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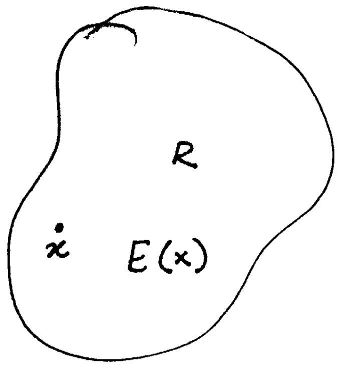
BASIC CONCEPTS OF APPROXIMATE ANALYSIS



R = Region or domain

B = Boundary

x = Point in region R

$E(x)$ = Exact response or solution to a given input

$L(x)$ = Differential Equations

APPROXIMATION: Assume that $E(x)$ can be approximated by a linear combination of suitably chosen coordinate functions : $\phi_1(x)$, $\phi_2(x)$, ... $\phi_n(x)$, i.e.,

$$E(x) \cong F(x, A) \longrightarrow (1)$$

Where,

$$F(x, A) = a_1 \phi_1(x) + a_2 \phi_2(x) + \cdots\cdots + a_n \phi_n(x) \longrightarrow (2)$$

In Eq. (2), A stands for a collection of parameters $(a_1, a_2, \ldots\ldots a_n)$ which are to be determined.

- $F(x, A)$ has to satisfy certain admissibility and compatibility conditions to approximate exact solution.
- And, we have to satisfy in some approximate sense:
  (1) the differential equation $L(E) = 0$
  (2) the boundary conditions $B(E) = 0$

CHOICE OF $F(x, A)$

- Two important factors
  - (a) Limit of Error: Measure (Norm) or tolerance of error permissible for the problem.
  - (b) Method of Approximation:
    - Point wise approximation
    - "r" points in the domain which satisfy $L(E)=0$
    - "s" points on the boundary which satisfy $B(E)=0$
    - Let, $r+s=n$

      ∴ End up with n linear equations which involve $a_1, a_2, \ldots\ldots a_n$
    - Solution of Equations give us (A)

- Choose $F(x, A)$ in the following ways:
  - (a) The differential equation is satisfied exactly in R and the a_i are selected to make F fit the boundary condition on B in the "best" way in some sense
    - Called the Boundary Method.
    - ϕ_i are chosen so that the trial function satisfies the differential equation at all interior points $(L(E)=0)$
    - Choose n points on the boundary such that $B(E)=0$

i.e., select $a_1, a_2 \ldots\ldots a_n$ such that $B(E) = 0$

- Reduced the size of the problem from 3D → 2D & 2D → 1D

(b) The boundary conditions are satisfied exactly on B and the a_i are selected so that F satisfies the differential equation in the interior R. in some "best" sense (norm).

- called INTERIOR METHOD
- e.g. include Rayleigh-Ritz, Methods of Weighted residuals (collocation, subdomain, Galerkin, least squares)

(c) Neither the differential equation is satisfied exactly in R nor the boundary conditions are satisfied exactly on B.

- The a_i are chosen to satisfy the differential equation at "r" points. and the boundary condition at "s" points.
- Called MIXED METHOD
- Eg. FINITE ELEMENT, FINITE DIFFERENCE } → METHODS

1.2 INTERIOR METHODS:

Method of Weighted Residuals

- Consider the governing differential equation for a problem

$$L(y) - g(x) = 0 \longrightarrow (1)$$

where, $y = f(x)$ is to be found (i.e, the response)

- Let the boundary conditions be given by

$$B(y) = 0 \longrightarrow (2)$$

- Note that $L(y)$ can be linear or nonlinear

e.g. $L(y) = C_1 y'' + C_2 y' + C_3 y$

If C_i are $f(y)$ → then $L(y)$ is nonlinear

- Assume an approximate trial solution y^* that satisfies the boundary conditions identically

$$y^*(x, A) = y^*(x, a_1, a_2, \ldots a_n) = \sum_{i=1}^{n} a_i \phi_i(x) \longrightarrow (3)$$

- When substituted into governing diff. eq. get differential residual

$$R(x) = L(y^*) - g(x) \longrightarrow (4)$$

- This residual vanishes for $y^* = y$

- Adjust a_i so that $R(x)$ stays close to zero in the domain.

$\therefore$ the integral of residuals $R(x)$ approximately weighted by function $W_i(x)$ vanishes over region.
i.e., for $x = [0, L]$

$$\int_0^L W_i(x)\, R(x)\, dx = 0 \longrightarrow (5)$$

for $i = 1, 2, \ldots n$

- For each i, the solution obtained for the integral gives a set of n analytical expressions that can be solved for a_i

- Depending on the weightage system used
  i) collocation method
  ii) subdomain method
  iii) least square method
  iv) Galerkin's method

1.2.1 Collocation method

- Select as many points in the interior region $[0, L]$ as parameters a_i
- Adjust the values of a_i to make the residual zero at these points.
- Thus the weighing function is:

$$W_i(x) = \delta(x - x_i) \quad \text{for} \quad i = 1, 2 \dots n$$

where, $\delta(x - x_i)$ is the delta function that satisfies

$$\int_{-\infty}^{+\infty} \delta(x - x_i)\, R(x)\, dx = R(x_i)$$

- Equation (5) reduces to

$$\int_0^L W_i(x)\, R(x)\, dx = R(x_i) = 0 \quad \text{for } i = 1, 2, \dots n \quad \longrightarrow (7)$$

- Thus, in the collocation method we select as many locations as a_i, and then adjust the values of a_i until the residual vanishes at these locations
- <u>Presumption</u>: residual does not get very far from zero in between the locations

Example:

$$y'' + y + x = 0 \longrightarrow \text{D.E.}$$

$$y(0) = y(1) = 0 \longrightarrow \text{B.C.}$$

- Try a one family parameter approx. sol^n

$$y^*(x, A) = a_1 x(1-x) = a_1 x - a_1 x^2$$

This is an admissible solution since it satisfies the boundary conditions.

- $$y' = a_1 - 2a_1 x$$

  $$y'' = -2a_1$$

  $$\therefore R(x) = -2a_1 + a_1 x(1-x) + x$$

- You can select onlye one point in the region $[0, 1]$.
  - Pick $x = 1/2$

  $\therefore$ residual must vanish here

  $$\therefore R(1/2) = -2a_1 + a_1 \frac{1}{4} + \frac{1}{2} = 0$$

  $$\therefore a_1 = \frac{2}{7} = 0.2857$$

  $\therefore$ solution is $y = \frac{2}{7} x(1-x)$

$\therefore\ y = 0.2857\, x(1-x)$

$\therefore\ y_{max}\Big|_{x=1/2}$

Exact solution:

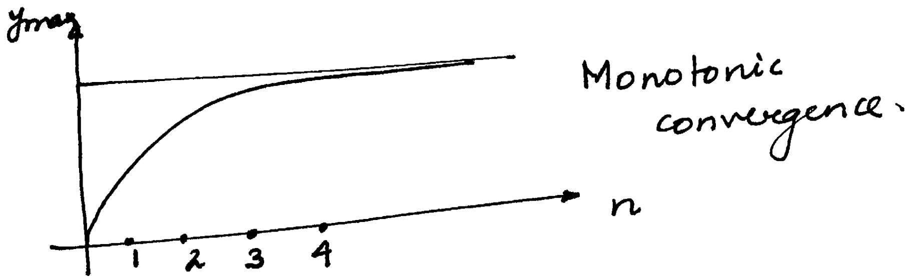
$$y = \frac{\sin x}{\sin 1^\circ} - x$$

y_{max} occurs @ $x = 0.57$

- Try 2 or 3 parameter family approximations

$$y^*(x, A) = a_1 x(1-x) + a_2 x^2 (1-x)$$

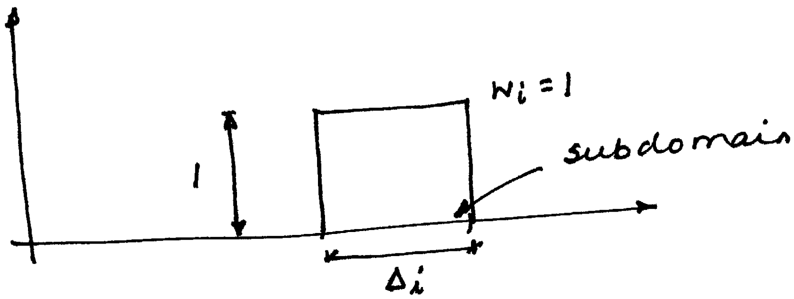
$$y^*(x, A) = a_1 x(1-x) + a_2 x^2 (1-x) + a_3 x^3 (1-x)$$



1.2. SUBDOMAIN METHOD

- Divide the region or domain into many segments or subdomains as there are a_i and choose these parameters so that the average values of the residual over each segment

$$\int_{\Delta i} R(x)\, dx \Big/ \int_{\Delta i} dx = 0$$



or we have

$$W_i(x) = 1 \qquad \forall \; x \text{ in subdomain } i$$

$$W_i(x) = 0 \qquad \forall \; x \text{ outside subdomain } i$$

where $i = 1, 2 \dots . n$

$$\therefore \int_0^L W_i(x)\, R(x)\, dx = \int_{\Delta i} R(x)\, dx = 0$$

$$\forall \; i = 1, 2, \dots . n$$

- Here we have to integrate the residual, which can get complicated. In such cases, use numerical integration.

Example:

$$y'' + y + x = 0 \longrightarrow L(y) = 0$$

$$y(0) = y(1) = 0 \longrightarrow B(y) = 0$$

$$y^*(x, A) = a_1 x(1-x) \longrightarrow \text{Assume.}$$

One parameter a_1

$\therefore$ whole region is one subdomain

$$0 \leq x \leq 1$$

$$R(x) = -2a_1 + a_1 x(1-x) + x$$

$$\therefore \int_0^1 R(x)\, dx = -2a_1 x + a_1 \frac{x^2}{2} - a_1 \frac{x^3}{3} + \frac{x^2}{2} \Big|_0^1 = 0$$

$$\therefore a_1 = \frac{3}{11} = 0.2757$$

$\therefore$ approximate solution is

$$y \approx \frac{3}{11} x(1-x) = 0.2757\, x(1-x)$$

1.2.3 LEAST SQUARES METHOD

- parameter a_i are chosen to minimize the square of the residual over the region.

$$0 \leq I = \int_0^L p(x) R^2(x)\, dx \longrightarrow (1)$$

weightage square error.

For this to be minimum. $\frac{\partial I}{\partial a_1} = 0$ for $i = 1, 2, \dots n$

- Generally $p(x) = 1$

$$\therefore \frac{\partial I}{\partial a_i} = \int_0^L R(x) \cdot \frac{\partial R(x)}{\partial a_i}\, dx = 0 \quad \text{for } i = 1, 2 \dots n$$

$\therefore$ weighing function is $w_i(x) = \frac{\partial R(x)}{\partial a_i}$

Example: Same as before.

$$L(y) = y'' + y + x = 0$$

$$B(y) = y(0) = y(1) = 0$$

Let $y^* = a_1 x(1-x)$

$$\therefore R(x) = -2a_1 + a_1 x(1-x) + x$$

$$\therefore \frac{\partial R}{\partial a_1} = -2 + x(1-x)$$

$$\therefore \int_0^1 \left(-2a_1 + a_1 x(1-x) + x\right)\left(-2 + x(1-x)\right) dx = 0$$

$$\therefore \quad a_1 = 0.2753$$

$$\therefore \quad y = 0.2753\, x(1-x)$$

1.2.4 GALERKIN'S METHOD

- In this method, the weightage function is taken

$$W_i(x) = \phi_i(x) \quad \text{for} \quad i = 1, 2, \ldots n$$

$\longrightarrow$ coordinate functions

- $\therefore$ weighted residual equation is

$$\int_0^L W_i(x) R(x)\, dx = \int_0^L \phi_i(x) R(x)\, dx$$

- $\therefore$ this method requires orthogonality of $R(x)$ with each of the coordinate functions $\phi_i(x)$

EXAMPLE:

$L(y) = y'' + y + x = 0$

$B(y) = y(0) = y(1) = 0$

Let $y^*(x, A) = a_1 x(1-x)$

$\therefore R(x) = -2a_1 + a_1 x(1-x) + x$

Let

$W_1(x) = x(1-x)$

$$\therefore \int_0^L W_i(x)\, R(x)\, dx = \int_0^1 \phi_1(x)\quad R(x)\, dx$$

$$= \int_0^1 \left[-2a_1 + a_1 x(1-x) + x\right] \quad x(1-x)\, dx \quad = 0$$

$$\therefore\ a_1 = \frac{5}{18} = 0.2778$$

approx. solution is $y = \frac{5}{18} x(1-x) = 0.2778\, x(1-x)$

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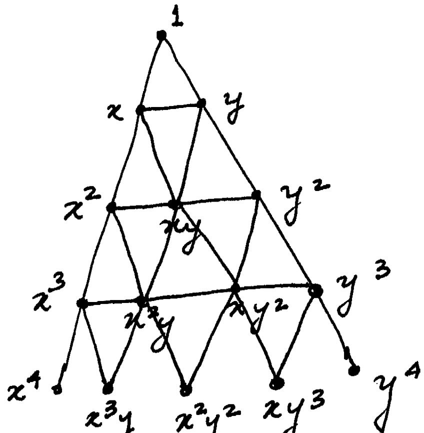
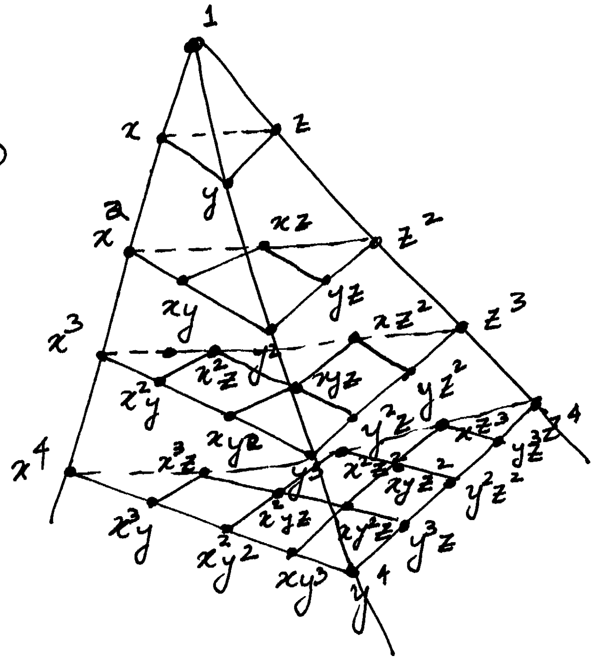
Pascal's triangle:

2D



3D

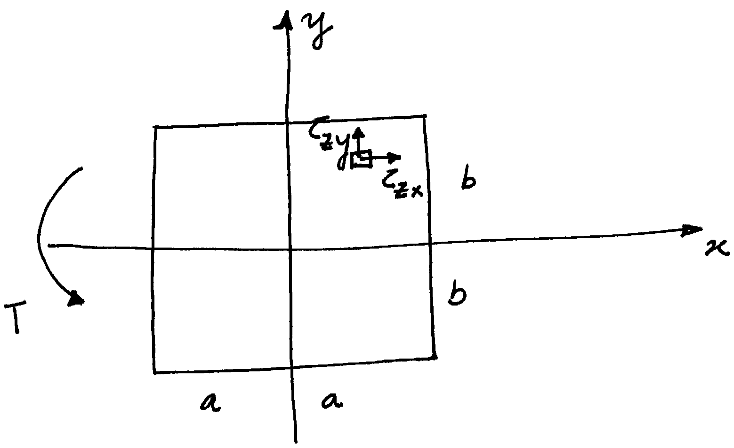


Linear: $a_0 + a_1 x + a_2 y$

Quadratic: $a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2$

Quadratic: $a_0 + a_1 x + a_2 y + a_3 z + a_4 x^2 + a_5 xy + a_6 yz + a_7 z^2 + a_8 xz$

Consider a 2D problem:



Poisson's equation: $-k\nabla^2\phi = f$ on Region R

$\phi = 0$ on Boundary B

For the torsion problem: ϕ = stress function

$f = 2$ (twist per unit length) $= 2\cdot\theta$

$k = \dfrac{1}{G}$

$$\tau_{zx} = G\theta \cdot \frac{\partial \phi}{\partial y}$$

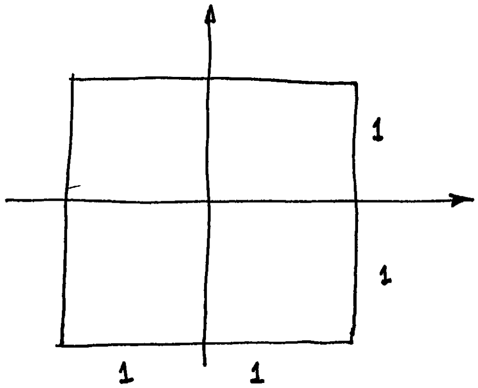
$$\tau_{zy} = G\theta\left(-\frac{\partial \phi}{\partial x}\right)$$

$$\rightarrow \quad \tau = \sqrt{\tau_{zx}^2 + \tau_{zy}^2}$$

τ_{max} also

$$T = 2G\theta \iint \phi \, dx\, dy$$

Let us look at:



$\nabla^2\phi = -2$ on R

$\phi = 0$ on B

$$\begin{Bmatrix} x = \pm 1 \\ y = \pm 1 \end{Bmatrix}$$

Look at the construction of the admissible polynomial

Since there is symmetry, ϕ should contain even powers of x and y.

$$\phi(x, y, A) = a_0 + a_1(x^2+y^2) + a_2 x^2 y^2 + a_3(x^4+y^4) + a_4(x^4y^2+x^2y^4) + \cdots\cdots$$

Let us consider the first three terms

$$\phi = a_0 + a_1(x^2+y^2) + a_2 x^2 y^2$$

@ $x = \pm 1$, $\phi = 0$ $\quad \therefore \quad a_0 + a_1 + (a_1+a_2)y^2 = 0$

@ $y = \pm 1$, $\phi = 0$ $\quad \therefore \quad a_0 + a_1 + (a_1+a_2)x^2 = 0$

$$\therefore \; a_0 + a_1 = 0 \quad \& \quad a_1 + a_2 = 0 \quad \Bigg\} \quad \therefore \; a_0 = a, \quad a_1 = -a_0 = -a, \quad a_2 = -a_1 = a$$

$$\therefore \; \phi(x, y, A) = a\,(1 - x^2 - y^2 + x^2 y^2)$$
$$= a\,(1 - x^2)(1 - y^2)$$
$$= a\,\phi_1(x, y)$$

$$\therefore \; \phi = a_1 \phi_1(x,y) + a_2 \phi_2(x,y) + a_3 \phi_3(x,y) + \cdots + a_n \phi_n(x,y)$$

where,

$$\phi_1(x,y) = (1 - x^2)(1 - y^2)$$
$$\phi_2(x,y) = x^2 \phi_1(x,y)$$
$$\phi_3(x,y) = y^2 \phi_1(x,y)$$
$$\phi_4(x,y) = x^2 y^2 \phi_1(x,y)$$
$$\vdots$$
$$\phi_n(x,y) = x^{2i} y^{2i} \phi_1(x,y)$$

You can also build a two parameter family approx.
- consider 5 terms in the beginning.

- Collocation method:

$$R(x_i, y_i) = 0 \quad \longrightarrow \quad i = 1 \text{ for one parameter}$$

- Subdomain method:

$$4 \int_0^1 \int_0^1 R(x,y)\, dx\, dy = 0 \quad \longrightarrow \quad \text{for one parameter}$$

- Least squares method:

$$\int_{-1}^{1} \int_{-1}^{1} R \cdot \frac{dR}{da} \cdot dx\, dy = 0 \quad \longrightarrow \quad \text{for one parameter}$$

- Galerkin's method:

$$\int_{-1}^{1} \int_{-1}^{1} \phi_1(x,y)\, R(x,y)\, dx\, dy = 0 \quad \longrightarrow \quad \text{for one parameter}$$

As the function ϕ becomes complex, we have to reesort to numerical integration

1.2.5 NUMERICAL INTEGRATION

- Numerical evaluation can proceed in two ways:
  - function is evaluated at equal spaced points
  - If there are n points, then a polynomial of order (2n-1) can be passed through n values.
  - gives rise to trapezoidal rule if n=2 and Simpson's rule if n=3.
  - known as Newton-Cotes quadrature
  - general formula in interval [a, b] is

$$\int_a^b f(x)\,dx = (b-a)\sum_{i=1}^{n} H_i\, f(x_i)$$

where, $i = 1, 2, \ldots\ldots n$ denotes sampling points and H_i are weighting coefficients

n	H_1	H_2	H_3	H_4
2	1/2	1/2		
3	1/6	2/3	1/6	
4	1/8	3/8	3/8	1/8

- The second approach to numerically integrate is to LOCATE THE SAMPLING POINTS such as to achieve the best accuracy.

  - The interval will be changed from $[a, b]$ to $[-1, 1]$ by transformation of variable.

$$a \leq x \leq b \implies -1 \leq z \leq 1$$

$$\therefore z = \frac{2x - (a+b)}{b-a} \implies x = \frac{(b-a)z + (a-b)}{2}$$

  - Define new function $f(z)$ so that

$$f(x) = f\left(\frac{(b-a)z + (a+b)}{2}\right) = f(z)$$

  - Evaluate a line integral as follows:

$$I = \int f(x)\,dx = \int_{-1}^{1} f(z)\,dz = \sum_{i=1}^{n} H_i\, f(z_i)$$

n = number of sampling points

H_i = weighting coefficients

z_i = coordinates of sampling points

  - This is known as <u>Gauss-Quadrature</u> formula.

  - For any given n, H_i and z_i can be obtained from Tables.

- For n sampling points, we have $2n$ unknowns (f_i & z_i) for which a polynomial order $(2n-1)$ can be constructed and integrated exactly
  - For $n=3$, a polynomial of degree 5 can be integrated exactly using this method
  - Sampling points are

$$z_1 = -0.77459 \qquad z_2 = 0 \qquad z_3 = 0.77459$$

$$\& \quad H_1 = 0.55555 \qquad H_2 = 0.8888 \qquad H_3 = 0.55555$$

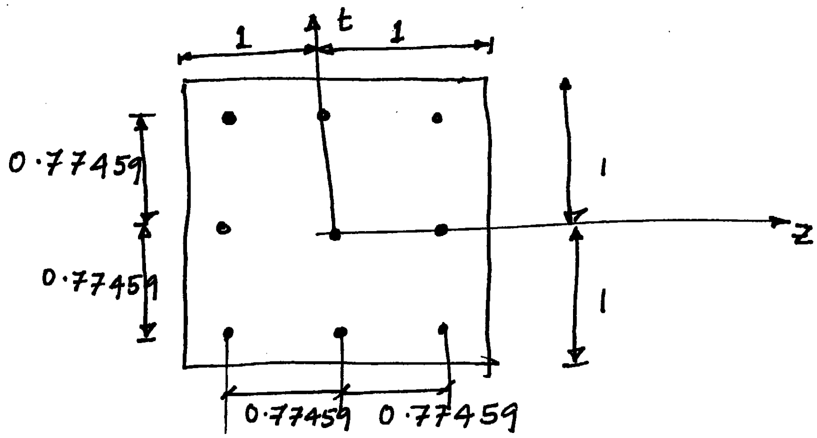
$$\therefore I = 0.5555\, f(-0.77459) + 0.88888\, f(0) + 0.5555\, f(0.77459)$$

- For a 2-D area, we integrate as follows:

$$I = \int_{-1}^{1}\int_{-1}^{1} f(z,t)\,dz\,dt = \int_{-1}^{1} \sum_{i=1}^{n} H_i\, f(z_i, t)\,dt$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} H_i\, H_j\, f(z_i, t_j)$$

e.g. for $n=3$, we select 9 points in the ~~using the~~ directions z and t shown below:



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1.3 APPLICATION OF WEIGHTED RESIDUALS FOR Natural Boundary Value Problem:

- So far, we have seen problems with forced boundary conditions, i.e., the dependent variable at the boundary has forced values (zero or otherwise).
  - These are called essential boundary conditions

- But there might be other boundary conditions, which involve the derivatives of the dependent variable w.r.t the independent variable.
  - Such types of boundary conditions are called natural or unessential boundary conditions.

- What happens if the problem involves both "forced" and natural boundary conditions?

u = dependent variable (for example, displacement)

x = independent variable (e.g. coordinates of point)

$L(u) = f \longrightarrow$ differential equation to be satisfied in the region R.

$B(u) = g \longrightarrow$ differential equation to be satisfied on the boundary B

$f, g \longrightarrow$ functions of x, or constants, or zero.

Let, $u^*(x, A) = \sum_{i=1}^{n} a_i \phi_i(x)$ be the approx. trial solution

- If u^* is substituted into the equations $L(u) = f$ and $B(u) = g$, we will obtain two residuals

$$R_L = R_L(x, A) = L(u^*) - f \longrightarrow \text{interior residual}$$

$$R_B = R_B(x, A) = B(u^*) - g \longrightarrow \text{boundary residual.}$$

- The residuals R_L and R_B are functions of x and $a_1, a_2, \ldots\ldots a_n$.
  - We must now find the values of a_i such that the exact solution u and the approximate solution u^* are "close" in some sense.

1.3.1 COLLOCATION METHOD.

For n different values of x, the residuals are set to zero as follows:

$$R_L(x_i, A) = 0 \qquad \text{for } i = 1, 2, \ldots\ldots j-1$$

$$R_B(x_i, A) = 0 \qquad \text{for } i = j, j+1 \ldots, n$$

Thus $(j-1)$ points are selected in the region R and the remaining points are selected on the B.

1.3.2 LEAST SQUARES METHOD.

- The a_i are chosen to minimize the integral of the square of the residual.
  - A large W_B makes R_B more important than R_L.
  - Integral of the square of the residuals

$$I = \int_R \left[R_L(x, A)\right]^2 dR + W_B \int_B \left[R_B(x, A)\right]^2 dB$$

  - For this to be minimum, we must satisfy:

$$\frac{\partial I}{\partial a_i} = 0 \qquad \text{for } i = 1, 2, \ldots\ldots n$$

  - Note that W_B can be so selected that the dimensions of the two terms are the same.
  - W_B is applied to R_B, which can be applied arbitrarily.
  - A large W_B makes R_B more important than R_L

1.3.3 GALERKIN'S METHOD.

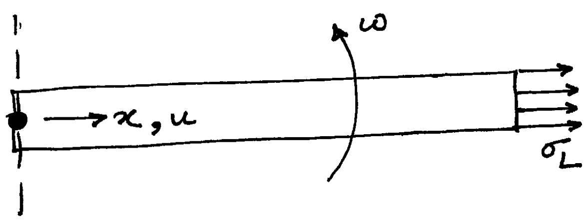
- We select "weight functions," $W_i(x)$ and set the weighted averages of the residual to zero.
- For $i = 1, 2, \ldots\ldots n$

$$R_i = \int_R W_i(x)\, R_L(x, A)\, dR$$

where, $$W_i(x) = \frac{\partial u^*}{\partial a_i} = \phi_i(x)$$

1.3.4 Example.

- uniform axial bar of cross-section area, A elastic modulus E, and mass density ρ, that rotates with an angular velocity ω about the end $x = 0$



- The governing d.e. is

$$E \frac{d^2u}{dx^2} + \rho\, \omega^2 x = 0 \longrightarrow \forall\, 0 < x < L$$

- The forced boundary condition is

$$u(0) = 0$$

- The natural boundary condition is:

$$E\left(\frac{du}{dx}\right)_{x=L} = \sigma_L$$

- Let the trial function be:

$$u^* = a_1 x + a_2 x^2$$

This satisfies the forced boundary condition.

Note: The trial function should be chosen to satisfy the forced boundary conditions

-

$$R_L = E\frac{d^2u^*}{dx^2} + \rho\omega^2 x$$

$$\boxed{R_L = E\cdot(2a_2) + \rho\omega^2 x.}$$

- $$R_B = E\left.\frac{du^*}{dx}\right|_{x=L} - \sigma_L$$

$$\boxed{R_B = E(a_1 + 2a_2 L) - \sigma_L}$$

- COLLOCATION METHOD SOLUTION
  - we can select upto two points: one in R and one in B
  - Let us arbitrarily select $x = L/3$ to evaluate R_L
  - and $x = L$ to evaluate R_B

$$\therefore\ R_L \Big|_{x=L/3} = 0 \implies 2Ea_2 + \rho\omega^2 L/3 = 0$$

$$R_B \Big|_{x=L} = 0 \implies E(a_1 + 2a_2 L) - \sigma_L = 0$$

Solve for a_1, a_2

$$a_1 = \frac{\sigma_L}{E} + \frac{\rho\omega^2 L^2}{3E} \quad \& \quad a_2 = -\rho\frac{\omega^2 L}{6E}$$

— x —

- LEAST SQUARES METHOD SOLUTION:

Assume $w_B = \frac{1}{L}$ → something simple...

$$I = \int_R R_L^2 \, dR + w_B \int_B R_B^2 \, dB$$

$$\therefore\ I = \int_0^L \left(2Ea_2 + \rho\omega^2 x\right)^2 A\,dx + \frac{1}{L}\left[E(a_1 + 2a_2 L) - \sigma_L\right]^2 A$$

Take $\frac{\partial I}{\partial a_1} = 0 \quad \therefore\ a_1 + 2La_2 = \sigma_L / E$

$$\frac{\partial I}{\partial a_2} = 0 \quad \therefore\ 2a_1 + 8La_2 = \frac{2\sigma_L}{E} - \frac{\rho\omega^2 L^2}{E}$$

$$\therefore\ a_1 = \frac{\sigma_L}{E} + \frac{\rho\omega^2 L^2}{2E} \quad \& \quad a_2 = -\frac{\rho\omega^2 L}{4E}$$

— x —

GALERKIN'S METHOD

$$R_i = \int_V W_i \, R_L(x, A) \, dR$$

$$= \int_0^L W_i \left(E \frac{d^2u^*}{dx^2} + \rho \omega^2 x \right) A \, dx$$

$$= \underbrace{\int_0^L W_i \left(E \frac{d^2u^*}{dx^2} \right) A \, dx}_{} + \int_0^L W_i \cdot \rho \omega^2 x \, A \, dx$$

Integrate 1st term by parts

$$\therefore \int_0^L W_i \left(E \frac{d^2u^*}{dx^2} \right) A \, dx = EA \left[W_i \frac{du^*}{dx} \right]_0^L - EA \int_0^L \frac{dW_i}{dx} \cdot \frac{du^*}{dx} \cdot dx$$

where, $W_i = u$ & $dv = \dfrac{d^2u^*}{dx^2}$

$$\therefore R_i = \underbrace{EA \left[W_i \frac{du^*}{dx} \right]_0^L}_{\text{boundary condition terms.}} - EA \int_0^L \frac{dW_i}{dx} \cdot \frac{du^*}{dx} \cdot dx + \int_0^L W_i \cdot \rho \omega^2 x \, A \, dx$$

$$\underbrace{EA \, W_i \frac{du^*}{dx} \bigg|_{x=L}}_{\text{natural boundary condition stated earlier.}} - EA \, W_i \frac{du^*}{dx} \bigg|_{x=0}$$

$$\therefore R_i = W_i \sigma_L A \Big|_{x=L} - EA \left[W_i \frac{du^*}{dx} \right]_{x=0}$$

$$- EA \int_0^L \frac{dW_i}{dx} \cdot \frac{du^*}{dx} \cdot dx + \int_0^L W_i \cdot \rho \omega^2 x \cdot A dx$$

But $\quad W_1 = \dfrac{\partial u^*}{\partial a_1} = x \longrightarrow$ substitute in $R_1 = 0$

$\quad W_2 = \dfrac{\partial u^*}{\partial a_2} = x^2 \longrightarrow$ substitute in $R_2 = 0$

$$\therefore a_1 = \frac{\sigma_L}{E} + \frac{7}{12} \frac{\rho \omega^2 L^2}{E} \quad \text{and} \quad a_2 = - \frac{\rho \omega^2 L}{4E}$$

VARIATIONAL METHOD

- Calculus of variation is concerned primarily with the extreme value of functions and function of functions - called functional.

  - Technique concerned with the determination of a function $y = u(x)$ that makes stationary a certain definite integral

    $J(u)$ = functional = function of a function

    $$J(u) = \int_a^b u(x)\, dx$$

    where,

    $u(x)$ → continuously differentiable function

  - In calculus of variations, we determine what conditions are imposed on $y = u(x)$ such that $J(u)$ is either a minimum or maximum, i.e., stationary.

  - This is referred as the extremum principle. These conditions result in the governing differential equations, called Euler equations.

THE VARIATION SYMBOL δ

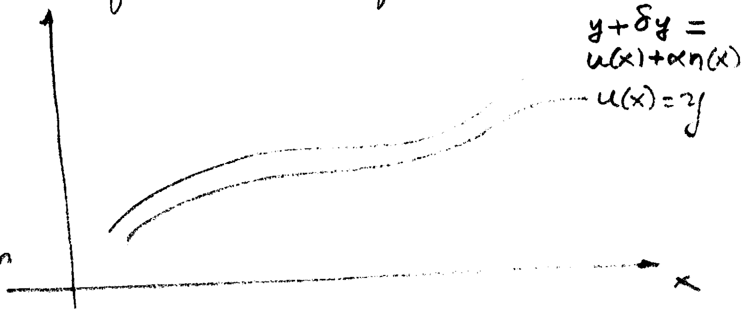
- FUNCTION $y = u(x)$ augmented by infinitesimal function $\alpha\eta(x)$

where, $\alpha \rightarrow$ infinitesimal number

$\eta(x) \rightarrow$ continuously differentiable function

$$y = u(x) + \alpha\,\eta(x)$$

$$+\delta y = \alpha\,\eta(x)$$



First variation of function

$$\delta y = \alpha\ \eta(x)$$

$$\delta y' = \frac{d}{dx}\left(\alpha\,\eta(x)\right) = \alpha\,\eta'(x) \quad = \delta\left(\frac{dy}{dx}\right)$$

$$\delta y'' = \alpha\,\eta''(x) \quad = \delta\left(\frac{d^2y}{dx^2}\right)$$

The operators δ and $\frac{d}{dx}$ are interchangeable.

- F is a functional $\rightarrow$ function of y, y', etc. that are already functions of x

$$F = F(x, y, y')$$

- If y is subjected to the variation δy, then F goes through a variation ΔF

$$\Delta F = F(x, y + \alpha\eta, y' + \alpha\eta') - F(x, y, y')$$

$\therefore$ Using Taylor series expansion of the first term

$$\Delta F = \delta F + \frac{1}{2!} \delta^2 F + \ldots\ldots$$

where $\delta F = \frac{\partial F}{\partial y} \cdot \delta y + \frac{\partial F}{\partial y} \cdot \delta y'$ — First Variation

$$\delta^2 F = \frac{\partial^2 F}{\partial y^2} \cdot \delta y^2 + 2 \frac{\partial^2 F}{\partial y \partial y'} \cdot \delta y \, \delta y' + \frac{\partial^2 F}{\partial y'^2} \cdot \delta y'^2$$ ← Second Variation

Sidenote:

Laws of δ variation are analogous to those for differentiation:

$$\delta(F_1 F_2) = F_1 \delta F_2 + F_2 \delta F_1$$

$$\delta(F_1/F_2) = (F_2 \delta F_1 - F_1 \delta F_2)/F_2^2$$

Further, the variational operator can be interchanged with partial differential operators:

$$\frac{d}{dx}(\delta y) = \delta \frac{dy}{dx}$$

- So, when we apply an infinitesimal variation to the function y in the form of δy.
  - → It produces a variation in the corresponding functional F, which is equal to $\Delta F = \delta F + \frac{1}{2!} \delta^2 F + \ldots$
  - → Now, the functional has become stationary, or achieved an extremum, if for any and all arbitrary δy ⟶ ΔF is always ≥ 0

$$\Delta F = \delta F + \frac{1}{2!}\delta^2 F + \cdots \geq 0$$

If the variation δy is infinitesimal, then if $\delta y \longrightarrow 0$ then $\Delta F \longrightarrow 0$

$\therefore\ \delta F = 0 \longrightarrow$ for stationary or extremum & the sign of the first non-zero $\delta^n F$ determines whether it is a minimum or max.

If $\delta^2 F > 0 \longrightarrow$ minimum

$\delta^2 F < 0 \longrightarrow$ maximum

- DEVELOPMENT OF EULER EQUATIONS

Consider the problem of finding $y = u(x)$ in the region R $[x_0, x_1]$ such that the following functional is minimized:

$$J(y) = \int_{x_0}^{x_1} F(x, y, y', y'')\, dx$$

$$\therefore\ \delta J = \int_{x_0}^{x_1} \delta F\, dx$$

$$\delta F = \frac{\partial F}{\partial y} \cdot \delta y + \frac{\partial F}{\partial y'} \cdot \delta y' + \frac{\partial F}{\partial y''} \cdot \delta y''$$

$$\delta J = \int_{x_0}^{x_1} \left(\frac{\partial F}{\partial y} \cdot \delta y' + \frac{\partial F}{\partial y'} \cdot \delta y' + \frac{\partial F}{\partial y''} \cdot \delta y'' \right) \cdot dx$$

Integrate by parts: $\int_{x_0}^{x_1} u\,dv = uv \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} v\,du$

$$\boxed{\therefore \int_{x_0}^{x_1} \frac{\partial F}{\partial y'} \cdot \delta y' = \frac{\partial F}{\partial y'} \cdot \delta y \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \cdot \delta y}$$

And, $\int_{x_0}^{x_1} \frac{\partial F}{\partial y''} \cdot \delta y'' = \frac{\partial F}{\partial y''} \cdot \delta y' \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) \cdot \delta y'$

$$\boxed{\int_{x_0}^{x_1} \frac{\partial F}{\partial y''} \cdot \delta y'' = \frac{\partial F}{\partial y''} \cdot \delta y' \Big|_{x_0}^{x_1} - \frac{d}{dx} \frac{\partial F}{\partial y''} \cdot \delta y \Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} \frac{d}{dx^2}\left(\frac{\partial F}{\partial y''}\right) \cdot \delta y}$$

$$\therefore \delta J = \int_{x_0}^{x_1} \delta y \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d}{dx^2}\left(\frac{\partial F}{\partial y''}\right) \right)$$

$$+ \frac{\partial F}{\partial y'} \cdot \delta y \Big|_{x_0}^{x_1} + \frac{\partial F}{\partial y''} \cdot \delta y' \Big|_{x_0}^{x_1} - \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) \cdot \delta y \Big|_{x_0}^{x_1}$$

Since $\delta J = 0$

<u>Euler equation:</u>

$$\underbrace{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d}{dx^2}\left(\frac{\partial F}{\partial y''}\right)}_{\text{differential equation for region } [x_0, x_1]} = 0$$

and <u>Boundary conditions:</u>

Forced:

either, $\delta y = 0$ @ x_0, x_1

$\delta y' = 0$ @ x_0, x_1

Natural:

or, $\dfrac{\partial F}{\partial y'} = 0$ @ x_0, x_1

$\dfrac{\partial F}{\partial y''} = 0$ @ x_0, x_1

$\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y''}\right) = 0$ @ x_0, x_1

- The Euler equation is the governing differential equation of the problem.

- If $F = F(x, y, y')$ then the corresponding Euler equation is:

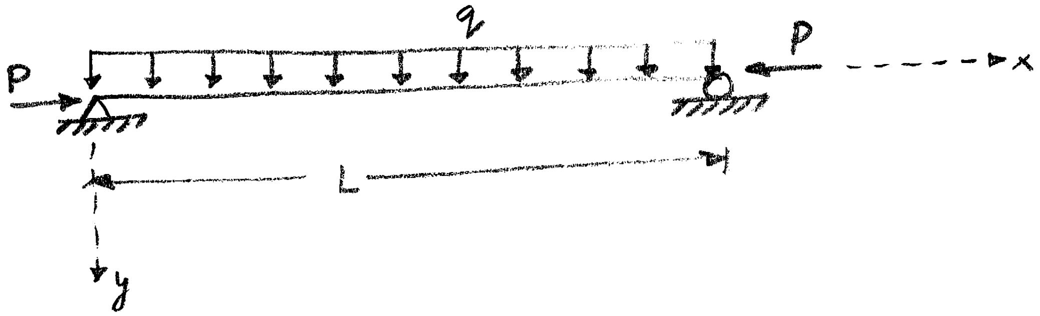
$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0}$$

- If $F = \left(x, y, u, v, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right)$, i.e. F depends on two dependent variables (u, v) then the corresponding Euler equations are:

$$\boxed{\begin{aligned} \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) &= 0 \\ \frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right) &= 0 \end{aligned}}$$

EXAMPLE:

- Beam-column subjected to uniform lateral load q and axial load P. It is simply supported.



- The potential energy functional of the beam-column can be derived as

$$V(y) = \frac{1}{2} EI \int_0^L (y'')^2 dx - \frac{1}{2} P \int_0^L (y')^2 dx - q \int_0^L y \, dx$$

$$\therefore V(y) = \int_0^L F(x, y, y', y'') \, dx$$

where, $F = \frac{EI}{2}(y'')^2 - \frac{P}{2}(y')^2 - qy$

The boundary conditions are $y(0) = 0$, $y(L) = 0$

$\therefore$ $\boxed{\delta y = 0 \text{ @ } x = 0 \text{ and } x = L \text{ are the forced b.c.}}$

& Moment @ $x = 0$ and $x = L$ are equal to zero.

$$y''(0) = y''(L) = 0$$

which imply that $\frac{\partial F}{\partial y''} = EIy'' = 0$ @ $x = 0$ & $x = L$ are the natural boundary conditions.

- The Euler equation for the problem, which yields a minimum $V(y)$ to assure equilibrium is:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

But,

$$F = \frac{EI}{2}(y'')^2 - \frac{P}{2}(y')^2 - qy$$

$$\therefore \boxed{\frac{\partial F}{\partial y} = -q} \qquad \boxed{\frac{\partial F}{\partial y'} = -Py'}$$

$$\boxed{\frac{\partial F}{\partial y''} = EIy''} \qquad \boxed{\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = -Py''}$$

$\therefore$ Euler equation is:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0$$

Becomes:

$$-q + Py'' + EIy^{IV} = 0$$

$$\therefore \boxed{y^{IV} + \frac{P}{EI}y'' = \frac{q}{EI}}$$

<u>Wow!</u>

Found the beam-column differential equation without a single <u>FBD</u>

- RAYLEIGH - RITZ

- general method for obtaining approximate solution of a problem, which admit the construction of functionals.
- Minimizing the functional establishes equilibrium and allows solving the problem.

$$J(y) = \int_a^b F(x, y, y')\,dx$$

Let $\bar{y} \longrightarrow$ exact solution

$\therefore \bar{J} = J(\bar{y})$ is minimum

If $y^* \longrightarrow$ admissible solution

$J^* = J(y^*)$

$$\boxed{J^* > \bar{J}}$$

$\longleftarrow$ Think about it!

If $J^* \xrightarrow{\text{close to}} \bar{J}$, then good approx.

- If there exists a convergent sequence $\{y_n^*\}$

then $\lim\limits_{n \to \infty} J_n^* = \bar{J}$ & $\lim\limits_{n \to \infty} y_n^* = \bar{y}$

$$y^* = \phi_0(x) + a_1 \phi_1(x) + a_2 \phi_2(x) + \cdots + a_n \phi_n(x)$$

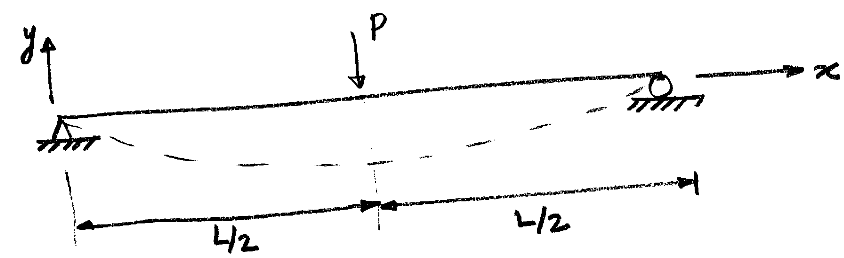
where, $a_1, a_2, \ldots a_n$ are approximating functions and $\phi_i(x)$ are approximating functions.

y^* becomes an admissible function
- It is continuously differentiable
- satisfies the boundary conditions
- and has order of the highest derivative in the d.e.

- Determine a_i such that

$$\left.\frac{\partial I}{\partial a_k}\right|_{(a_1, a_2, \ldots a_n)} = 0 = \int_a^b \left.\frac{\partial F}{\partial a_k}\right|_{a_1, a_2 \ldots a_n}$$

EXAMPLE:



$$\therefore \quad V = \int_0^L \frac{EI}{2}(y'')^2 dx - Py\left(\frac{L}{2}\right)$$

$$\therefore \quad F = \frac{EI}{2}(y'')^2 - Py\left(\frac{L}{2}\right)$$

Euler equation: $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) + \dfrac{d^2}{dx^2}\left(\dfrac{\partial F}{\partial y''}\right) = 0$

Boundary : $y(0) = y(L) = 0$

$y''(0) = y''(L) = 0$

Assume: $y^* = a_1 \sin\left(\dfrac{\pi x}{L}\right)$ → satisfies admissibility conditions

$$V = \int_0^L \frac{EI}{2} \times \left(-\frac{a_1 \pi^2}{L^2} \cdot \sin \frac{\pi x}{L} \right)^2 dx - Pa_1$$

$$\therefore \quad V = \frac{\pi^4 EI}{4L^3} a_1^2 - Pa_1$$

But $\frac{dV}{da_1} = 0$ $\quad \therefore a_1 = \frac{2PL^3}{\pi^4 EI} = 0.0205 \frac{PL^3}{EI}$

$$\therefore \quad \boxed{y^* = 0.0205 \frac{PL^3}{EI} \sin\left(\frac{\pi x}{L}\right)}$$

- Exact solution $y = \frac{PL^2 x}{16EI} - \frac{Px^3}{12EI}$ for $0 \leqslant x \leqslant \frac{L}{2}$

$$\therefore y_{max} = 0.0208 \frac{PL^3}{EI}$$

- Comparing y^*_{max} with y_{max} $\longrightarrow$ error only 1.6%

- Comparing the moment

$$M = -EI \frac{d^2 y}{dx^2}$$

$$\therefore \quad M^* = 0.2026 \, PL \sin \frac{\pi x}{L}$$

- $M^*_{max} = 0.2026 \, PL$ compare with $M_{max} = 0.25 PL$
- error about 19%. [WOW!]

<u>V. V. Imp:</u> Note that the error for M^* is much larger than the error for y^*. This is because it is a function of y''!

- The SOLUTION converges slowly for higher derivatives of y.

- Try a 3-parameter family approximation

$$y^* = a_1 \sin\frac{\pi x}{L} + a_2 \sin\frac{2\pi x}{L} + a_3 \sin\frac{3\pi x}{L}$$

$$\therefore \ V = \int_0^L \frac{EI}{2}(y'')^2 dx - Py\frac{L}{2}$$

$$= \frac{\pi^4 EI}{4L^3}\left(a_1^2 + 16a_2^2 + 8a_3^2\right) - P(a_1 - a_3)$$

$$\frac{\partial V}{\partial a_1} = 0 \longrightarrow \therefore a_1 = \frac{2PL^3}{\pi^4 EI}$$

$$\frac{\partial V}{\partial a_2} = 0 \longrightarrow \therefore a_2 = 0$$

$$\frac{\partial V}{\partial a_3} = 0 \longrightarrow \therefore a_3 = \frac{-2PL^3}{81\pi^4 EI}$$

$$\therefore y^* = \frac{2PL^3}{\pi^4 EI}\left(\sin\frac{\pi x}{L} - \frac{1}{81}\sin\frac{3\pi x}{L}\right)$$

$$\therefore y^*_{max} = 0.0208\frac{PL^3}{EI} \longrightarrow \text{v. good}$$

$$\& \ M^*_{max} = 0.2252 \ PL \longrightarrow \text{still 10\% error}$$

(WOW!).