i = \frac{1}{2}(x_2 y_3 - x_3 y_2) = \frac{\Delta}{2}$$

$$I_2 = \frac{1}{I_1} \iint_{A_i} x \, dA_i = \frac{1}{3}(x_2 + x_3)$$

$$I_3 = \frac{1}{I_1} \iint_{A_i} y \, dA_i = \frac{1}{3}(y_2 + y_3)$$

$\therefore$ The element body force load vector is given by:

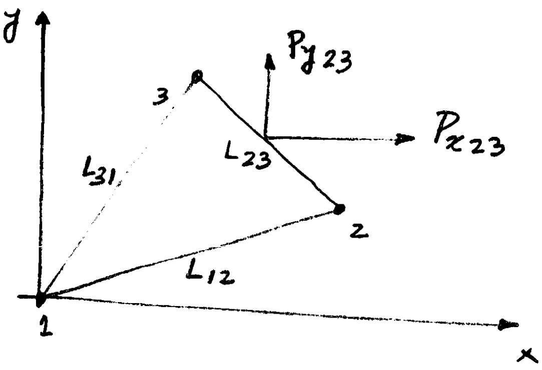
$$\boxed{\{Q\}_i = \left([A]_i^{-1}\right)^T \{\bar{Q}\}_i} \qquad \{Q\}_i = \frac{h\Delta}{6} \begin{Bmatrix} B_x \\ B_y \\ B_x \\ B_y \\ B_x \\ B_y \end{Bmatrix}$$

(2) DISTRIBUTED BOUNDARY FORCES:

i.e., the external boundary tractions (or loads) acting on the elemental boundaries.

$$\{P\}_i = \begin{Bmatrix} P_x \\ P_y \end{Bmatrix}_{23}$$

per unit surface area 23
in x- and y- directions



- Let us consider edge joining nodes 2 and 3 of the element has boundary forces only.

Thus, we can write the boundary force vector as:

$$\{P\}_{23} = \begin{Bmatrix} P_{x23} \\ P_{y23} \end{Bmatrix}$$

- The displacement field $\{\phi(x,y)\}_i = [\bar{N}]_i [A]_i^{-1} \{u\}_i$

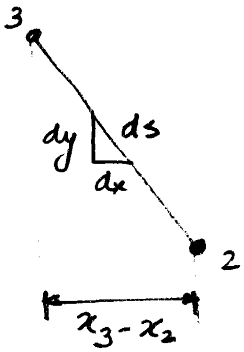
Work done by external forces = W_e

$$= \int_S \{u\}_i^T \left([A]_i^{-1}\right)^T [\bar{N}]_i^T \begin{Bmatrix} P_{x23} \\ P_{y23} \end{Bmatrix} h \cdot ds_i$$

$$= \{u\}_i^T \left([A]_i^{-1}\right)^T \int_S [\bar{N}]_i^T \begin{Bmatrix} P_{x23} \\ P_{y23} \end{Bmatrix}_i \cdot h\, ds_i$$

where the differential length ds_i is related to dx by

$$ds_i = \frac{L_{23}}{x_3 - x_2} dx$$



Substituting $[\bar{N}]_i$ and ds_i
and carrying out the multiplication
and integration over dx (between x_2 and x_3)

$$\{\bar{P}\}_{i23} = h\, L_{23} \begin{Bmatrix} P_{x23} \\ \frac{1}{2} P_{y23}(x_3 + x_2) \\ \frac{1}{2} P_{x23}(y_3 + y_2) \\ P_{y23} \\ \frac{1}{2} P_{x23}(x_3 + x_2) \\ \frac{1}{2} P_{y23}(y_3 + y_2) \end{Bmatrix}$$

$$\therefore \{P\}_{i23} = [A]_i^{-1} \{\bar{P}\}_{i23}$$

- Similar expressions can be derived for boundary forces along edges L_{12} and L_{31}. We just change the indices 23 to 12 and 31, respectively
- Thus, if we have boundary forces along edges L_{12} and L_{31}, then to obtain $\{P\}_i$, we have to perform a summation

$$\left\{\overline{P}\right\}_i = h\,L_{23} \begin{Bmatrix} P_{x23} \\ \frac{1}{2} P_{y23}(x_3+x_2) \\ \frac{1}{2} P_{x23}(y_3+y_2) \\ P_{y23} \\ \frac{1}{2} P_{x23}(x_3+x_2) \\ +\frac{1}{2} P_{y23}(y_3+y_2) \end{Bmatrix} + h\,L_{12} \begin{Bmatrix} \\ \\ \\ \end{Bmatrix} + h\,L_{31} \begin{Bmatrix} \\ \\ \\ \end{Bmatrix}$$

corresponding terms in 12 & 31

Finally, $\left\{P\right\}_i = \left[A\right]_i^{-1} \left\{\overline{P}\right\}_i$

Total Force vector

$$\left\{F\right\}_i = \left\{Q\right\}_i + \left\{P\right\}_i$$

- GENERAL STEPS OF ANALYSIS

(1) Formulation of element stiffness matrices and load vectors

(2) Assemblage of system stiffness matrix and system load vector

(3) Computation of nodal displacements

(4) Computation of element stresses and strains.

- VARIOUS STEPS OF ELEMENT STIFFNESS MATRIX FORMULATION

(1) Choose the trial "displacement field"

$$\{\phi(x,y)\}_i = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}_i = [\bar{N}]_i \{a\}_i$$

where, $\{a\}_i \rightarrow$ generalized coordinates

$[\bar{N}]_i \rightarrow$ Shape function matrix w.r.t. $\{a\}_i$

(2) Develop relationship between generalized coordinate vector and nodal displacement vector $\{\delta\}_i$

$$\{\delta\}_i = [A]_i \{a\}_i$$

Invert to get :

$$\{a\}_i = [A]_i^{-1} \{\delta\}_i$$

Substituting in (1) gives:

$$\{\phi(x,y)\}_i = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}_i = [\bar{N}]_i [A]_i^{-1} \{\delta\}_i$$

$$= [N]_i \{\delta\}_i$$

(3) Express strains in terms of element nodal displacements

$$\{\varepsilon\}_i = [\bar{B}]_i \{a\}_i = [B]\{\delta\}_i$$

strain-displacement transformation matrices

(4) Express stresses in terms of the element nodal displacements

$$\{\sigma\}_i = [D]_i [B]_i \{\delta\}_i = [DB]_i \{\delta\}_i$$

stress-displacement transformation matrix

(5) Using either Principal of Virtual Work or Minimization of Potential Energy, formulate the generalized element stiffness matrix

$$[\bar{K}]_i = \iiint_V [\bar{B}]_i^T [D]_i [\bar{B}] \, dV$$

(6) Formulate the element stiffness matrix

$$[K]_i = \left([A]_i^{-1}\right)^T [\bar{K}]_i [A]_i^{-1}$$

$$[K]_i = \left([A]_i^{-1}\right)^T \left(\iiint_V [\bar{B}]_i^T [D]_i [\bar{B}] \, dV\right) [A]_i^{-1}$$

(7) Formulate the <u>generalized</u> element load vector $\{\bar{F}\}_i$

$$\{\bar{F}\}_i = \iiint_V [\bar{N}]_i \{q(x,y)\}_i \, dV \quad \text{for body forces}$$

and $$\{\bar{F}\}_i = \iiint_V [\bar{N}]_i^T \{p(x,y)\}_i \, dA \quad \text{for boundary distributed forces}$$

Where, A = area of the element boundary on which $p(x,y)_i$ so the integration has to be carried over all 3 edges.

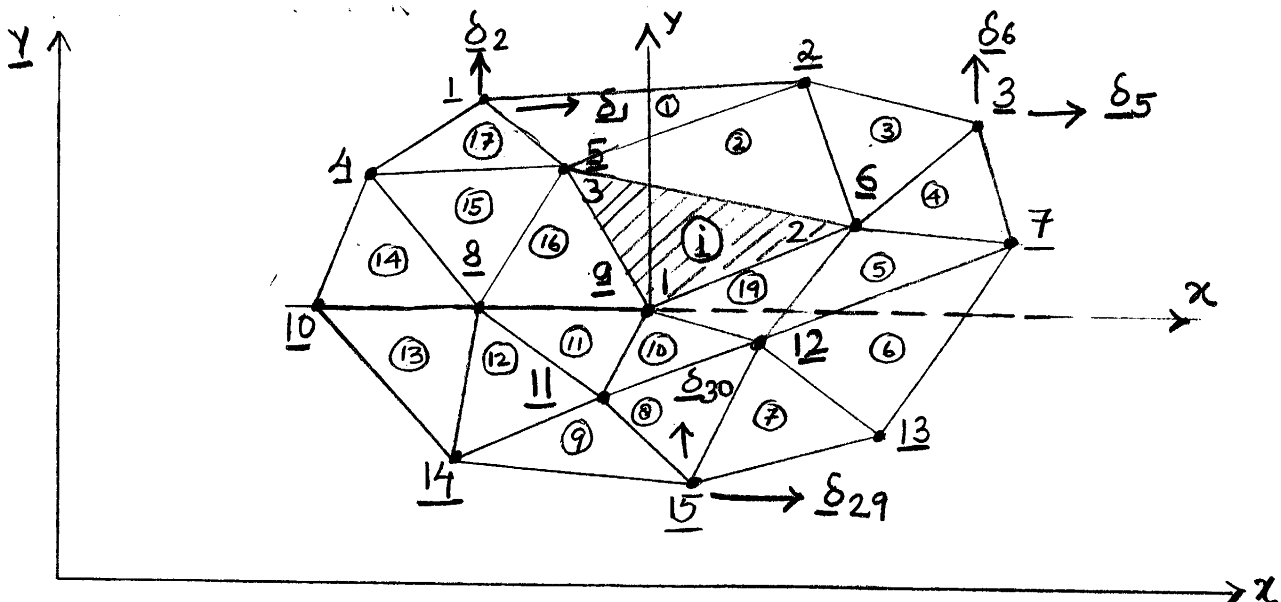
(8) Formulate the element load vector $\{F\}_i$

$$\{F\}_i = \left([A]_i^{-1}\right)^T \{\bar{F}\}_i$$

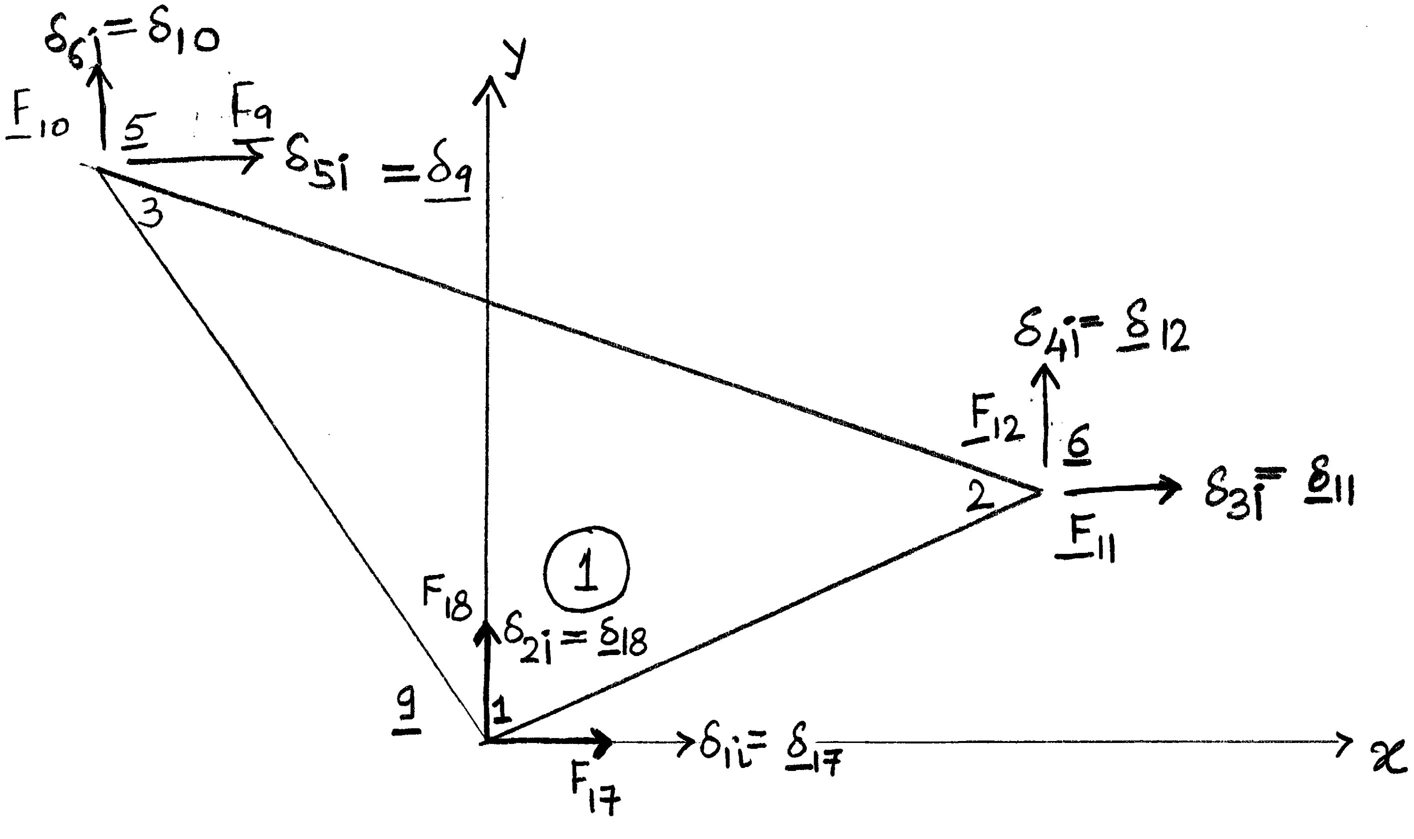
- Thus, to obtain the element stiffness matrix $[K]_i$ and element load vector $\{F\}_i$, we need to know the following element matrices : $[\bar{N}]_i$, $[A]_i$, $[B]_i$, $[D]_i$ and $[\bar{F}]_i$

CE595

ASSEMBLING THE SYSTEM STIFFNESS MATRIX



Finite Element Idealization of a Plane Problem



Element and Structural Numbering Systems

- As shown in the figure, the correspondence between the element and the system nodal displacements is as follows :

$$\begin{Bmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \\ \delta_{4i} \\ \delta_{5i} \\ \delta_{6i} \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \underline{\delta}_{17} \\ \underline{\delta}_{18} \\ \underline{\delta}_{11} \\ \underline{\delta}_{12} \\ \underline{\delta}_{9} \\ \underline{\delta}_{10} \end{Bmatrix}$$

- Between the element and structural nodes.

$$\begin{Bmatrix} F_{1i} \\ F_{2i} \\ F_{3i} \\ F_{4i} \\ F_{5i} \\ F_{6i} \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \underline{F}_{17} \\ \underline{F}_{18} \\ \underline{F}_{11} \\ \underline{F}_{12} \\ \underline{F}_{9} \\ \underline{F}_{10} \end{Bmatrix}$$

- Generally, the ith element loads make up only a part of the total structure loads.

- The total structural load at a node will include contributions from all elements including the node.

- There are fifteen structural nodes $\underline{n_s}$ and three element nodes $\underline{n_e}$.

- The structural nodal displacement numbering system

$$\left\{\begin{matrix} \underline{\delta_1} \\ \underline{\delta_2} \\ \underline{\delta_3} \\ \vdots \\ \underline{\delta_{2ns}} \end{matrix}\right\} \qquad\qquad \left\{\begin{matrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \\ \vdots \\ \delta_{6i} \end{matrix}\right\}$$

- The element nodal displacement numbering system

- For this problem, we have fifteen structural nodes with total thirty D.O.F.

- Each element has three nodes and a total of six D.O.F.

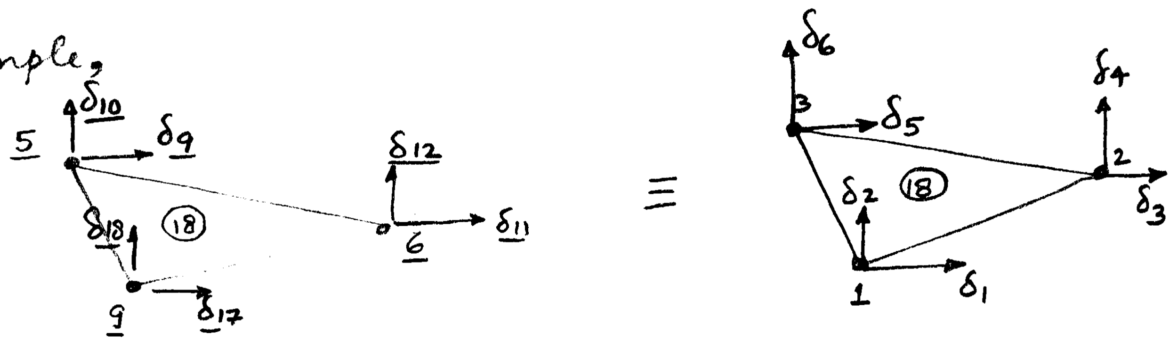
- For example for the ith element, we have a correspondence between element nodes (1, 2, 3) and system nodes ($\underline{9}$, $\underline{6}$, $\underline{5}$).

- ELEMENT CONNECTIVITY

ELEMENT NO	NODE 1	δ_{1i}	δ_{2i}	NODE 2	δ_{3i}	δ_{4i}	NODE 3	δ_{5i}	δ_{6i}
1	1	δ_1	δ_2	5	δ_9	δ_{10}	2	δ_3	δ_4
2	2	δ_3	δ_4	5	δ_9	δ_{10}	6	δ_{11}	δ_{12}
3	2	δ_3	δ_4	6	δ_{11}	δ_{12}	3	δ_5	δ_6
4	3	δ_5	δ_6	6	δ_{11}	δ_{12}	7	δ_{13}	δ_{14}
5	6	δ_{11}	δ_{12}	12	δ_{23}	δ_{24}	7	δ_{13}	δ_{14}
6	7	δ_{13}	δ_{14}	12	δ_{23}	δ_{24}	13	δ_{25}	δ_{26}
7	12	δ_{23}	δ_{24}	15	δ_{29}	δ_{30}	13	δ_{25}	δ_{26}
8	12	δ_{23}	δ_{24}	11	δ_{21}	δ_{22}	15	δ_{29}	δ_{30}
9	11	δ_{21}	δ_{22}	14	δ_{27}	δ_{28}	15	δ_{29}	δ_{30}
10	9	δ_{17}	δ_{18}	11	δ_{21}	δ_{22}	12	δ_{23}	δ_{24}
11	9	δ_{17}	δ_{18}	8	δ_{15}	δ_{16}	11	δ_{21}	δ_{22}
12	8	δ_{15}	δ_{16}	14	δ_{27}	δ_{28}	11	δ_{21}	δ_{22}

AND SO ON...

For example,



Element No.	1	2	3	4	5	6	← local d.o.f.
	9		6		5		
(18)	17	18	11	12	9	10	← System d.o.f

The element has behavior $\{F\}_i = [K]\{\delta\}_i$

where $[K] = \begin{bmatrix} K_{11} & \cdots & K_{16} \\ \vdots & & \vdots \\ K_{61} & \cdots & K_{66} \end{bmatrix}$ relating local forces to local displacement

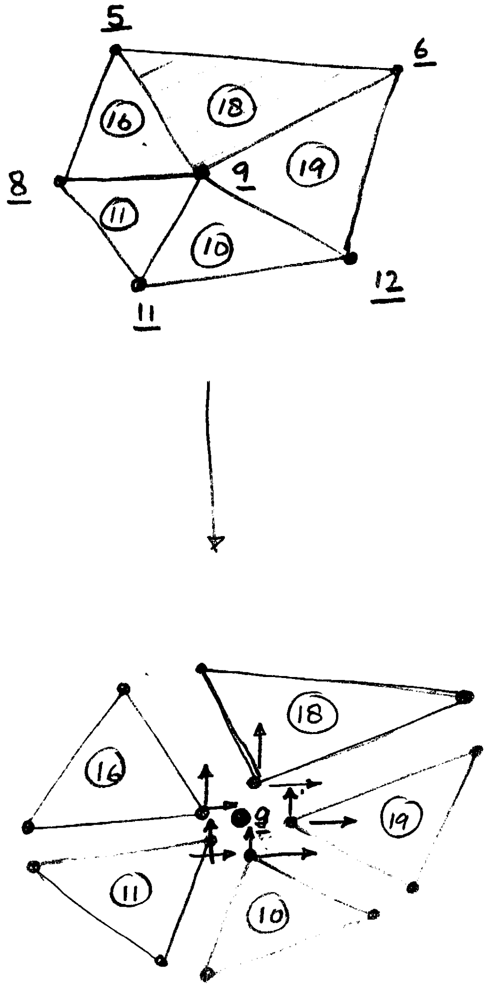
$K_{i,j}\Big|_{local} \xrightarrow{\text{goes to}}$ global stiffness matrix appears in the term $K_{m,n}$

where, m is the global d.o.f $\equiv$ local d.o.f i

n is the global d.o.f $\equiv$ local d.o.f. j

i.e. $K_{1,5}\Big|_{local} \longrightarrow K_{17,9}\Big|_{global}$

$K_{2,6}\Big|_{local} \longrightarrow K_{18,10}\Big|_{global}$



$$F_{\underline{9}x} = F_{\underline{9}x}^{(18)} + F_{\underline{9}x}^{(19)} + F_{\underline{9}x}^{(10)} + F_{\underline{9}x}^{(11)} + F_{\underline{9}x}^{(16)}$$

```
FOR e = 1, numel              <-- sum over all elements
  For i = 1, numdof(e)        <-- sum over all d.o.f
    For j = 1, numdof(e)      --> local d.o.f.
      ii := map(e, i)         } --> assign ii from map
      jj := map(e, j)         }     element connectivity array
      K(ii, jj) := K(ii, jj) + k(e, i, j)    --> Assembly Step
    CONTINUE
    F(ii) := F(ii) + f(e, i)
  CONTINUE
CONTINUE
```