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- From Abebooks
- skimmed.
- Liepmann – noise from δ* variations ← Chap 7

lai: noise rev.

AGARDograph

- lots of math.
- p.56 – wall pressure fluct. related to radiated noise

MECHANISM OF NOISE GENERATION IN THE TURBULENT BOUNDARY LAYER

by

JOHN LAUFER
JOHN E. FFOWCS WILLIAMS
STEPHEN CHILDRESS

– ch. 8 describes Laufer expts

– nothing on wall temp effects on noise

LAUFER 1964

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NORTH ATLANTIC TREATY ORGANIZATION
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MECHANISM OF NOISE GENERATION IN THE TURBULENT BOUNDARY LAYER

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John Laufer
John E. Ffowcs Williams
Stephen Childress

November 1964

This is one of a series of NATO-AGARD publications by the Fluid Dynamics Panel. Professor Wilbur C. Nelson of the University of Michigan is the Editor.

SUMMARY

The first chapter briefly states the problem and gives its historical development; the second reviews some basic notions of classical acoustics with special emphasis on the sound field produced by elementary sound sources, while the third one contains the generalized wave equation governing the pressure field radiated by nonuniform nonstationary flow. Subsequent chapters describe the methods proposed by the various authors for finding a solution to the radiation problem. The experimental approach is discussed.

SOMMAIRE

Le premier chapitre décrie brièvement le problème et conte son développement historique; le deuxieme chapitre rappelle quelques notions de base en acoustique classique et, en particulier, le champ sonore produit par des sources sonores elementaires, alors que le troisième contient les equations d'onde generalisées qui governent le champ de pression radié par un écoulement nonuniforme et nonstationnaire. Les chapitres suivants décrivent les methodes proposées par les differents auteurs pour trouver une solution au problème de radiation. L'approche experimentale est discutée.

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PREFACE

The subject matter treated here is a comparatively new and rapidly expanding one. The bulk of the published works on boundary layer noise appeared in the last four or five years. Under these circumstances the writer of a monograph faces the risk of becoming rapidly out-dated. Recognising this situation, the authors did not try to cover all the available literature on boundary layer noise production in detail - except by giving an extensive list of references - but rather they attempted to pick out those works that in their opinion were most instrumental in generating new ideas and approaches to this difficult problem. The arrangement of the chapters reflects this point of view.

Chapters 1, 2 and 3 are introductory ones: the first chapter briefly states the problem and gives its historical development; the second reviews some basic notions of classical acoustics with special emphasis on the sound field produced by elementary sound sources, while the third one contains the generalized wave equation governing the pressure field radiated by a nonuniform, nonstationary flow. The subsequent chapters describe then the methods proposed by various authors for finding a solution to the radiation problem.

Undoubtedly, the theory that exerted the most profound influence on the subject is Lighthill's acoustic analogy. Although the theory has been described in a number of papers by Lighthill himself and by several subsequent workers, it has often been misinterpreted and misused. Chapter 4 attempts to restate again the analogy, summarizing all the assumptions involved and endeavors to point out the advantages and disadvantages of the approach. In the same chapter Ribner's interpretation of the source term in the analogy is touched upon, but only briefly.

Chapter 5 describes the application of the analogy to the boundary layer problem. The importance of the size of the boundary surface compared to the radiation wavelength is emphasized.

Chapter 6 introduces a new approach proposed by Phillips to treat the radiation problem for high convection velocities. Some comparisons are made between his theory and the consequences Ffowcs Williams drew from Lighthill's analogy applied to higher convection speeds.

A separate section, Chapter 7, is devoted to a third method of attack to study aerodynamic noise first proposed in an unpublished work by Liepmann. This approach, although virtually unexplored, offers a possibility of relating the radiated noise to flow parameters familiar in the study of incompressible flows. It is this aspect, with its promise of straight-forward experiments, that led the authors to devote to it a complete chapter, although much of the chapter is of a general illustrative nature.

The last chapter deals with the experimental investigations concerning boundary layer noise. It is to be noted that the considerable wealth of information on the fluctuating velocity field within a boundary layer and on the pressure fluctuations over the solid surface adjacent to the layer obtained at subsonic speeds have not been included. There are several indications, both theoretical and experimental, that such subsonic fields radiate very small noise indeed and have no practical significance. For this reason the chapter concentrates primarily on the supersonic boundary layer problem.

The monograph is designed for those who are familiar with classical acoustics and fluid mechanics in addition to some basic notions of turbulence: the concepts of correlation and spectrum functions. Wherever a more detailed treatment of a particular question exists in the literature, an attempt was made to give adequate references. The bibliography does not include all the papers on aerodynamic noise but only those restricted to boundary layers. In this sense it is hopefully fairly complete.

Portions of this AGARDograph were prepared at the Jet Propulsion Laboratory, California Institute of Technology, and the encouragement and cooperation of Dr. William H. Pickering is greatly appreciated. One of us (J.E.F.W.) gratefully acknowledges the sponsorship of the Bureau of Ships' Fundamental Hydromechanics Research Program, administered by the David Taylor Model Basin.

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John Laufer, John E. Ffowcs Williams
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CHAPTER 1

GENERAL CONSIDERATIONS

1.1 Statement of the Boundary Layer Radiation Problem

In the present monograph we will consider a turbulent boundary layer over a rigid wall and will examine the time-dependent pressure field outside of the layer. Within the layer the fluctuations may be described primarily in terms of vorticity and entropy modes and, to a lesser extent, sound modes. Outside of the layer the first two modes decrease rapidly in amplitude, so that, at a sufficiently large distance from the shear zone (several wavelengths away), one expects to find fluctuations only in the sound mode present. These fluctuations are usually referred to in the literature as aerodynamic noise. The current ideas on the mechanism of production, nature and intensity of the aerodynamic noise are the subject of the present work.

There are two fundamentally new features that arise in this problem, not present in an incompressible turbulent field. Once the compressibility of the fluid is taken into account, a disturbance from a source will propagate at a finite speed and will influence the flow field over a finite distance in a given time. This means that, in calculating the flow properties at a given point and time, it will now be necessary to known the behavior of the disturbance source at a certain earlier time. This fact must be reflected in the statistical description of the fluctuating flow field: in order to calculate the pressure fluctuations emanating from a turbulent shear field, certain statistical quantities, such as the space-time correlation functions, have to be known within the shear zone.

The second new feature in a compressible turbulence is the fact that a new form of energy loss, in addition to the dissipation, appears in the problem. This is simply the energy radiated away from the turbulent field. Just how important such a process is could turn out to be one of the most important questions connected with high Mach number flows. If indeed the energy radiation were so intense that it, together with the dissipation rate, exceeded the rate of turbulence energy production, turbulence could not exist at very large Mach numbers. This question is discussed in Chapter 8.

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1.2 Methods of Solution

Mathematical tools to handle radiation problems have been well developed in the field of electromagnetic, acoustic and nonstationary supersonic theories. In all of these cases, however, the radiation source could be defined and separated from the radiation field it generates. One of the basic difficulties in the aerodynamic noise problem is the fact that the separation of the flow field from the noise it produces is no longer obvious in the sense that the operation of a mechanical acoustic emitter may be separated from the sound field it creates. Perhaps the greatest merit of Lighthill's acoustic analogy, described in Chapter 4, is the successful arrangement of this problem by formulating it in such a manner that a classical treatment is made possible. The analogy, of course, does have certain limitations, which were discussed by Lighthill himself.

A less formal approach proposed by Liepmann, on the other hand, attacks the aerodynamic noise problem from a different point of view. It introduces for the generation process certain acoustical models which presumably retain the main features of the physical problem but at the same time treat it in precise mathematical terms with explicit results. The physical meaning of these results can then be clearly interpreted. The pitfalls of this approach are quite obvious; nevertheless, with proper physical intuition considerable progress can be made with this method, as will be seen in Chapter 7.

The fundamental difficulty of the problem is well reflected in the rather controversial nature of its historical development. Next to the problem of hydrodynamic stability, few questions in fluid mechanics have excited as much controversy as has this one.

1.3 Historical Background

The aerodynamic noise problem is another example in the history of development of fluid mechanics, the conception and formulation of which belongs clearly to a single individual. While the name of Prandtl is associated with the boundary layer theory, Taylor's to the statistical turbulence theory, similarly Lighthill's name belongs to the general field of turbulence-generated sound radiation.

It has been a little over a decade since Lighthill published his famous paper, "On Sound Generated Aerodynamically." The work stimulated a great number of workers in the field to extend, re-examine and apply his theory. Of particular interest is the work of Proudman[69] who considered the noise produced by an isotropic turbulent field, the only case in which the radiation intensity can be explicitly calculated. Later, Curle[11] applied the Lighthill analogy to the boundary layer, but it was Phillips[60] who actually evaluated the surface integral derived by Curle. His rather surprising result that an incompressible, homogeneous turbulent boundary layer does not emit sound brought about a considerable controversy (see, for instance, Reference 44). It is generally accepted now that, within the framework of his assumptions, Phillips's conclusion is correct (see Chapter 5). During the controversy, some interesting points concerning the region of validity of Curle's formulation have been brought out and clarified by Powell[65] and Ffowcs Williams[22].

Parenthetically, one might mention that, along with attempts to estimate the far field intensity[44,49] considerable effort has been spent in the calculation of the pressure fluctuations in the near field[9,34,46,47]. For this one has available the great wealth of experimental information concerning the wall pressure fluctuations[4,28,29,32,70,76,81] that has recently appeared in the literature.

In 1959 Ribner proposed an alternative formulation of Lighthill's acoustic analogy, expressing the sound sources in terms of simple sources rather than quadrupoles[74]. The work produced criticism from many sides[44,66], but recently Lighthill has pointed out the basic soundness of the approach even though he questioned on several grounds the advantages of this formulation[45].

The extension of the acoustic analogy to high speeds was first attempted by Lilley[49]; his approach, however, was unconvincing. It was Phillips who advanced a completely new formulation of the problem[61], the most important result of which turned out to be the concept of "eddy-Mach wave" radiation in a supersonic flow. As far as his detailed analysis is concerned, certain questions have been raised and are discussed in Chapter 6. Somewhat later Ffowcs Williams has shown that the eddy-Mach wave radiation can be deducted by a proper extension of the Lighthill analogy to high speeds[24].

Quite independently from the development described above, Liepmann forwarded a completely different approach that attracted very little attention, mainly because it was not published in the open literature.

He has suggested to express the radiated pressure intensity in terms of fluctuations in the boundary layer displacement thickness. His work has inspired the concept of constructing physically realistic models of turbulent sound sources that can be handled mathematically. Chapter 7 was written in the spirit of this approach.

It is interesting to observe that, while the boundary layer radiation problem occupied the interest of a large number of theoretical workers, very few experimental papers were published on the subject. In 1960 Wilson considered the sound radiated from the turbulent flow produced around a rotating cylinder; this, however, is not exactly a boundary layer noise problem. Probably, the first measurements of boundary layer noise were made by Laufer[38] in studying the turbulence levels in supersonic wind tunnels. Later on, he extended these experiments to examine the statistical nature of the radiation field[40]. At present, these are the only measurements which can serve as a guide to the analytical studies.

CHAPTER 2

THE CLASSICAL RADIATION PROBLEM

2.1 Introduction

Before presenting the general problem of sound generation by turbulence, it is appropriate to review briefly the classical treatment of radiation due to elementary sources. The purpose of the discussion is twofold: first, to present the governing equations of acoustics, so that subsequently they may be compared to the more general equations; this will provide us with a logical basis for Lighthill's acoustic analogy. Second, to examine the near and far fields generated by elementary sources; this will enable us to construct an appropriate acoustic model that should contain the main features of the turbulent radiation.

2.2 Governing Equations

We will consider a compressible, inviscid, nonconducting fluid. It is well known that in such a fluid small perturbations propagate with a characteristic velocity and obey the wave equation. Since we are interested primarily in the generation of these waves, we shall exhibit in the equations of motion the fluctuating body forces, ρF_i , and masses, ρm , producing the perturbation fields. Our main goal is to show the wave character of the perturbations, and to determine the structure and intensity of radiation for a given distribution of fluctuating masses and body forces.

The equations expressing the conservation of mass and momentum for the fluid described above have the following form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = \rho m \tag{2-1}$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} + \frac{\partial p}{\partial x_i} = \rho F_i \tag{2-2}$$

In what follows we may, with no loss of generality, take the fluid as a whole to be at rest, its ambient pressure and density being, respectively, p_0 and ρ_0 . We shall further assume deviations from the ambient state are small:

$$u_i = u_i' , \quad p = p_0 + p' , \quad \rho = \rho_0 + \rho'$$

where $\frac{u'}{a_0}, \frac{p'}{p_0}, \frac{\rho'}{\rho_0} \ll 1$ and a_0 is the sound velocity of the medium. Making these substitutions in Equations (2-1) and (2-2), and neglecting higher order terms, we obtain

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = \rho_0 m \tag{2-3}$$

$$\frac{\partial \rho_0 u_i}{\partial t} = -\frac{\partial p}{\partial x_i} + \rho_0 F_i \qquad (2\text{-}4)$$

where henceforth, for convenience, the primes will be omitted.

These equations, together with the isentropic relation between the pressure and density fluctuation, $p' = a_0^2 \rho'$, enable us to calculate the velocity and pressure fields, for example, outside of a closed, bounded region of space produced by a known distribution of m and F_i over the interior.

We now take the time derivative of (2-3) and the divergence of (2-4) and subtract

$$\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial^2 p}{\partial x_i \partial x_i} = \frac{\partial \rho_0 m}{\partial t} - \frac{\partial \rho_0 F_i}{\partial x_i}$$

With the use of the isentropic relation, this may be rewritten

$$\frac{1}{a_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x_i \partial x_i} = \frac{\partial \rho_0 m}{\partial t} - \frac{\partial \rho_0 F_i}{\partial x_i} \qquad (2\text{-}5)$$

It is thus seen that the pressure perturbations obey an inhomogeneous wave equation with a known source distribution

$$\frac{\partial \rho_0 m}{\partial t} - \frac{\partial \rho_0 F_i}{\partial x_i} .$$

2.3 General Solution

If this function is distributed in an infinite volume V bounded internally by a surface S, the solution of (2-5), usually called the Kirchhoff solution, may be written down explicitly (Ref.79, p.429)

$$p(\underset{\sim}{x}, t) = \frac{1}{4\pi}\int_V \left[\frac{\partial \rho_0 m}{\partial t} - \frac{\partial \rho_0 F_i}{\partial \eta_i}\right] dV(\underset{\sim}{\eta}) +$$

$$+ \frac{1}{4\pi}\int_\Sigma \left\{\frac{1}{r}\left[\frac{\partial p}{\partial n}\right] - [p]\frac{\partial}{\partial m}\left(\frac{1}{r}\right) - \frac{1}{a_0 r}\frac{\partial r}{\partial n}\left[\frac{\partial p}{\partial t}\right]\right\} dS(\underset{\sim}{\eta}) \qquad (2\text{-}6)$$

Here r is the distance between a volume or surface element at $\underset{\sim}{\eta}$ to point $\underset{\sim}{x}$ where the pressure is to be determined; n is the outward normal from $\underset{\sim}{S}$, and the square brackets indicate that the quantities within them are taken at the retarded time $t - (r/a_0)$.

If no internal surfaces are present, Equation (2-6) becomes simply

$$p(\underset{\sim}{x}, t) = \frac{1}{4\pi}\int \frac{1}{r}\left[\frac{\partial \rho_0 m}{\partial t} - \frac{\partial \rho_0 F_i}{\partial \eta_i}\right] dV(\underset{\sim}{\eta}) \qquad (2\text{-}7)$$

Thus, in order to determine the pressure at any point in the field, an *a priori* knowledge of the time rate of change of the mass flow fluctuations and of the body force gradients throughout the volume V is necessary. The nature of radiation produced by these two volume integrals will be examined later in this chapter.

2.4 The Boundary Value Problem

In many instances the radiation problem is more amenable to solution as a boundary value problem. If, for instance, the sources are concentrated within a surface S, outside of which the field is homogeneous, then the field is determined uniquely by its properties on this surface together with possibly a local condition at infinity. It is convenient to define at this point a velocity potential ϕ such that

$$\frac{\partial \phi}{\partial x_i} = u_i \tag{2-8}$$

ϕ can be shown to be related to the pressure as

$$p = -\rho_0 \frac{\partial \phi}{\partial t} \tag{2-9}$$

to satisfy within the source-free volume the wave equation

$$\frac{1}{a_0^2}\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x_i \partial x_i} = 0 \tag{2-10}$$

The solution of (2-10) which describes, for example, the sound produced by small pulsations of S, is completely determined by the usual radiation condition at infinity (waves are entirely outgoing with respect to S) and by the requirement that the medium follow the motion of the surface S :

$$u_n = \frac{\partial \phi}{\partial n} \equiv G \quad \text{on} \quad S \tag{2-11}$$

The function G may be found from the motion of the surface. To be consistent we must assume that the velocity on S remains small (in the acoustic sense), in which case $G = \partial f/\partial t$, where $f(\underset{\sim}{x}, t) - m = 0$ defines the surface S.

2.5 Structure of the Radiation Field

Either of the aforementioned methods can of course be applied in solving for the emission of sound by radiators, and it usually depends on the particular problem which is the more convenient one to use. As we will see later, Lighthill formulated the aerodynamic noise problem in such a way that the first method of solution was the logical one. He was able to express the radiated pressure field in terms of an explicit volume integral in a form similar to Equation (2-7). The main difficulty of his formulation lies in the evaluation of the integral. Liepmann, on the other hand, proposed the second method of attack. In this approach the difficulty reappears as an unknown boundary condition.

In this section we will consider some simple examples of mass-like and force-like sources representing emission from a point and which can be easily handled mathematically. With the method described in the previous section, we will obtain the desired expressions for their radiation fields and compare them with the more general, continuous distribution afforded by Equation (2-7). On the basis of such a comparison, we will be able to make conjectures concerning the nature of the radiation field of more complicated sources.

(a) *Mass-Like Emitters*

Suppose that through a spherical control surface a mass of fluid is ejected uniformly and radially at a time-dependent rate. The acoustic field is represented by a solution of the wave equation having the form

$$\phi = -\frac{q(t - r/a_0)}{4\pi r} \tag{2-12}$$

where, by the boundary condition (2-11), q is related to the instantaneous mass flux at the sphere $r = r_0$, say $q_0(t)$, as follows:

$$q_0(t) = q(t - r_0/a_0) + \frac{r_0}{a_0}\frac{\partial q(t - r_0/a_0)}{\partial t}$$

Suppose that associated with the time variation of mass flux there is a characteristic frequency ω, so that $q_0(t)$ is a function of $t\omega = t^*$. According to the last equation, therefore, there is a dimensionless parameter $r_0\omega/a_0$ contained in the boundary condition. Moreover, if $r_0\omega/a_0 \ll 1$, $q_0(t) = q(t)$ approximately, and the boundary $r = r_0$ may be said to lie in the *near field*. Similarly, if $q(t)$ so determined is inserted into Equation (2-12), there emerges a second dimensionless parameter, representing the characteristic distance of an observer from the boundary divided by the wavelength a_0/ω. If this number is large compared to one, the observer may be said to be situated in the *far field* of the acoustic radiation. (If a random distribution of sources over a region is under consideration, it will also be useful to define the far field as that part of the sound where the above condition holds, and also which is far from the sources in units of a typical correlation length). With $q = q_0$ the pressure may be obtained from Equation (2-9).

$$p = \frac{\rho_0}{4\pi r}\frac{\partial q_0(t - r/a_0)}{\partial t}$$

Thus the pressure at time t and at a distance r from the source is determined by the rate of change of the flux through a sphere of sufficiently small radius r_0, evaluated at the retarded time $t - r/a_0$.

If this simple source were not concentrated within the control surface but distributed over a volume V, then the pressure field would be given by a volume integral identical to the first right-hand side term of Equation (2-7), where $\partial\rho_0 m/\partial t$ corresponds to the source strength per unit volume.

(b) *Force-Like Emitters*

Let us consider now a solid spherical surface executing an oscillating motion in the x_i direction. Clearly a spherical harmonic of the first order of the form

$$\phi = -\frac{\partial}{\partial x_i}\frac{A_i(t - r/a_0)}{r}$$

will satisfy the necessary boundary conditions, at least in the near field. For the components $A_i(t)$ one obtains, using (2-11),

$$\frac{\partial f}{\partial t} = -2\frac{n_i A_i}{r_0^3} = n_i u_i(t)$$

and

$$\phi = -\frac{r_0^3}{2}\frac{\partial}{\partial x_i}\frac{u_i(t - r/a_0)}{r}$$

Clearly, the potential field is caused by a dipole. The pressure field, according to Equation (2-9), will have the form

$$p = \frac{\rho_0 r_0^3}{2}\frac{\partial}{\partial x_i}\frac{1}{r}\frac{\partial u_i(t - r/a_0)}{\partial t}$$

Now the total external force acting on an oscillating sphere is[37]

$$D_i = 2\pi\rho_0 r_0^3 \frac{\partial u_i}{\partial t}$$

Consequently, the emitted pressure may be written in terms of this force

$$p = \frac{1}{4\pi}\frac{\partial}{\partial x_i}\frac{D_i(t - r/a_0)}{r} + 0\left(\frac{r_0}{a_0}\right)^2 \qquad (2\text{-}13)$$

It is seen, therefore, that the acoustic dipole field is equivalent to a field produced by a concentrated fluctuating force, the dipole strength being equal to the force D_i . (For a more general treatment, see Lamb).

On the basis of this argument, we may interpret the second term of the right-hand side of Equation (2-7) as a volume distribution of dipoles of strength $\rho_0 F_i$. Thus, in a uniform acoustic medium which contains distributed mass-like and force-like sources, the radiated pressure field can be interpreted as that of simple poles and dipoles (provided the force-like sources have no spacial gradients; this point will be further discussed later).

We will examine further the nature of these two types of radiation.

2.6 The Stokes Effect

Let us introduce the following dimensionless quantities:

$$t^* = \frac{a_0 t}{\lambda}, \qquad x^* = \frac{x}{l}, \qquad \lambda^* = \frac{\lambda}{l}, \qquad \lambda = a_0/\omega$$

where λ and l are the characteristic wavelength and linear dimension (size of the radiator) in the problem. Introducing these quantities into Equations (2-12) and (2-13), we obtain

$$\text{for a simple source:} \qquad p_s^* = \frac{1}{4\pi r^*}\frac{\partial q^*}{\partial t} \tag{2-14}$$

$$\text{for a dipole:} \qquad p_d^* = \frac{1}{4\pi r^{*2}}\left[\frac{r^*}{\lambda^*}\frac{\partial n_i D_i^*}{\partial t^*} + n_i D_i^*\right] \tag{2-15}$$

It is to be noted here that for a fixed λ^*, while due to a simple source the pressure varies like $1/r^*$ over the whole medium, due to a dipole it behaves differently in the near and far field, namely,

$$\text{for} \qquad \frac{r^*}{\lambda^*} \ll 1, \qquad p_d^* \sim \frac{n_i D_i^*}{4\pi r^{*2}} \tag{2-16}$$

$$\text{for} \qquad \frac{r^*}{\lambda^*} \gg 1, \qquad p_d^* \sim \frac{1}{4\pi r^{*2}}\frac{r^*}{\lambda^*}\frac{\partial n_i D_i^*}{\partial t^*} \tag{2-17}$$

Assuming now that the pressure fluctuations near the two radiators are approximately the same, Figure 1 indicates the pressure amplitude distributions with distance for a fixed λ^*.

We note in Figure 1, first, that the behavior of the pressure fluctuation in the far field is similar in the two cases (see also equations (2-14) and (2-17)), second, that the amplitude of the pressures in the far field is much smaller for the dipole. Thus the dipole is a less efficient radiator.

This fact was well recognized by Stokes who explains it in the following way[78]. In the region $1 \ll r^* \ll \lambda^*$, the fluid elements follow the motion of the sphere almost instantaneously; thus they execute a simple reciprocating motion producing negligibly small compression and dilatation. At larger distances, however, the fluid elements will lag behind the sphere motion, producing compression and rarefaction waves that will propagate outward. This outward propagation is therefore caused simply by the time lag between emission and fluid readjustment far from the source. Thus an appreciable fraction of the energy imparted from the body to the fluid will go into the reciprocating motion, which renders the radiation inefficient.

If we consider now the more general case of distributed sources, the conclusions are similar: the radiation from a distributed simple source field of non-zero strength is more efficient than that of a distributed dipole field. However, if the total source strength of the simple poles is zero (as it is for the dipole field), this is not true anymore; in fact, in this case the far field radiation is equivalent for the two types of sources.

2.7 Sources of Higher Order

There are many instances when the fluctuating force per unit volume $\rho_0 F_i$ is a divergence of a stress tensor T_{ij} ,

$$\rho_0 F_i = \frac{\partial T_{ij}}{\partial x_j} \tag{2-18}$$

If we substitute this into Equation (2-2), then it is easy to show that the volume integral for the pressure (Equation (2-7)) will contain a double divergence term. Or, if we consider a concentrated stress S_{ij} over a small volume V , the pressure fluctuation may be expressed in the form

$$p = \frac{V}{4\pi} \frac{\partial^2}{\partial x_i \partial x_j} \frac{S_{ij}(t - r/a_0)}{r} \tag{2-19}$$

Carrying out the differentiation, we obtain the dimensionless equation

$$p^* = \frac{1}{4\pi r^{*3}} \left(3n_i n_j S_{ij}^* + \frac{3r^*}{\lambda^*} \frac{\partial n_i n_i S_{ij}^*}{\partial t^*} + \frac{r^{*2}}{\lambda^{*2}} \frac{\partial^2 n_i n_j S_{ij}^*}{\partial t^{*2}} \right) \tag{2-20}$$

Comparing again the near and far field form of the pressure fluctuations, and using the same argument as in the previous section, one finds that the ratio of the strength of the radiation to the fluctuation in the near field will be even smaller than in the dipole case. Thus, the quadrupole process is less efficient than the dipole one, particularly at the larger acoustic wavelengths.

CHAPTER 3

THE GENERALIZED WAVE EQUATION

3.1 General Considerations

In order to fix our ideas, we shall consider a uniform velocity field in which a finite turbulent region having a known mean velocity distribution moves at a certain speed. We assume initially that no solid surface is present. We shall be interested in the possible production of sound by turbulence and in the structure of the acoustic field throughout the uniform region. In principle at least, one could proceed to solve the complete set of equations describing the flow and calculate the noise field. Obviously, there could be no hope of real success by such procedure, not only because of the complicated initial value problem which must be solved, but also because there is as yet no physical basis upon which to build a theory of the noise-producing mechanism, that is, of the turbulence itself. One possible alternative method of attack is to borrow ideas from a related discipline and focus attention on the acoustical problem, and upon the structure of the sound field alone, and try to set up the problem analogous to that of a noise field due to acoustic sources indicated in the previous chapter. In other words, one would like to express the equation of motion in the form of an inhomogeneous wave equation in which a certain mechanism within the turbulent field provides the source term.

3.2 Derivation of the Wave Equation

Although in the present work we will consider aerodynamic sound generation only, that is, the generation of sound by a fluctuating vorticity field, it will be constructive when setting up the general equations to include other sources as well, so that one may make some comparison between the nature of various ways of generating sound. We will therefore write the conservation laws in a form which will include a source density, m , external fluctuating body forces, F_i , and heat sources Q . Assuming that the gas is compressible, viscous, heat-conducting and perfect, the conservation laws have the following form:

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{\partial u_i}{\partial x_i} = m \qquad (3\text{-}1)$$

$$A_i = -\frac{1}{p}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} + F_i \qquad (3\text{-}2)$$

where $A_i = \frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}$ and τ_{ij} is the viscous stress tensor

$$\rho T\frac{DS}{Dt} = \Phi - \frac{\partial}{\partial x_i} k \frac{\partial T}{\partial x_i} + \rho Q \qquad (3\text{-}3)$$

Here T is the temperature, Φ the dissipation function and S the entropy. We now write the second law of thermodynamics in the form

$$\frac{DS}{Dt} = \frac{C_v}{T}\frac{DT}{Dt} - \frac{R}{\rho}\frac{D\rho}{Dt} \qquad (3\text{-}4)$$

where C_v is the specific heat of constant volume, and R the specific gas constant. By using the equation of state

$$p = \rho RT \qquad (3\text{-}5)$$

and remembering that

$$R = C_p - C_v , \qquad \gamma = C_p/C_v$$

Equation (3-4) may be rewritten

$$\frac{1}{C_p}\frac{DS}{Dt} = \frac{1}{\gamma p}\frac{Dp}{Dt} - \frac{1}{\rho}\frac{D\rho}{Dt} \tag{3-6}$$

We take now the total derivative of (3-1) and (3-6) and the divergence of (3-2)

$$\frac{D}{Dt}\frac{1}{\rho}\frac{D\rho}{Dt} = -\frac{D}{Dt}\frac{\partial u}{\partial x_i} + \frac{Dm}{Dt} \tag{3-7}$$

$$\frac{D}{Dt}\frac{1}{C_p}\frac{DS}{Dt} = \frac{D}{Dt}\frac{1}{\gamma p}\frac{Dp}{Dt} - \frac{D}{Dt}\left(\frac{1}{\rho}\frac{D\rho}{Dt}\right) \tag{3-8}$$

$$-\frac{\partial}{\partial x_i} a^2 \frac{\partial p}{\partial x_i} = \frac{\partial A_i}{\partial x_i} - \frac{\partial}{\partial x_i}\frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} - \frac{\partial F_i}{\partial x_i} \tag{3-9}$$

where the isentropic speed of sound $a^2 = \gamma p/\rho$ has been introduced. Combining (3-7) and (3-8)

$$\frac{D}{Dt}\frac{1}{\gamma p}\frac{Dp}{Dt} = \frac{D}{Dt}\frac{1}{C_p}\frac{DS}{Dt} - \frac{D}{Dt}\frac{\partial u_i}{\partial x_i} + \frac{Dm}{Dt}$$

and adding it to (3-9) we obtain

$$\frac{D}{Dt}\frac{1}{p}\frac{Dp}{Dt} - \frac{\partial}{\partial x_i}\frac{a^2}{p}\frac{\partial p}{\partial x_i} = \gamma\frac{\partial A_i}{\partial x_i} - \gamma\frac{D}{Dt}\frac{\partial u_i}{\partial x_i} - \gamma\frac{D}{Dt}\frac{1}{C_p}\frac{DS}{Dt} -$$

$$-\gamma\frac{\partial}{\partial x_i}\frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} + \gamma\frac{Dm}{Dt} - \gamma\frac{\partial F_i}{\partial x_i} \tag{3-10}$$

This equation in a slightly different form was first obtained by Phillips[61]. Introducing the pressure perturbation $p' = p - p_0$, the equation may be rewritten

$$\left\{\frac{D^2}{Dt^2} - \frac{\partial}{\partial x_i} a^2 \frac{\partial}{\partial x_i}\right\}\frac{p'}{p_0} = \gamma\frac{D}{Dt}\left(m - \frac{\partial u_i}{\partial x_i} - \frac{1}{C_p}\frac{DS}{Dt}\right) + \gamma\frac{\partial}{\partial x_i}\left(A_i - F_i - \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}\right) \tag{3-11}$$

The pressure perturbations in an inhomogeneous medium are thus governed by an equation of this form (it is not the only form; see, for instance, Lighthill's formulation in Chapter 4). The equation brings to light two basic difficulties not present in the classical radiation problem dealt with in Chapter 2.

Inhomogeneity. Since the turbulent field usually is inhomogeneous, both in mean velocity and temperature, the usual wave operator is replaced by a more general one. If the right-hand side is known and if the variable coefficients of the partial differential equation are prescribed, the structure of the sound may be studied in various ways, and (3-11) provides then the starting point for several theories. However, even in the case of a linear inhomogeneous equation, general solutions which could be of interest here are known only under quite restrictive conditions. Thus, in view of the somewhat speculative nature of the simplifications that must be made, a satisfactory and general discussion of the effect of inhomogeneities is quite difficult.

Nonlinearity. The terms on the right-hand side of Equation (3-11) are grouped in such a way that purely formalistically one can recognize "mass-like" and "force-like" sources as in classical acoustics. For instance, it is conjectured that the dilatation term, $\partial u_i/\partial x_i$, and the entropy production within the turbulent field would generate noise as does a mass fluctuation in an acoustic medium: in effect, like a simple source type of radiation. On the other hand, fluctuations in acceleration and in shearing stress are "force-like", they produce dipole or higher order radiation. This rather superficial argument, however, has to be examined more carefully.

The terms in question are in general implicit functions of the pressure and therefore should be considered as external forcing functions acting on the wave equation. Clearly, certain assumptions will have to be introduced in order to decouple these terms from the radiated pressure field. It should be emphasized, however, that this does not necessarily mean a linearization of the equations.

The following chapters will present simplifications introduced by various authors in an attempt to eliminate some of these difficulties.

CHAPTER 4

LIGHTHILL'S ACOUSTIC ANALOGY

4.1 Introduction

It was shown in the previous chapter that the equation governing the pressure fluctuations is too difficult to solve in general. Certain simplifications will be introduced to make the problem tractable. It soom becomes obvious that a conventional linearization procedure that is to neglect terms, which are in some sense "second order", is inadequate. Certain terms that we know to be important in turbulence production, and which might also be essential for the noise generation, would disappear as a result of such a linearization. We will discuss in this chapter an approximation introduced by Lighthill, who first formulated successfully the aerodynamic noise problem. His formulation is based upon an acoustic analogy that is an exact one, the approximation being introduced only after the analogy has been established. In order to demonstrate the generality of the analogy, we will follow Lighthill's deductive approach, instead of applying his approximations directly to the general equations obtained in the previous chapter.

4.2 The Acoustic Analogy

Lighthill succeeded in formulating the problem by a seemingly simple rearrangement of the equations of motion, but, in fact, he was motivated by an ingenious and useful idea: to draw an analogy between acoustical and aerodynamic radiation. As was shown in Chapter 2, the density (or pressure) fluctuations in a uniform acoustic medium at rest obey the well-known inhomogeneous wave equation

$$\frac{\partial^2 \rho}{\partial t^2} - a_0^2 \frac{\partial^2 \rho}{\partial x_i \partial x_i} = \text{source terms} \tag{4-1}$$

One may pose the question: is it possible to express the density fluctuations in a turbulent fluid in motion by an equation of a similar form? We may proceed by considering the conservation laws of mass and momentum in a viscous, compressible gas:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 \tag{4-2}$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \tag{4-3}$$

where τ_{ij} is the viscous stress tensor. By taking the time derivative of (4-2), the divergence of (4-3), and subtracting the two, we obtain

$$\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial^2 p}{\partial x_i \partial x_i} = \frac{\partial^2 (\rho u_i u_j - \tau_{ij})}{\partial x_i \partial x_j} \tag{4-4}$$

We may rewrite (4-4) in a purely formal manner*

$$\frac{\partial^2 \rho}{\partial t^2} - a_0^2 \frac{\partial^2 \rho}{\partial x_i \partial x_i} = \frac{\partial^2}{\partial x_i \partial x_j} [\rho u_i u_j - \tau_{ij} + (p - a_0^2 \rho)\delta_{ij}] \tag{4-5}$$

If the right-hand side of this equation is considered to be a forcing term, Equations (4-1) and (4-5) are indeed of the same form. Thus, *the density fluctuations in the real fluid may be identified with those occurring in a uniform acoustic medium at rest produced by the source terms* $\partial^2 T_{ij}/\partial x_i \partial x_j$ where

$$T_{ij} = \rho u_i u_j - \tau_{ij} + (p - a_0^2 \rho)\delta_{ij} \tag{4-6}$$

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* Lighthill chose the density rather than the pressure as the primary variable mainly because the forcing term is then the double divergence of an effective stress tensor leading to a solution in the form of a quadrupole field; if instead the pressure field is sought, the forcing term has a more complicated structure.

We may consider the turbulent fluctuations as some equivalent external forcing function acting on a free system which, in this case, is the uniform acoustic medium at rest. This is in essence Lighthill's acoustic analogy.

4.3 Remarks on the Analogy

It should be pointed out that up to now no simplifying assumptions have been introduced explicitly into the equations. Thus, the analogy describes not only the generation of sound at its source, but also accounts for certain global features of the sound field, e.g., the effect of an overall convection of the turbulence with respect to the fluid at infinity; it is then merely required that the analogy be applied in a coordinate system fixed to the free stream. It also includes dissipation by viscosity and by conduction.

However, the analogy does have certain limitations. In the extreme case when the generated sound field is so strong that it interacts with and alters the turbulent field itself, the analogy has no obvious meaning. What then are the conditions under which the acoustic analogy holds in the sense that it is suggestive of the analytical properties of the full solution? This question, in spite of its relevance, has not been treated adequately in the literature. A few comments are therefore appropriate. Vaguely speaking, Lighthill's acoustic analogy will be useful only when the process of generation of sound by a flow is in some sense separable from the sound itself. To put this another way, there should exist a clean division of the flow into near and far fields, a separation that implies corresponding division of the labor of constructing a solution of (4-5).

The inhomogeneous wave equation (4-5) may be put into an integral form using Kirchhoff's solution (Chapter 2). For simplicity we assume at present that no internal surfaces exist in the field, in which case the density fluctuations $\rho' = \rho - \rho_0$ may be written

$$\rho'(\underset{\sim}{x},t) = \frac{1}{4\pi a_0^2}\int_V \frac{1}{r}\left[\frac{\partial^2 T_{ij}}{\partial\eta_i\partial\eta_j}\right] dV(\underset{\sim}{\eta}) \qquad (4\text{-}7)$$

Without making any statements regarding the stress tensor T_{ij} , we have merely transformed the partial differential equation (4-5) into an integral equation (4-7). Clearly, we may consider (4-7) a solution of our problem only if the sound field has no contribution to the stress tensor T_{ij} . This is not precisely the case. Thus, in order to consider T_{ij} an *external* forcing function independent of the sound field - the basis of the acoustic analogy - we have to introduce certain assumptions.

4.4 Lighthill's Approximation

There are two assumptions introduced by Lighthill at this point which we will discuss subsequently.

4.4.1 Low Mach Number Flows

Let us examine the consequences of this assumption. First of all, under this condition the mean temperature or density gradients in the flow field may be neglected, and therefore $\rho \simeq \rho_0$. It follows that density fluctuations at constant pressure produced by these gradients are small compared to the fluctuations associated with the sound field; thus ρ' is closely balanced by p'/a_0^2 . The last term in (4-6) may therefore be neglected.

The remaining two terms contain velocity fluctuations. These fluctuations are either part of the vorticity field of the turbulence, u_{vi} , or part of the sound field generated by the turbulence, u_{si} . We may thus write formally

$$u_i = u_{vi} + u_{si}$$

The viscous term of Equation (4-7) can now be written in two parts. The term containing u_{si} is responsible for the dissipation of sound energy by viscosity. This, however, is a very slow process, as shown by Kirchhoff, and becomes important only at extremely large values of r/λ , at distances outside the region of our interest. The second term containing the u_{vi} fluctuations, when compared with the Reynolds stress fluctuations $\rho u_i u_j$, are also negligibly small in low Mach number flows. Equation (4-7) can now be written in the form

$$\rho'(\underset{\sim}{x},t) = \frac{1}{4\pi a_0^2} \int_V \frac{1}{r} \left[\frac{\partial^2 \rho_0 u_i u_j}{\partial \eta_i \eta_j} \right] dV(\underset{\sim}{\eta}) \tag{4-8}$$

4.4.2 Small Refraction Effects

The stress $\rho_0 u_i u_j$ again has contributions from the vorticity and sound fields. Following the usual acoustic approximation, the pure sound term $\rho_0 u_{si} u_{sj}$; can be neglected. This leaves the pure vorticity term $\rho_0 u_{vi} u_{vj}$ and the mixed terms in the volume integral. Within the framework of Lighthill's approximation only the pure vorticity term is retained. The neglect of the mixed term is difficult to justify without adequate experimental evidence. In the presence of large mean velocity gradients, terms of the form

$$\frac{\partial}{\partial \eta_i} u_{si} \frac{\partial \bar{u}_i}{\partial \eta_j}$$

may indeed be important. They refer to refraction of sound within the turbulent region by the mean velocity, $\bar{u}_i$.

4.5 Nature of the Source Term

Under the assumptions described above, we propose that

$$T_{ij} \simeq \rho_0 u_i u_j$$

where u_i and u_j refer to the mean and fluctuating velocities of zero divergence (no sound is included). Referring now to the discussion in Chapter 2, we recognize that the volume integral of Equation (4-7) represents the gradient of a force-like source, more specifically, a distributed quadrupole source with a strength T_{ij} (see Equation 2-20). Furthermore, we note that the radiation of such a source is a very inefficient one, especially at large wavelengths, since the ratio of the far-field to near-field pressure fluctuations contains the factor $1/\lambda^{*2}$. This inefficiency is the result of the Stokes cancellation effect in the near field, this cancellation being more effective for the higher order source type of radiation.

4.6 Evaluation of the Volume Integral

Equation (4-7) is rewritten to indicate explicitly the argument of

$$\rho'(\underset{\sim}{x},t) = \frac{1}{4\pi a_0^2}\int_V \frac{\partial^2 T_{ij}(\underset{\sim}{\eta},t - r/a_0)}{\partial\eta_i\partial\eta_j}\frac{dV(\underset{\sim}{\eta})}{r} \tag{4-9}$$

where

$$r^2 = (x_i - \eta_i)(x_i - \eta_i)$$

Now the quantity of interest is the mean square of the density fluctuation, which is proportional to the sound intensity. This can be written as

$$\overline{\rho'^2(\underset{\sim}{x})} = \frac{1}{16\pi^2 a_0^4}\iint_{V^2} \frac{\partial^2 T_{ij}(\underset{\sim}{\eta},t - r/a_0)}{\partial\eta_i\partial\eta_j}\frac{\partial^2 T_{kl}(\underset{\sim}{\eta}',t - r'/a_0)}{\partial\eta_k'\partial\eta_l'}\frac{dV(\underset{\sim}{\eta})dV(\underset{\sim}{\eta}')}{rr'} \tag{4-10}$$

where $\underset{\sim}{\eta}$ and $\underset{\sim}{\eta}'$ are two points in the turbulent field where the correlations are taken.

It is seen that, in order to evaluate the sound intensity, the space-time correlation of the double space derivatives of the Reynolds stresses, T_{ij}, have to be calculated (or measured) and integrated over the volume of the turbulent field. This would be an extremely difficult, if not impossible, task indeed.

Under certain conditions Equation (4-10) can be somewhat simplified. Suppose the difference in the retarded times $r/a_0 - r'/a_0 = l/a_0$ is small compared to the characteristic period $1/\omega$ of the turbulence, and to the time of emission r/a_0; that is

$$\frac{l\omega}{a_0} \ll 1 \qquad \text{and} \qquad \frac{l}{r} \ll 1 \tag{4-11}$$

The first condition is usually met in low Mach number flows which we consider, while the second condition is satisfied if the observation distance is large compared to the extent of the turbulent source. This last one, however, is not always easy to satisfy for practical reasons (signal levels too low at large distances). With this in mind we may proceed by applying the divergence theorem twice to the integral of Equation (4-9) (see the details in Reference 77). One obtains then the following form

$$\rho'(\underset{\sim}{x},t) = \frac{1}{4\pi a_0^2}\frac{\partial^2}{\partial x_i\partial x_j}\int_V \frac{T_{ij}(\underset{\sim}{\eta},t - r/a_0)}{r}\,dV(\underset{\sim}{\eta}) \tag{4-12}$$

Carrying out the double differentiation under the integral sign, one obtains three terms that fall off as the inverse first, second and third power of r (see also Equation 2-20). For large distances from the turbulence source (for large r^*/λ^*) the r^{-1} term will dominate, therefore

$$\rho'(\underset{\sim}{x},t) = \frac{1}{4\pi a_0^4} \int_V \frac{(x_i - \eta_i)(x_j - \eta_j)}{r^3} \frac{\partial^2 T_{ij}(\underset{\sim}{\eta}, t - r/a_0)}{\partial t^2} dV(\underset{\sim}{\eta}) \qquad (4\text{-}13)$$

and the mean square density will have the form

$$\overline{\rho'^2(\underset{\sim}{x})} =$$

$$\frac{1}{16\pi^2 a_0^8} \iint_{V^2} \frac{(x_i-\eta_i)(x_j-\eta_j)(x_k-\eta_k')(x_l-\eta_l')}{r^3 r'^3} \overline{\frac{\partial^2 T_{ij}(\eta, t-r/a_0)}{\partial t^2} \frac{\partial^2 T_{kl}(\underset{\sim}{\eta}', t-r'/a_0)}{\partial t^2}} dV(\underset{\sim}{\eta}) dV(\underset{\sim}{\eta}') \qquad (4\text{-}14)$$

Introducing the inequalities (4-11), Equation (4-14) simplifies to

$$\frac{\partial}{\partial \eta} \overline{\rho'^2(\underset{\sim}{x})} = \frac{1}{16\pi^2 a_0^8} \frac{x_i x_j x_k x_l}{r^6} \int_V \overline{\frac{\partial^2 T_{ij}(0, t - r/a_0)}{\partial t^2} \frac{\partial^2 T_{kl}(\underset{\sim}{\eta}', t - r'/a_0)}{\partial t^2}} dV(\underset{\sim}{\eta}') \qquad (4\text{-}15)$$

The notation $\frac{\partial}{\partial \underset{\sim}{\eta}}$ is introduced for the contribution to the sound field from unit volume of turbulence centered at $\underset{\sim}{\eta}$. Thus, the complexity of the integral has been considerably reduced; nevertheless its exact evaluation still represents a formidable task. However, an order of magnitude estimate of the sound intensity emanating from the volume V can now be performed and, in fact, the greatest practical value of the Lighthill approach is the fact that it provides the basis of such an estimate (Chapter 6).

4.7 Ribner's Simple-Source Theory

While the physical basis of the acoustic analogy of Lighthill is relatively easily understood using his mathematical formalism, his interpretation of the effective sources as quadrupoles is a much more subtle one and more difficult to comprehend. This prompted Ribner to propose a different interpretation, but still within the framework of the acoustic analogy[74]. In Ribner's picture the sound generation may be interpreted in terms of simple sources produced by fluctuations in "pseudo sound" pressure; that is, in an incompressible flow approximation of the pressure. His formal expression for the radiation intensity is mathematically equivalent to that of Lighthill. Its evaluation is just as difficult a task as, for instance, Equation (4-15). For details the reader is referred to the original work of Ribner.

4.8 A Mechanical Model

As a further aid to the understanding of the reasoning behind Equation (4-5), Liepmann has described a simple mechanical model which, although it by no means furnishes an exact analogy, illustrates the way in which sound is generated by a flow.

This model starts from the observation that the role of the pressure force in *incompressible* flows is analogous to the role of the force exerted by the string of a pendulum. Indeed, in either case the equations follow from Hamilton's principle and the tension in the one, and the pressure in the other, enter the variational problem as Lagrange multipliers*. These forces then do no work and may be interpreted physically as *reactions*, against the gravitional and centrifugal forces in the pendulum problem, and against a change of volume of a fluid element in the hydrodynamic problem.

To introduce the "sound field" into the model we now imagine the string to be extensible and elastic with a certain spring constant, so that in the absence of lateral motion of the bob, the length of the string can be made to oscillate about some mean value. In practice, both modes of oscillation will occur, the length of the string with some characteristic frequency ω_s and the pendulum near the frequency ω_p. There is also a third parameter in the "boundary conditions", namely the amplitude, which we now fix. If $\omega_s \gg \omega_p$, i.e., if the string is essentially inextensible, the appropriate expansion procedure is obvious. If ω_s and ω_p are comparable, however, no obvious division of the two modes is possible. On the other hand, we are certainly able to write down an equation for the length l of the string in the form

$$\ddot{l} + k(l - l_0) = f(l, \theta, \dot{\theta}) \tag{4-16}$$

in which the right-hand side depends on l as well as on the angular displacement θ of the string and its first derivative with respect to time.

We are thus led to consider the following correspondence between this mechanical system (or, in fact, any similar system exhibiting two coupled modes) and the problem of aerodynamic noise. The motion of the bob may be identified with the turbulent fluctuations of the medium, having a certain characteristic amplitude (u' normalized by the mean value $\bar{u}$) and frequency (e.g. $L/\bar{u}$, where L is a characteristic eddy dimension). In place of ω_s we now take a/L, where a is the speed of sound. The perturbation for small ω_p/ω_s now becomes the perturbation for small Mach number M, and Equation (4-16) states the "acoustic analogy" for this mechanical system. The acoustic analogy therefore emerges here as an expression of a basic division of the computation of noise, a division which is exact in the case of small M. The model also predicts that at sufficiently high speeds the coupling between turbulence and sound is not obtainable by conventional perturbation methods, at least provided the turbulence level† scales with $\bar{u}$.

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* For a derivation of the equations for incompressible flow from Hamilton's principle, see, e.g., Sommerfield, *Mechanics of Deformable Bodies*, Chapter III, Academic Press, 1950.

† It should be noted that this mechanical model does not contain a length analogous to the characteristic wavelength of turbulent fluctuations. Since the notion of eddy convection does not enter, without further refinements the mechanism of noise generation at high speeds cannot be properly represented (cf. Chapters 6 and 7).

CHAPTER 5

THE ACOUSTIC ANALOGY APPLIED TO THE TURBULENT BOUNDARY LAYER

5.1 Curle's Analysis

The problem of aerodynamic noise generation in the presence of solid boundaries was first examined by Curle[11]. He showed how rigid surfaces were acoustically equivalent to dipoles of strength equal to the aerodynamic surface stresses. Within the framework of Lighthill's analogy, these surface stresses appeared to dominate the radiation field - because they represent dipoles of inherently greater radiation efficiency than the equivalent quadrupoles of the turbulent flow. This was Curle's approach, an eminently successful one in accounting for the noise produced by unstable flows in the vicinity of small surfaces. The theory, being an exact extension of the acoustic analogy, is applicable to the boundary layer problem also, but there one must exercise a certain amount of caution, since the surface dipoles are of zero strength in many instances of practical interest. It is easy to appreciate a proof of this property if one is first reminded of the way in which the general solution of the forced wave equation should be applied. This point is best illustrated by noting an equivalence between certain integrals over a closed surface Σ, and those over a volume V bounded by that surface.

Consider the volume integral

$$\int_V \left[\frac{\nabla^2\rho}{r}\right] dV(\underset{\sim}{\eta}) \qquad (5\text{-}1)$$

where the square brackets, [ ], indicate a retarded time, $t - r/a_0$, to be operative.

This can be expanded by completing divergences and applying the divergence theorem to introduce surface integrals over Σ.

$$\int_V \left[\frac{\nabla^2\rho}{r}\right] dV(\underset{\sim}{\eta}) = \int_\Sigma \left\{\frac{1}{r}\left[\frac{\partial\rho}{\partial n}\right] - \frac{\partial}{\partial n}\left(\frac{1}{r}\right)[\rho] + \frac{1}{a_0 r}\frac{\partial r}{\partial n}\left[\frac{\partial\rho}{\partial t}\right]\right\} dS(\underset{\sim}{\eta}) +$$

$$+ \int_V [\rho]\,\nabla^2\left(\frac{1}{r}\right) dV(\underset{\sim}{\eta}) + \int \frac{1}{a_0^2}\left[\frac{1}{r}\frac{\partial^2\rho}{\partial t^2}\right] dV(\underset{\sim}{\eta}) \qquad (5\text{-}2)$$

$\partial/\partial n$ is the derivative with respect to the outward normal to the surface at $\underset{\sim}{\eta}$.

Now $\nabla^2 (1/r)$ is equal to $-4\pi\delta(\underset{\sim}{r})$, so that the second term on the right-hand side is non-zero only if the observation point, $\underset{\sim}{r} = 0$, is enclosed by the surface Σ . The integrand of the first term on the right-hand side we shall denote X , and subtract the third term from each side to obtain the relation

$$\int_V \left[\nabla^2\rho - \frac{1}{a_0^2}\frac{\partial^2\rho}{\partial t^2}\right]\frac{dV(\eta)}{r} = -\frac{1}{a_0^2}\int_V \left[\frac{\partial^2 T_{ij}}{\partial\eta_i\partial\eta_j}\right]\frac{dV(\underset{\sim}{\eta})}{r}$$

$$= \int_\Sigma [X]\, dS(\underset{\sim}{\eta}) - \int_V [\rho]4\pi\delta(\underset{\sim}{r})\, dV(\underset{\sim}{\eta}) \qquad (5\text{-}3)$$

The first equality defines Lighthill's stress tensor, T_{ij} , cf. Equation (4-5). This equation can then be written in two forms, the first one applicable whenever the observation point, $\underset{\sim}{r} = 0$, is enclosed by the surface Σ ;

$$\rho(\underset{\sim}{x}, t) = \frac{1}{4\pi a_0^2}\int_V \left[\frac{\partial^2 T_{ij}}{\partial\eta_i\partial\eta_j}\right]\frac{dV(\underset{\sim}{\eta})}{r} + \frac{1}{4\pi}\int_\Sigma [X]\, dS(\underset{\sim}{\eta}) \qquad (5\text{-}4)$$

The second form is that applicable when the observation point is excluded from the volume V, and shows how the source system enclosed by the surface Σ makes *no* contribution at the point $\underset{\sim}{r} = 0$.

$$0 = \frac{1}{4\pi a_0^2}\int_V \left[\frac{\partial^2 T_{ij}}{\partial\eta_i\partial\eta_j}\right]\frac{dV(\underset{\sim}{\eta})}{r} + \frac{1}{4\pi}\int_\Sigma [X]\, dS(\underset{\sim}{\eta}) \qquad (5\text{-}5)$$

Curle's equation can be derived from these by completing the divergence in the volume integral and applying the divergence theorem.

$$\int_V \left[\frac{\partial^2 T_{ij}}{\partial\eta_i\partial\eta_j}\right]\frac{dV(\underset{\sim}{\eta})}{r} = \frac{\partial^2}{\partial x_i\partial x_j}\int_V \left[\frac{T_{ij}}{r}\right] dV(\underset{\sim}{\eta}) +$$

$$+ \int_\Sigma \left\{\frac{1}{r}\left[\frac{\partial T_{in}}{\partial\eta_i}\right] - \frac{\partial}{\partial\eta_i}\left(\frac{1}{r}\right)[T_{in}] + \frac{1}{a_0 r}\frac{\partial r}{\partial\eta_i}\left[\frac{\partial T_{in}}{\partial t}\right]\right\} dS(\underset{\sim}{\eta}) \qquad (5\text{-}6)$$

The suffix n is used to indicate the direction of the outward normal at the surface. The momentum relation equates $\partial T_{in}/\partial\eta_i$ to

$$\left\{-\frac{\partial(\rho u_n)}{\partial t} - a_0^2\frac{\partial\rho}{\partial n}\right\}.$$

Writing T_{ij} as $\rho u_i u_j + p_{ij} - a_0^2 \rho \delta_{ij}$, Equation (5-6) can be re-expressed

$$\frac{1}{4\pi a_0^2} \int\limits_V \left[\frac{\partial^2 T_{ij}}{\partial\eta_i \partial\eta_j}\right] \frac{dV(\underset{\sim}{\eta})}{r} = \frac{1}{4\pi a_0^2} \frac{\partial^2}{\partial x_i \partial x_j} \int\limits_V \left[\frac{T_{ij}}{r}\right] dV(\underset{\sim}{\eta}) -$$

$$- \frac{1}{4\pi} \int\limits_\Sigma [X] \, dS(\underset{\sim}{\eta}) - \frac{1}{4\pi a_0^2} \int\limits_\Sigma \left[\frac{\partial(\rho u_n)}{\partial t}\right] \frac{dS(\underset{\sim}{\eta})}{r} +$$

$$+ \frac{1}{4\pi a_0^2} \frac{\partial}{\partial x_i} \int\limits_\Sigma \left[\rho u_i u_n + p_{in}\right] \frac{dS(\underset{\sim}{\eta})}{r} \qquad (5\text{-}7)$$

The general equation of the acoustic analogy can then be written by inserting this expression into Equation (5-4)

$$\rho(\underset{\sim}{x}, t) = \frac{1}{4\pi a_0^2} \frac{\partial^2}{\partial x_i \partial x_j} \int\limits_V \left[\frac{T_{ij}}{r}\right] dV(\underset{\sim}{\eta}) - \frac{1}{4\pi a_0^2} \frac{\partial}{\partial x_i} \int\limits_\Sigma \left[P_i - \rho u_i u_n\right] \frac{dS(\underset{\sim}{\eta})}{r} -$$

$$- \frac{1}{4\pi a_0^2} \int\limits_\Sigma \frac{\partial}{\partial t} [\rho u_n] \frac{dS(\underset{\sim}{\eta})}{r} \qquad (5\text{-}8)$$

The radiation is seen to be zero if $\underset{\sim}{x}$, the observation point where $r = 0$, is excluded from V and Σ , a point evident from Equations (5-5) and (5-7).

P_i is the force exerted on the fluid by the surface in the x_i direction and is equal to $- p_{in}$. Curle's equation is the particular value of Equation (5-8) when the normal velocity at the surface is set equal to zero.

A comparison of this general result with that of Lighthill's theory, Equation (4-12), allows us to draw an important conclusion. It is evident that, in the presence of boundaries, sound is generated, not only by quadrupoles of strength T_{ij} distributed throughout the turbulent region, but also by dipoles distributed over Σ , of strength equal to the rate of change of momentum through Σ , and finally by simple sources at the surface, of strength equal to the rate of change of mass flux through Σ . Our knowledge of the increasing inefficiency of the more complex sources allows us to rate these three contributions in an ascending order of importance, but we shall see how such an argument sometimes proves to be misleading.

Before going on to discuss the general theory it is worth emphasizing one point which is often misunderstood. On a rigid surface the dipole strengths are related to forces applied on the fluid by the surface. It is sometimes argued that the radiation from these sources must be zero since the points of application of the forces do not move, so that the forces do no work. The point is easily answered when one appreciates that the whole theory is in the form of an acoustic analogy so that the source system is essentially an equivalent one, which need not correspond directly to any real source of acoustic energy. The dipole energy is in fact extracted from the turbulent volume and the equation could have been written in a way to stress that point because

$$\int_{\Sigma} P_i dS = \int_{V} \frac{\partial(\rho u_i)}{\partial t} dV , \qquad (5\text{-}9)$$

a relation that follows by applying the divergence theorem to the momentum equation. But, however the term is expressed, the situation is precisely analogous to a surface distribution of radiating dipoles and this is an important concept in the theory of aerodynamic sound production.

5.2 The Reflection Property of a Solid Surface

The mechanism of noise generation in the turbulent boundary layer is our particular concern here and the question of how sound is generated on an infinite, flat, rigid surface by boundary layer turbulence, is then an important one. It happens to be an instance when the major surface dipoles vanish, so that the theory must be extended considerably beyond Curle's analysis before it becomes really useful. This vanishing property has been argued on several grounds, most arguments being based on the equations of incompressible fluid motion. Phillips[57] showed how the instantaneous surface integral of pressure, which is the integrated dipole strength, vanished under those conditions, but it was Powell[65] in a paper later to be emphasized by Ffowcs Williams[22] and Meecham[56] who first gave a rigorous argument on the exact role of the surface pressure field. Since Powell's argument is also the simplest, it is with that that we deal here. Consider the situation illustrated in Figure 2. The real flow, including the turbulence and the observation point, is supposed to lie within a volume V enclosed by a surface Σ , which is, in part, coincident with the rigid surface. The equation relating the real equivalent sources to the sound heard at $(\underset{\sim}{x}, t)$ is then Equation (5-8). Below the surface is supposed an image flow, an exact specular reflection of the real flow, enclosed by the surface Σ' . It is immaterial whether this image is physically realizable or not, since it makes precisely *no* net contribution to the sound heard at the observation point $(\underset{\sim}{x}, t)$ which lies outside the surface Σ' . This point follows directly from Equations (5-5) and (5-7).

The sound heard at $(\underset{\sim}{x}, t)$ can then be written as the sum of that generated by the source system enclosed by the surface Σ and the zero contribution of that enclosed by Σ' . The surface velocity v_n is zero for the rigid surface, so that Equation (5-8) gives the total radiation field to be

$$\rho(\underset{\sim}{x}, t) = \frac{1}{4\pi a_0^2} \frac{\partial^2}{\partial x_i \partial x_j} \left\{ \int_V \left[\frac{T_{ij}}{r}\right] dV + \int_{V'} \left[\frac{T_{ij}}{r}\right] dV \right\} -$$

$$- \frac{1}{4\pi a_0^2} \frac{\partial}{\partial x_i} \left\{ \int_\Sigma \left[\frac{P_i}{r}\right] dS + \int_{\Sigma'} \left[\frac{P_i}{r}\right] dS \right\} \quad (5\text{-}10)$$

We recall that P_i was defined (page 22) to be the force in the i direction exerted on the fluid by unit area of surface. The tangential force is clearly equal for both the real and image flow but the normal force has exactly opposite sign across the boundary. The "normal", or pressure, dipoles therefore annihilate each other, leaving only the viscous dipoles whose axes lie in the surface. The summation over i should then be restricted to those directions, and this we achieve by replacing i by the suffix μ, a notation implying that viscous effects alone are operative. Equation (5-10) thus reduces exactly to

$$\rho(\underset{\sim}{x}, t) = \frac{1}{4\pi a_0^2} \frac{\partial^2}{\partial x_i \partial x_j} \int_{V+V'} \left[\frac{T_{ij}}{r}\right] dV - \frac{1}{2\pi a_0^2} \frac{\partial}{\partial x_\mu} \int_\Sigma \left[\frac{P_\mu}{r}\right] dS \quad (5\text{-}11)$$

We recall that T_{ij} in V' is not equal to T_{ij} in V, but is that associated with the specular reflection of T_{ij} in the surface. Similarly, r, and consequently the retarded time, differ in the two regions.

In the absence of viscosity, the second term in Equation (5-11) is zero and we see how the surface integral of Equation (5-8) has become exactly equivalent to the integral of the image quadrupoles distributed through the volume V'. We have, in fact, repeated Powell's argument that in the absence of viscous stresses a plane rigid surface acts like a passive reflector. The reflection property is distorted by any viscous terms which are reinforced by their images.

5.3 Radiation by Flows Near "Small" Surfaces

Bearing in mind the possibility of a vanishing dipole strength, we can now consider more specific applications of the general aerodynamic noise theory. If would seem appropriate to deal first with the question of sound generation by flows near surfaces small compared to the acoustic wavelength. It is there that Curle's extension of Lighthill's theory is straightforward. Equivalent dipoles associated with surface stresses overwhelm quadrupoles distributed throughout the unsteady flow, so that the pertinent equation is that involving the surface integrals of Equation (5-8).

$$\rho(\underset{\sim}{x}, t) \simeq - \frac{1}{4\pi a_0^2} \int_\Sigma \frac{\partial}{\partial t} \left[\rho u_n\right] \frac{dS(\underset{\sim}{\eta})}{r} - \frac{1}{4\pi a_0^2} \frac{\partial}{\partial x_i} \int_\Sigma \left[P_i - \rho u_i u_n\right] \frac{dS(\underset{\sim}{\eta})}{r} \quad (5\text{-}12)$$

Whenever the surface is rigid the normal velocity u_n is zero, so that this equation reduces to precisely Curle's result, which, when expressed in its far-field form, shows how the radiation strength increases with the rate of change of surface stresses.

$$\rho(\underset{\sim}{x}, t) \simeq \frac{1}{4\pi a_0^3} \int_{\Sigma} \frac{\partial r}{\partial x_i} \left[\frac{\partial P_i}{\partial t}\right] \frac{dS(\underset{\sim}{\eta})}{r} \tag{5-13}$$

The normal stresses induce dipoles whose axes lie perpendicular to the surface. These normal stresses, P_n, are likely to be the most significant since they contain the effects of both viscosity and pressure. The fluctuating pressure, which increases in direct proportion to a typical dynamic head, $\frac{1}{2} \bar{\rho} U^2$, exceeds the fluctuating viscous stresses by a factor of the order of a typical Reynolds number. Consequently, Equation (5-13) can be approximated by

$$\rho(\underset{\sim}{x}, t) \simeq \frac{-1}{4\pi a_0^3} \int_{\Sigma} \frac{(x_n - \eta_n)}{r^2} \left[\frac{\partial \rho}{\partial t}\right] dS(\underset{\sim}{\eta}) \tag{5-14}$$

The differentiation with respect to time is dimensionally equivalent to multiplication by a characteristic frequency, U/l, where l is a typical length. The integration introduces an l^2 in the dimensional treatment which, when applied to this equation, shows how the radiation density ρ' has the proportionality

$$\bar{\rho} \frac{l}{r} \frac{U^3}{a_0^3} \tag{5-15}$$

This radiation overwhelms the volume quadrupole contribution by a factor proportional to Mach number squared in intensity. This is evident from a comparison of the mean square density perturbations predicted here, $\bar{\rho}^2 (l^2/r^2)(U^6/a_0^6)$, with that predicted for the quadrupoles in the next chapter, $\bar{\rho}^2 (l^2/r^2)(U^8/a_0^8)$, (convection effects being set equal to zero).

This dimensional dependence has been observed in numerous experiments concerning Aeolian tones[57,17]. There, vortices are periodically shed from a cylinder lying normal to the direction of flow. As the vortices are shed the cylinder is subjected to a lifting force and this force gives rise to a dipole whose axis lies normal to the flow.

Should the cylinder be free to move, the periodic force will excite it into oscillation, and then the remaining terms of Equation (5-12) would have to be considered. Oscillatory motion would be mainly normal to the flow since the cylinder is subjected to a predominantly lifting force. If retarded time changes were ignored, the first integral of Equation (5-12) would vanish, since the outward normal surface velocity at one point is exactly cancelled by that at its diametrically opposite point on the cylinder. But, taking proper account of retarded time, one can show that term to be equivalent to an additional dipole field. The cancellation is incomplete, since the surface velocity has changed slightly in the time interval separating emission from any two opposite points. This change is then the strength of the additional equivalent dipole. The leading term of a Taylor expansion about any point in the cylinder, of diameter D, shows that strength to be

$$\frac{D}{a_0} \frac{\partial r}{\partial x_n} \frac{\partial^2}{\partial t^2} [\rho u_n] = \frac{D}{a_0} \frac{(x_n - \eta_n)}{r} \frac{\partial^2}{\partial t^2} [\rho u_n] \tag{5-16}$$

This term, being linear in the surface velocity, overwhelms the velocity term in the second integral of Equation (5-12) by a factor of the order of the ratio of a typical flow velocity to the vibrational surface velocity, generally a very large factor. The equation relevant to Aeolian tone generation by a vibrating cylinder is then

$$\rho(\underset{\sim}{x}, t) \simeq \frac{-1}{4\pi a_0^3} \int_{\Sigma} \frac{(x_n - \eta_n)}{r^2} \left[\frac{\partial p}{\partial t} - \frac{D}{2} \frac{\partial^2}{\partial t^2} (\rho u_n)\right] dS(\underset{\sim}{\eta}) \tag{5-17}$$

The surface pressure is, of course, related to the velocity v_n which it excites. Should the cylinder be unrestrained and under the assumption that gross flow is unaffected by the response of the cylinder, these two terms can be brought together very simply. The result is particularly simple in its far-field form when the surface is sufficiently small that retarded time differences are negligible.

The integral

$$\int_{\Sigma} \frac{(x_n - \eta_n)}{r} \, p \, dS(\underset{\sim}{\eta})$$

is the force per unit length exerted on the cylinder in the direction of sound emission, (that of the vector $(\underset{\sim}{x} - \underset{\sim}{\eta})$). Since the cylinder is subjected to a lifting force, F, this component can be designated $F \sin \theta$, $\theta = 0$, implying the observer to be downstream of the cylinder. The velocity V at which the cylinder responds to the lifting force is related to F through the equation of motion:

$$F = \rho_s \frac{\pi D^2}{4} \frac{\partial V}{\partial t} \tag{5-18}$$

where ρ_s is the density of the solid material. The force available for acceleration of the cylinder is reduced from that active on a fixed cylinder, F_f, say, by the force required to overcome the inertia of the virtual mass:

$$F = F_f - \rho \frac{\pi D^2}{4} \frac{\partial V}{\partial t} \tag{5-19}$$

The integral

$$\int_{\Sigma} \frac{(x_n - \eta_n)}{r} \, u_n \, dS(\underset{\sim}{\eta})$$

has the value $(\pi/2)$ VD $\sin \theta$, a relation that allows us to rewrite Equation (5-17) in a form more appropriate to the relatively simple problem of Aeolian tone generation by flow over a small unsupported cylinder excited into transverse vibration:

$$\rho(\underset{\sim}{x}, t) = \frac{1}{4\pi a_0^3 r} \sin \theta \left[\frac{\partial F_f}{\partial t}\right] \left\{1 - \frac{2\rho_s}{(\rho + \rho_s)}\right\} \tag{5-20}$$

We are, of course, discussing a rather hypothetical situation in our treatment of a purely two-dimensional flow. Pressures are unlikely to be fully correlated over the entire length of a cylinder, a feature of considerable importance in the practical problem. This is particularly true in considering the main effect of vibration, for the influence of coherent vibration in synchronizing the pressure field is possibly the most important aspect affecting radiation strength in air, a point first noted by Phillips. But, within the framework of our model, it is of interest to note that the effect of vibration seems to be unimportant in air since the density ratio $\rho_s/(\rho + \rho_s)$ is, to all intents and purposes, unity.

The first effect is one of impeding the generation of sound, and, if the response were increased by allowing the density of the cylinder to approach that of the fluid, the dipole strength would vanish completely. Further reduction of the density would cause the dipole to become reversed in sense, limiting to precisely the opposite of the rigid cylinder case should the cylinder become weightless.

One other aspect of considerable importance in underwater applications concerns the case when the rigid surface is maintained in a preferred position, or path, by a force proportional to its displacement from that path. Then the possibility of a resonance is raised, and, if the frequency of vortex shedding coincides with that resonance, a major noise source can occur. Such resonances were discussed by Lamb in connection with the scattering problem, and are dealt with more specifically by Fitzpatrick and Strasberg[25] in their summary article on sources of underwater sound.

Before leaving the topic of small surfaces immersed in unsteady flow, it is worth touching briefly on the subject of sound generation by bubbles entrained in turbulent liquids. It is often the case that the fluid entrained in bubbles is considerably more compressible than is that of the surrounding liquid. Such compressibility emphasizes the simple source terms of Equation (5-8) which is likely to overwhelm the other multipole terms. The details of the surface motion must be very complex indeed, being controlled by both compressible and surface tension forces. The characteristic frequency of the motion is also complex, since there often exists the possibility of resonance which destroys the Strouhal number dependence normally operative in flow noise problems. These difficulties detract from the value of an estimate of radiation strength based on dimensional analysis of the type that led to Equation (5-15). Perhaps the most reliable deduction that one can make on the importance of bubbles is that they represent fundamentally more efficient sources of acoustic energy than those otherwise present in flow. A detailed analysis could be based on Equation (5-8), though such an analysis would be extremely complex, but the dynamics of the interface plays such a major role as to rule out significant deductions of a general nature.

5.4. Radiation by Flows Near "Large" Surfaces

We turn now to consider the sound radiated by turbulent boundary layer flow formed on a plane surface of sufficient size that edge effects are negligible. More strictly, we return to the topic, to fill in some of the details, for this is a situation to which we have already applied the basic theory. We deduced, as Powell had done beforehand, that a plane rigid surface merely reflects the sound generated by equivalent quadrupoles in the turbulence. Reflection is distorted by the action of viscous stresses, but, since they could only become significant at relatively low Reynolds numbers, of the order of the flow Mach number, we ignore them. We consider the plane surface to be a perfect reflector.

The relevant aerodynamic noise equation is then the leading term of Equation (5-11) where the sound is shown to be the sum of that generated by the real quadrupoles in the volume V above the boundary and that of the image quadrupoles in the volume V′ below the boundary.

$$\rho(\underset{\sim}{x},t) = \frac{1}{4\pi a_0^2}\frac{\partial^2}{\partial x_i \partial x_j}\int_{V+V'}\left[\frac{T_{ij}}{r}\right] dV \qquad (5\text{-}21)$$

A comparison of this equation with the inviscid form of Equation (5-8) applied directly to the rigid surface, shows a certain equivalence between volume and surface integrals,

$$\frac{1}{4\pi a_0^2}\frac{\partial^2}{\partial x_i \partial x_j}\int_{V'}\left[\frac{T_{ij}}{r}\right] dV = \frac{1}{4\pi a_0^2}\frac{\partial}{\partial x_n}\int_{\Sigma}\left[\frac{p}{r}\right] dS \qquad (5\text{-}22)$$

It is tempting to suppose a general equivalence between surface and volume integrals of this type, and to seek a procedure for computing the radiation field in terms of equivalent surface stresses. Such surface stresses are easier to handle than the fluctuating Reynolds stresses of the volume integral, both experimentally and analytically. But no such equivalence is possible in the boundary layer application, and one is forced to admit that the leading terms of a multipole expansion of the radiation field are volume quadrupoles that cannot be synthesized from a surface stress system. The surface pressure is the sum of contributions from both real and image quadrupoles, which radiate towards the surface in phase so that their contributions add. On the other hand, radiation from the source and image combination to positions outside the boundary layer is markedly different from that of the real source system in isolation. The reason for this is that those quadrupoles with one and only one axis in a direction normal to the flow are opposed by their images so that the combination forms an octupole of basically lower efficiency. All other quadrupoles are reinforced by their images, so that they appear to be doubled in strength. These arguments hold whenever the boundary layer thickness is smaller than the acoustic wavelengths of interest, a common condition to which we restrict our present arguments.

The turbulent boundary layer is a flow in which high mean velocity gradients occur. Lighthill[43] showed how such gradients amplify lateral quadrupoles which can often be treated as the dominant sources of sound in turbulent shear flows. Such quadrupoles have one, and only one, axis normal to the flow, so that it is precisely those quadrupoles that dominate a free shear flow that degenerate into octupoles of negligible strength in the boundary layer case. From a practical standpoint, a large, and possibly dominant, part of the surface pressure field results from interaction of the boundary layer turbulence with the mean shear flow. That pressure field would now appear to be precisely the contribution which is *not* related to the radiated noise problem, for it is the field which opposes directly the radiation from the mean shear amplified lateral quadrupoles of the turbulent flow. This conclusion is in agreement with the directionality and strength of the sound field radiated by the wall pressure field reported in Reference 21, but there the all-important details of the cancellation were not considered. The conclusion that one is led to from the foregoing discussion, is that the only markedly directional radiation of the boundary layer turbulence is destroyed by the presence of the boundary. The resulting radiation will consequently be of an omnidirectional quadrupole character, whose intensity will increase proportionally to $(\bar{\rho}^2/\rho_0)U^3M^5$, a result that follows from the dimensional arguments of our next chapter. This conclusion is modified drastically by rapid convection of the turbulent stream, especially at supersonic speeds. The reflection argument remains unchanged, so that the proper analysis need consider only the flow in association with its image, but the radiation becomes extremely directional as Mach waves emitted by supersonically convected eddies begin to dominate the radiation field.

We conclude our present discussion of the boundary layer noise problem by making brief reference to two important effects. The first is that, although the main surface dipoles were shown to vanish on a large surface, they need not vanish near the edges of that surface or from smaller surfaces in contact with turbulent flow. Dipoles exist in those regions and, unless the area of the plane surface exceeds that of the edge region by a factor of the order M^2, they will be the dominant source of sound. A precise definition of the edge region is not possible within the present state of the art but is the subject of current theoretical study. Those dipoles will have the same general features as those already discussed in the earlier sections of this chapter so they need not be considered further.

5.5 Some Remarks on Flexible Surfaces

Our final point concerns the possibility of surfaces responding to stresses induced by turbulent flow and thereby creating an additional source of sound. For surfaces of limited extent, wall motion becomes equivalent to a simple source system of high acoustic efficiency and can quickly become the most important feature of the practical boundary layer noise problem. This aspect is dealt with in Reference 21 where the computations of radiated noise is based on the hypothesis that all the power flowing into the vibration is radiated as sound. The problem then reduces to one of estimating surface response to boundary layer turbulence and does not come within the scope of our present discussion of the acoustic analogy applied to the boundary layer. Should the surface be plane, large and constructed of a homogeneous material, the acoustic analogy can be applied directly. This problem has been dealt with by one of the authors* and we shall only summarize the results here, since the analysis is again more concerned with structural vibration than with the fluid mechanics. It appears that the role of the surface is precisely what it would be when excited by a purely acoustic field. The infinitely rigid surface reflects, and the infinitely limp surface reflects in the opposite sense, with the images having the negative of their value at the rigid condition. Our conclusion regarding the strong lateral quadrupoles would then be modified considerably since they would be the ones enhanced by their images. Internal damping of the surface tends to be a dissipative action which distorts the image system, but under all these conditions, both the simple source and dipole contributions vanish on the large surface so that the radiation maintains its quadrupole character of low radiation efficiency.

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* Ffowcs Williams, J.E., *Radiation from a Boundary Layer Formed on a Flexible Surface.* (Unpublished).

CHAPTER 6

NOISE GENERATION AT HIGH SPEED

6.1 Introduction

The mechanism of noise generation by turbulent flow has, so far, been described in terms of a straightforward acoustic analogy. An equivalent source system produces in an ideal fluid at rest the same sound as is generated in the real fluid by the turbulent flow. Certain properties of the equivalent sources defy a detailed description. Refractive effects are prime examples for their influence is only felt on the source terms if the stress tensor is admitted to contain elements of the acoustic field. In principle, however, the analogy is complete and its application is hampered more by a lack of turbulence knowledge than it is by difficulties of this type.

By identifying the equivalent aerodynamic sources as quadrupole Lighthill[41] inferred many important features of the radiation from concepts of classical acoustics. The extreme acoustic inefficiency of turbulence became obvious, for a quadrupole is a combination of two dipoles in opposition. Radiation results from imperfect cancellation of these dipoles which are themselves of low efficiency, being comprised of two mutually opposing simple sources. That aerodynamic sound could be highly directional was also evident, as was the property that radiation strength would increase as a high power of frequency, for both these are fundamental features of acoustic quadrupoles. One other point was obvious too but was an aspect that had escaped the detailed attention of early workers in acoustics. An aerodynamic source might well be in motion relative to the surrounding fluid. Such relative motion is known to increase the frequency of the radiated wave above that of the source by the doppler factor. Since quadrupole efficiency increases rapidly with increasing frequency, source convection must augment the radiation of aerodynamic sound. But the analogy could go very little further on this point because classical studies on the effect of source motion were concerned only with frequency changes. The theory developed by Lighthill to describe convective effects is more analogous to the Lienard-Wiechert theory of electromagnetic radiation and it is with this aspect of the theory that we shall deal now. Lighthill initially developed the theory to illustrate the effects of source motion at subsonic speeds. More recently[24] this work has been extended to supersonic situations where turbulent eddies are shown to emit Mach waves, analogous to the shock waves radiated by thin aerofoils flying supersonically, or to the Cherenkov radiation from ulta-relativistic particles. These Mach waves were first studied specifically by Phillips[61], whose approach was quite different to Lighthill's which predicts Mach waves as a particular instance of quadrupole emission at high speed. Phillips, on the other hand, considered these Mach waves alone. We shall outline both approaches here, starting with Lighthill's at low speed, then at supersonic speed and finally describe some aspects of the development of Phillips's theory although we shall discuss only the early sections, since there is some question as to the validity of the final result.

6.2 Lighthill's Theory of Convected Quadrupoles

The theoretical development of the equations governing aerodynamic noise production at high speeds is simpler if we restrict our discussion to regions sufficiently far away from the turbulent flow that the far field equations apply. The relevant theory can then be based on the final equation of Chapter 4, Equation (4-15). That will be taken as the starting point of this section. The mean square density fluctuation at the observation point, $\underset{\sim}{x}$, per unit volume of turbulence at the origin of coordinates is

$$\frac{\partial}{\partial \underset{\sim}{\eta}} \overline{\rho'^2}(\underset{\sim}{x}) = \frac{1}{16\pi^2 a_0^8} \frac{x_i x_j x_k x_l}{r^6} \int_V \overline{\frac{\partial^2 T_{ij}(0, t - r/a_0)}{\partial t^2} \frac{\partial^2 T_{kl}(\eta', t - r'/a_0)}{\partial t^2}} \, d\eta' \qquad (6\text{-}1)$$

In writing the mean square density at $\underset{\sim}{x}$ as a function independent of time, we have assumed statistical stationarity in time. The stress tensor correlation function forming the integrand of this equation can be rewritten under such stationary conditions as

$$\frac{\partial^4}{\partial \tau^4} T_{ijkl}\left(\underset{\sim}{\eta} = 0, \ \underset{\sim}{\eta}', \tau = \frac{r - r'}{a_0}\right) = \overline{\frac{\partial^2 T_{ij}(0, t - r/a_0)}{\partial t^2} \frac{\partial^2 T_{kl}(\eta', t - r'/a_0)}{\partial t^2}} \qquad (6\text{-}2)$$

where

$$T_{ijkl}(\underset{\sim}{\eta}, \underset{\sim}{\eta}', \tau) = \overline{T_{ij}(\underset{\sim}{\eta}, t) \, T_{kl}(\underset{\sim}{\eta}', t + \tau)} \qquad (6\text{-}3)$$

We can simplify this further by assuming a stationarity in space so that the correlation function is dependent only on the separation $(\underset{\sim}{\eta}' - \underset{\sim}{\eta})$ which is equal to η' at $\eta = 0$. Then Equation (6-1) has the simpler form

$$\frac{\partial \overline{\rho'^2}(\underset{\sim}{x})}{\partial \underset{\sim}{\eta}} = \frac{1}{16\pi^2 a_0^8} \frac{x_i x_j x_k x_l}{r^6} \int_V \frac{\partial^4}{\partial \tau^4} T_{ijkl}\left(\underset{\sim}{\eta}', \tau = \frac{r - r'}{a_0}\right) d\underset{\sim}{\eta}' \qquad (6\text{-}4)$$

This equation can be written in a still more simple form if one denotes the particular element of the correlation tensor T_{ijkl} , where the suffixes are all equal and indicate the direction of emission, by T_r . Then x_i, x_j, x_k and x_l are all equal to r , the distance travelled by the sound, and Equation (6-4) becomes

$$\frac{\partial \overline{\rho'^2}(\underset{\sim}{x})}{\partial \underset{\sim}{\eta}} = \frac{1}{16\pi^2 a_0^8} \frac{1}{r^2} \int_V \frac{\partial^4}{\partial \tau^4} T_r\left(\underset{\sim}{\eta}', \frac{\underset{\sim}{\eta}' \cdot \underset{\sim}{x}}{a_0 r}\right) d\underset{\sim}{\eta}' \qquad (6\text{-}5)$$

where use has been made of the fact that $r - r'$ may be rewritten as $\underset{\sim}{\eta} \cdot \underset{\sim}{x}/r$ whenever the correlation lengths are much smaller than the distance separating the observation point from the turbulence, i.e., $r/\eta' \gg 1$.

Whenever the turbulence time scale is large compared to the time taken by a sound wave to traverse an eddy, or a correlated region, the value of the correlation function will remain insensitive to the variation of retarded time. To all intents and purposes the integral of Equation (6-5) could then be regarded as an instantaneous one and such a step is often taken in approximate calculations of aerodynamic noise. But if the source system is in motion, much of the temporal variation observed at a point is due to the passage of turbulence past that point. The time derivative is then made up in part of a velocity times a space derivative, which integrates directly if retarded time is neglected. An approximate calculation based on the magnitude of such time derivatives would consequently overestimate the radiation, for the space derivatives make no contribution to the integral. In practice, turbulent eddies are often convected a distance many times their scale so that the time derivative is dominated by a part which makes no contribution to the radiation field, a point that renders Equation (6-5), in its present form, of little value in assessing the radiated sound. Lighthill overcame this difficulty by carrying out an axis transformation which emphasized that small part of the flow capable of radiation. This transformation is as follows.

Let $\underset{\sim}{\zeta}$ be a coordinate system which is in motion with the average convection speed of the turbulence. The convection velocity is assumed to be a solenoidal field of magnitude $a_0\underset{\sim}{M}$. Then define a function, $S_r(\underset{\sim}{\zeta},\tau)$, to be the correlation of the stress tensor in the moving reference frame.

$$\underset{\sim}{\eta}' = \underset{\sim}{\zeta} + a_0\underset{\sim}{M}\tau = \underset{\sim}{\zeta} + \underset{\sim}{M}\frac{\underset{\sim}{\eta}'\cdot\underset{\sim}{x}}{r} \tag{6-6}$$

$$S_r(\underset{\sim}{\zeta},\tau) = T_r(\underset{\sim}{\eta}',\tau) \tag{6-7}$$

It is a straightforward exercise (Reference 24, p.480) to show that the Jacobian of the transformation is $\left|1 - \underset{\sim}{M}\cdot\underset{\sim}{x}/r\right|$ and that

$$\frac{\partial T_r(\underset{\sim}{\eta}',\tau)}{\partial\tau} = \frac{1}{\left\{1 - \frac{\underset{\sim}{M}\cdot\underset{\sim}{x}}{r}\right\}}\frac{\partial S_r}{\partial\tau}(\underset{\sim}{\zeta},\tau) - a_0 M_i\frac{\partial}{\partial\zeta_i}S_r(\underset{\sim}{\zeta},\tau) \tag{6-8}$$

$$= \frac{1}{\left\{1 - \frac{\underset{\sim}{M}\cdot\underset{\sim}{x}}{r}\right\}}\frac{\partial S_r}{\partial\tau}(\underset{\sim}{\zeta},\tau) - a_0\frac{\partial}{\partial\zeta_i}\left\{M_i S_r(\underset{\sim}{\zeta},\tau)\right\} \tag{6-9}$$

The last term in Equation (6-9), being a divergence of a vector field, integrates directly to zero; consequently, it may be discarded in computing the moving axis form of Equation (6-5). That equation can then be written down immediately as

$$\frac{\partial\overline{\rho'^2}(\underset{\sim}{x})}{\partial\underset{\sim}{\eta}} = \frac{1}{16\pi^2 a_0^8}\frac{r^3}{\left|r - \underset{\sim}{M}\cdot\underset{\sim}{x}\right|^5}\int_V\frac{\partial^4}{\partial\tau^4}S_r\left(\underset{\sim}{\zeta},\frac{\underset{\sim}{\eta}\cdot\underset{\sim}{x}}{a_0 r}\right)d\underset{\sim}{\zeta} \tag{6-10}$$

The retarded time variation, $\underset{\sim}{\eta}'\cdot\underset{\sim}{x}/r$, can be expressed in terms of the moving cooordinate $\underset{\sim}{\zeta}$ by simply forming the scalar product $\underset{\sim}{\eta}'\cdot\underset{\sim}{x}$ from Equation (6-6).

$$\frac{\underset{\sim}{\eta}'\cdot\underset{\sim}{x}}{a_0 r} = \frac{\underset{\sim}{\zeta}\cdot\underset{\sim}{x}}{a_0\{r - \underset{\sim}{M}\cdot\underset{\sim}{x}\}} = \tau \tag{6-11}$$

Equation (6-10) then assumes its more significant form from which the high Mach number extensions can be readily derived.

$$\frac{\partial\overline{\rho'^2}(\underset{\sim}{x})}{\partial\underset{\sim}{\eta}} = \frac{1}{16\pi^2 a_0^8}\frac{r^3}{|r - \underset{\sim}{M}\cdot\underset{\sim}{x}|^5}\int_V \frac{\partial^4}{\partial\tau^4} S_r\left(\underset{\sim}{\zeta}, \frac{\underset{\sim}{\zeta}\cdot\underset{\sim}{x}}{a_0\{r - \underset{\sim}{M}\cdot\underset{\sim}{x}\}}\right) d\underset{\sim}{\zeta} \tag{6-12}$$

$$= \frac{1}{16\pi^2 a_0^8 r^2}\frac{1}{|1 - M\cos\theta|^5}\int_V \frac{\partial^4}{\partial\tau^4} S_r\left(\underset{\sim}{\zeta}, \frac{\underset{\sim}{\zeta}\cdot\underset{\sim}{x}}{a_0 r(1 - M\cos\theta)}\right) d\underset{\sim}{\zeta} \tag{6-13}$$

Here, $r(1 - M\cos\theta)$ has been written for $r - \underset{\sim}{M}\cdot\underset{\sim}{x}$. The angle θ is measured in such a way that $\theta = 0$ when the observer lies downstream of the convective flow.

In basing an estimate of radiation intensity on Equation (6-13) it soon becomes apparent that there exist two characteristic regimes of radiation. The first of these is the one encountered in low speed flows where retarded time changes are small compared to the natural, moving axis time scale of the turbulence. If L_r is the correlation scale of the turbulence in the direction of sound emission, and τ^* is its moving axis time scale, then provided that

$$\frac{L_r}{a_0|1 - M\cos\theta|} \ll \tau^* \tag{6-14}$$

retarded time effects are negligible and Equation (6-13) can be effectively approximated by

$$\frac{\partial\overline{\rho'^2}(\underset{\sim}{x})}{\partial\underset{\sim}{\eta}} = \frac{1}{16\pi^2 a_0^8 r^2}\frac{1}{|1 - M\cos\theta|^5}\int_V \frac{\partial^4}{\partial\tau^4} S_r(\underset{\sim}{\zeta}, 0)\, d\underset{\sim}{\zeta} \tag{6-15}$$

Where this equation holds, the situation is one where convected quadrupoles emit waves in a relatively classical manner. Frequencies are shifted by the doppler factor $\{1 - M\cos\theta\}$ (this was the time scale expansion of Equation (6-9)) and subsonic convection is seen to augment the radiation efficiency; the factor, $|1 - M\cos\theta|^{-5}$, having an average value in excess of unity. This radiation regime is seen to apply at very high convection speeds since the inequality of (6-14) is satisfied there too. At supersonic convection speeds where $M\cos\theta \gg 1$, the observer hears the quadrupole sound in reverse time, since nearer parts of the quadrupole emit at an earlier time, quite contrary to the low speed situation.

6.3 Extension of Lighthill's Formulation

When the inequality (6-14) is not satisfied a different situation exists which can occur at all supersonic convection speeds whenever $(1 - M\cos\theta)$ approaches zero. The situation is most straightforward at that condition for the observer is able to hear each constituent element of the quadrupole separately. At low speeds an equivalent quadrupole is of low efficiency because the cancellation of its constituent elements is almost complete. At higher convection speeds the radiation time scale is increased and the elements can change more in the interval separating their emission times, so the cancellation becoming less complete results in an enhanced acoustic output. This is the Stokes effect that accounts for the $(1 - M\cos\theta)$ factors in Equation (6-15). But the increase in efficiency is strictly limited, for the quadrupole can at no time radiate more than its constituent simple sources. It could do this only when the cancellation was completely absent, a condition experienced whenever $M\cos\theta = 1$. Then the quadrupole is approaching the observer with precisely the speed of sound. The near elements of the quadrupole emit and continue to move with the sound wave they generate. The other quadrupole elements never overtake this wave and are therefore quite unable to make their presence felt and all mutual cancellation ceases. Since the simple source emission is far more efficient than that of a quadrupole, strong radiation must occur at the Mach angle, $\theta = \cos^{-1}M^{-1}$. The details of the radiation at this condition were first studied by Phillips, who showed how eddies convected supersonically gave rise to Mach waves. It was later[24] that Lighthill's equations were transformed into a system capable of dealing with these waves although, in retrospect, the transformation is extremely simple. It is most easily performed by recognizing that in Equation (6-13), the retarded time τ could well replace ζ_r as an independent variable, ζ_r being the component of $\underset{\sim}{\zeta}$ in the radiation direction. τ and ζ_r are related through Equation (6-11).

$$\tau = \frac{\zeta_r}{a_0(1 - M\cos\theta)} \tag{6-16}$$

$$\frac{\partial}{\partial\tau} = a_0(1 - M\cos\theta)\frac{\partial}{\partial\zeta_r} \tag{6-17}$$

The volume element $d\underset{\sim}{\zeta}$ can be re-written

$$d\underset{\sim}{\zeta} = a_0(1 - M\cos\theta)d\tau d\underset{\sim}{\zeta_s}$$

where $d\underset{\sim}{\zeta_s}$ is an element of area perpendicular to the radiation direction.

The radiation equation can then be re-expressed in a form devoid of possible singularities at the Mach wave condition:

$$\frac{\partial\overline{\rho'^2}(\underset{\sim}{x})}{\partial\underset{\sim}{\eta}} = \frac{1}{16\pi^2 a_0^3 r^2}\int_S\int_\tau \frac{\partial^4}{\partial\zeta_r^4} S_r\left(\zeta_r = a_0\tau(1 - M\cos\theta), \underset{\sim}{\zeta_s}, \tau\right) d\tau d\underset{\sim}{\zeta_s} \tag{6-18}$$

If we neglect all but the Mach waves, $(1 - M\cos\theta) = 0$ and $\underset{\sim}{\zeta_r} = 0$, so that the radiation is described by the equation

$$\frac{\partial\overline{\rho'^2}(\underset{\sim}{x})}{\partial\underset{\sim}{\eta}} = \frac{1}{16\pi^2 a_0^3 r^2}\int_S\int_\tau \frac{\partial^4}{\partial\zeta_r^4} S_r(\underset{\sim}{\zeta_s}, \tau)\, d\tau d\underset{\sim}{\zeta_s} \tag{6-19}$$

Application of these equations is strictly limited by our presently restricted knowledge of turbulent flows, particularly at high speed. However, a crude dimensional analysis readily yields the characteristic dependence of the radiation intensity on typical flow parameters. Such a technique is clearly useful where the flow fields remain geometrically similar. At low speeds, similarity is usually established in both jet and boundary layer flows but the situation is more complex at high speed. At low supersonic speeds the region of flow capable of generating Mach waves, i.e., that region moving supersonically relative to the observer at rest in the uniform flow enclosing the turbulence, is increasing in scale with increasing Mach number and the dimensional arguments based on similar systems cannot apply. More refined arguments are possible, but very few examples have so far been attempted. In jet or rocket exhausts, similarity of the Mach wave producing flow seems to be established at nozzle exit Mach numbers near three, for the important regions of intense turbulence near the nozzle evidently move supersonically above that speed and emit intense Mach waves. The supersonic boundary layer is more complicated, for the region of intense turbulence travelling supersonically relative to the free stream is continually changing, even up to Mach numbers near five and one must be cautious in applying the dimensional arguments based on geometrical similarity.

6.4 Dimensional Considerations

The dimensional arguments are straightforward. The correlation function S_r, being quadratically dependent on the turbulence stress tensor T_{ij}, will increase in level like $\bar{\rho}^2 U^4$ where $\bar{\rho}$ and U are respectively the typical density and velocity of the turbulent flow. Differentiation with respect to time is dimensionally equivalent to multiplication by a typical frequency which will have the characteristic Strouhal proportionality U/l, where l is a characteristic turbulence length scale. Differentiation with respect to the correlation variable, ζ_r, is carried out symbolically by simply dividing by l and integration is the converse of this operation. Applying these techniques to the quadrupole equation, (6-15), one predicts that the mean square radiation density will have the characteristic dependence.

$$\overline{\rho'^2} \sim \bar{\rho}^2 \frac{U^8}{a_0^8} \frac{l^2}{r^2} |1 - M\cos\theta|^{-5} \sim \bar{\rho}^2 M^8 \frac{l^2}{r^2} |1 - M\cos\theta|^{-5} \qquad (6\text{-}20)$$

provided, of course, the inequality of (6-14) is obeyed. Near the singularity of this equation Mach wave emission dominates the radiation field and its strength is derived by applying the same dimensional arguments to Equation (6-19).

$$\overline{\rho'^2}\ (M\cos\theta = 1) \sim \bar{\rho}^2 \frac{U^3}{a_0^3} \frac{l^2}{r^2} \sim \bar{\rho}^2 M^3 \frac{l^2}{r^2} \qquad (6\text{-}21)$$

Both these formulae can be combined in one more uniformly valid form as has been done in Reference 20. There it was suggested that ϵ, the ratio of the root mean square turbulence level to mean velocity, plays an important part in establishing the speed range over which Mach wave emission is dominant. The form given in that reference has the value of Equation (6-20) far away from the Mach wave condition and that of Equation (6-21) when $M\cos\theta$ is equal to unity.

$$\overline{\rho'^2} \sim \bar{\rho}^2 \frac{l^2}{r^2} M^3 \left\{\frac{\epsilon^2 M^2}{(1 - M\cos\theta)^2 + \epsilon^2 M^2}\right\}^{5/2} \tag{6-22}$$

The same dimensional arguments, when applied to Phillips's equations, do not reproduce this result; the difference is probably due to the singular nature of his final result, and that we shall now discuss in more detail.

6.5 Phillips's Theory of Mach Wave Generation

Phillips's theory[61] is based on the leading term of an asymptotic expansion solution to the equations describing a model shear flow. In presenting the theory Phillips described apparent shortcomings of the acoustic analogy, which was at that time regarded as essentially limited to flows of low Mach number. His novel approach to the problem overcame (at least in principle) some of those shortcomings of which the refraction problem was possibly uppermost in his mind. His solution promised an important advance, for not only did it display refractive effects brought about by changes in both mean flow velocity and temperature, but it was then the only solution available which gave any indication of the situation existing at Mach numbers higher than unity.

Phillips based his theory on the equations of continuity and momentum, the equations leading to Lighthill's acoustic analogy. However, Phillips required more specific details of the fluid state and these he obtained by confining his attention to a perfect gas. He arranged those equations in a form equivalent to Equation (3-10).

$$\frac{D^2}{Dt^2} \log\left(\frac{p}{p_0}\right) - \frac{\partial}{\partial x_i}\left\{a^2 \frac{\partial}{\partial x_i} \log\left(\frac{p}{p_0}\right)\right\} = \gamma \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + \gamma \frac{D}{Dt}\left(\frac{1}{C_p}\frac{DS}{Dt}\right) - \gamma \frac{\partial}{\partial x_i}\left(\frac{1}{\rho}\frac{\partial}{\partial x_j}\tau_{ij}\right) \tag{6-23}$$

The right-hand side of this expression may be regarded as the source terms of the modified wave equation which constitutes the left-hand side. The first term represents the sources arising from turbulent velocity fluctuations, while the remaining two deal with the effects of entropy and viscosity which are both assumed negligible. Were a general solution to this equation available, it would offer two significant advantages over Lighthill's theory. The first is that major convective and refraction effects have been incorporated into the left-hand side of the equation and need not be sought in a modified source term as in the acoustic analogy. On the other hand it must not be thought that the problem would have been avoided completely. The velocity term on the right-hand side is also subject to acoustic influence, the neglect of which induces an error of the order u'/u, where u' is the acoustic velocity fluctuation and u the typical turbulence level. The error of ignoring the acoustic term in Lighthill's stress tensor is of the order ρ'/ρ, which exceeds the error in Phillips's term by a factor proportional to the mean flow Mach number, a significant fraction at the high supersonic speeds studied by Phillips. The second advantage is that changes in the flow temperature responsible for refraction of sound would play a natural part in the theory, again an aspect important in high Mach number flows.

It is not surprising that the known solutions to such a general equation are few and far between, a point that led Phillips to seek an approximate solution at very high Mach numbers, a regime quite beyond the scope of other currently available theories. In seeking this solution, Phillips confined his attention to those refractive effects associated with changes in the mean flow properties. He selected a particular class of mean flows, that of two-dimensional free shear layers. To simplify the analysis he also neglected the second derivative of acoustic velocity so that the temperature across the layer was, in effect, assumed to vary linearly with cross-stream position. This step, coupled with a slight variable modification, allowed him to write the relevant form of Equation (6-23) as a modified wave equation. That modified equation can be written out explicitly, although this was not done by Phillips.

$$\left[\left\{\frac{\partial}{\partial t} + \bar{u}_1(x_3)\frac{\partial}{\partial x_1}\right\}^2 - \bar{a}^2(x_3)\nabla^2\right]\bar{a}(x_3)\log\left(\frac{p}{p_0}\right) = \gamma\bar{a}(x_3)\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} \qquad (6\text{-}24)$$

x_3 is the coordinate normal to the shear layer and $\bar{u}_1(x_3)$ is the mean velocity. The other variables have their obvious meaning, already defined in Chapter 2. The equation treated by Phillips was this one, but expressed in terms of generalized Fourier transforms of the normalized parameters. The normalization was based on expressing the time scale t as $\tau L/U$, the space coordinates, x_i as $y_i L$, the velocities u_i as $v_i U$ and the local speed of sound a as $A(y_3)a_0$. τ, y_i, v_i and $A(y_3)$ are then non-dimensional variables based on the mean velocity differential, $2U$, across a shear layer of width $2L$. a_0 is the speed of sound in the uniform flow on the positive y_3 side of the layer and the characteristic Mach number of the flow is U/a_0. The generalized Fourier transforms of the normalized right-hand side of Equation (6-24) is defined as $\Gamma(y_3,\underset{\sim}{k},n)$ and that of the left-hand side variable, $A(y_3)\log(p/p_0)$, as $\bar{\omega}(y_3,\underset{\sim}{k},n)$. $\underset{\sim}{k}$ is a non-dimensional wave number vector in the plane of the shear layer, $\underset{\sim}{k}(k_1,k_2)$, and n a non-dimensional frequency. The suffix on the y_3 is dropped and k is written for $|\underset{\sim}{k}|$ in Phillips's restatement of Equation (6-24):

$$\frac{d^2\bar{\omega}}{dy^2}(y,\underset{\sim}{k},n) + \left\{\frac{M^2}{A^2}(n + Vk_1)^2 - k^2\right\}\bar{\omega}(y,\underset{\sim}{k},n) = -\frac{M^2}{A^2}\Gamma(y,\underset{\sim}{k},n) \qquad (6\text{-}25)$$

This equation can then be rearranged in a form more suitable to a solution by an asymptotic technique. We follow Phillips in this rearrangement but express the technique in rather a different way. We do this to facilitate comparison with other high speed theories later on.

$$q(y,k_1,n) \quad \text{is written for} \quad \frac{[n + k_1V(y)]}{A(y)} \qquad (6\text{-}26)$$

k_1 is assumed to be greater than n so that q has a single zero at $y = Y$, say, and is positive for all values of y in excess of that value.

$$q(Y,k_1,n) = 0 \qquad (6\text{-}27)$$

Now we define a positive parameter ξ^2 as

$$\xi^2(y,k_1,n) = 2\int_Y^y q(y,k_1,n)\,dy \tag{6-28}$$

and we rearrange Equation (6-25) in the form

$$\frac{\xi^{3/2}}{q^{3/2}}\left\{\frac{d^2\bar{\omega}}{dy^2} + q^2\left\{M^2 - \frac{k^2}{Q^2}\right\}\bar{\omega}\right\} = \frac{\xi^{3/2}}{q^{3/2}}k^2\left(1 - \frac{q^2}{Q^2}\right)\bar{\omega} - \frac{\xi^{3/2}}{q^{3/2}}\frac{M^2}{A^2}\Gamma(y,\underset{\sim}{k},n) \tag{6-29}$$

where Q is the limiting value of q as y approaches infinity. q effectively assumes the value Q immediately outside the shear layer. This can be rewritten right away as

$$\frac{d^2\left\{\frac{q^{1/2}}{\xi^{1/2}}\bar{\omega}\right\}}{d\xi^2} + \left\{M^2 - \frac{k^2}{Q^2}\right\}\xi^2\left\{\frac{q^{1/2}}{\xi^{1/2}}\bar{\omega}\right\} = \frac{\xi^{3/2}}{q^{3/2}}k^2\left(1 - \frac{q^2}{Q^2}\right)\bar{\omega} - \frac{\xi^{3/2}}{q^{3/2}}\frac{M^2\Gamma}{A^2} \tag{6-30}$$

Phillips employed an approximate analysis, the exact nature of which we shall discuss later, in order to solve (6-30). The first term on the right-hand side was shown to make no contribution to the limiting solution, which was dependent only on the value the second term assumed at the critical layer, $y = Y$. At the critical layer, $q = 0$, so that the frequency n is related to the wave number k_1 by the convection velocity V. Since the terms on the right-hand side may again be regarded as source terms, it becomes evident that at very high Mach number the development of the turbulent structure during its travel downstream is of no consequence in the radiation problem. Such development is essentially tied up with frequencies and wave numbers not related by the convection velocity, for they characterize the convection of a rigidly frozen pattern. This feature of the Mach wave radiation was later to be re-emphasized by application of Lighthill's equation to high speed flow and is quite contrary to the situation existing at lower convection speeds. This point is readily appreciated when one considers that the radiation integral of Equation (6-15) would assume the value zero if the turbulence did not develop during its convective motion downstream, for the time derivative operative there is that in a reference frame moving with the turbulence.

The solution with the above properties is obtained by Phillips as an asymptotic approximation for large values of the parameter $\{M^2 - k^2/Q^2\}$, a parameter which in the later developments of his theory is replaced by simply M^2. The approximate spectrum function, incorporating the last simplification, is then valid pointwise in wave number space, as $M \to \infty$. However, the intensity of the radiated noise involves an integral over all wave numbers contributing to the Mach wave radiation, a range which includes wave numbers for which the expansion parameter is in fact arbitrarily small. Therefore, the validity of Phillips's final results necessitates also that the solution be *uniformly valid* in wave number space, to a degree which allows an expansion of the intensity in terms of the pointwise approximation alone. Moreover, the pointwise approximation in the region outside the shear zone varies as $\{M^2 - k^2/Q^2\}^{-5/4}$ near the critical wave numbers (cf. Phillips's Equation 5-6), so that the intensity, without further strong and unnatural restrictions on the source spectrum Γ, is not bounded. Physically, the difficulty is not easily explained since the wave numbers which contribute to the divergent part lie in a range which decreases with increasing Mach number. On the other hand, a comparison of his analysis with related asymptotic expansions suggests that the questions we raise are standard ones in the mathematical theory, and lead to the hope that the source of the difficulty, and an improved approximate solution, will emerge from a refinement of the same basic approach.*

With this strong reservation on the validity on the final result, we abandon our discussion of Phillips's theory, even though some of the physical ideas are complementary to those described and the work is of considerable intrinsic interest.

CHAPTER 7

ACOUSTICAL MODELS

7.1 Preliminary Remarks

In the preceding chapters the term *acoustic analogy* has been used consistently to describe the reduction of our problem to essentially the solution of the classical wave equation with a random forcing term. This analogy has been found in many instances to reflect the essential fact that the sound produced by an aerodynamic flow may be computed from an independent knowledge of an incompressible motion. On the basis of its derivation, therefore, the acoustic analogy is likely to provide the most satisfactory and complete theory when the right-hand side of the fundamental equation

$$\frac{\partial^2 \rho}{\partial t^2} - a_0^2 \nabla^2 \rho = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \tag{7-1}$$

may be correctly described as a forcing term which is essentially independent of the sound field. In practice, of course, we encounter physical problems, such as the generation of noise in the turbulent boundary layers at high speeds, where the division of the solution implied by (7-1) is probably not the most natural, and where computation of the T_{ij}'s from experimental data or otherwise is exceedingly difficult. In these complex situations involving high speeds, large gradients in mean speed and density, etc., the usefulness of (7-1) rests largely on the possibility

* In this connection we mention two recent additions to the literature, which deal with the noise generated in a two-dimensional supersonic shear flow. In a paper presented at the Sixth Symposium on Advanced Problems in Fluid Mechanics held at Zakopane, Poland in 1963, Lilley has described a modification of Phillips's procedure which leads to an improved estimate of the intensity of the radiated noise. Also, an extension of Lighthill's theory to this problem has been discussed in a recent paper by Ffowcs Williams (to appear in the Journal of Fluid Mechanics).

of deriving partial information, e.g. of a dimensional nature, which is virtually independent of the statistical structure and level of the turbulent fluctuations in the quadrupoles. Therefore, in a sense the large amount of information contained in the exact forcing term is not used and any model equation of the form

$$\frac{\partial^2 \rho}{\partial t^2} - a_0^2 \nabla^2 \rho = f \qquad (7\text{-}2)$$

where f is a random function having certain broad constraints, will yield substantially the same answers. In view of these facts there emerge two obvious questions: First, for a given problem to find for the equivalent forcing function in the model equation a representation which is, in some sense, appropriate to the available "input" as derived from measurements on the turbulence; second, for a given representation of the input, to relate it in a precise way to the "output", i.e., the noise in the far field. In the present chapter, under the general heading of "acoustical models", we propose to examine several simple examples where the first of these equations is of definite interest, where the second can be answered in detail.

The proposal that an acoustical model of the generation process might be useful in some problems was originally put forward in an unpublished paper by Liepmann*. Liepmann proposed to divide up the calculations of boundary layer noise into two parts. In the first part, the effect of the boundary layer on the acoustic medium is calculated as if the boundary layer were absent, its effect being the same as if the wall were flexible, vibrating in a random way. Therefore, this part of the problem explores the "piston" action of the boundary layer on the outer flow, and the forcing term appears in the boundary condition at the wall on the normal component of the acoustic velocity. In the second part, the boundary condition is related to the "displacement effect" of the boundary layer, in a way which is well known in classical stationary viscous flows and also useful in the present study. In this division of the problem, the principal unknown becomes the fluctuation in an equivalent displacement thickness of the boundary layer, which then takes the place of the quadrupole density in (7-1). The possible advantage gained here is that in certain instances this displacement thickness might be simpler to measure than the quadrupole density, and is in fact known once the instantaneous integrals of density and tangential velocity fluctuations, as well as their product, over the boundary layer are known.

The examples discussed by Liepmann suggest that there are likely to be other cases where his general viewpoint may be useful, and it is in this spirit that the present chapter was written. We stress at the outset that, so long as the wave equation (7-2) forms the basis for the model, the way in which the forcing term appears is immaterial and the mathematical equivalence of the results with those derived from a strict application of (7-1) is complete (cf. Section 7.5). However, the fact that we now place the unknown in the boundary condition leads to estimates on the noise through a different sequence of steps, and it is hoped that this alternative analysis will aid in the understanding of the essential ideas, even in those cases where it offers no intrinsic advantage over the quadrupole theory.

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* Liepmann, H.W., *On the Acoustic Radiation from Boundary Layers and Jets*, 1954 (Unpublished).

7.2 The Displacement Effect

The concept of displacement thickness is a useful concept in viscous flow theory, not only because, as a parameter which can be measured directly, it is useful to the experimentalist, but also because of its appearance in the fundamental theory of laminar boundary layers. For the purposes of the present chapter, only the following well known result from the steady two-dimensional theory need be mentioned. Consider a boundary layer on a semi-infinite plane wall. In the region outside a boundary layer, that is, in the potential flow, the effect of the boundary layer on the uniform stream is the same as if the boundary layer were removed and the exterior flow determined by the condition

$$v_w = U_\infty \frac{d\delta^*}{dx} \tag{7-3}$$

at the wall. In Equation (7-3), U_∞ and v_w are, respectively, the free-stream speed and the normal component of the potential flow due to the displacement effect, the latter evaluated at the wall; δ^* is the boundary layer displacement thickness, and x is the distance along the wall. It is assumed in (7-3) that separation of the boundary layer does not occur. In physical terms it may be said that the curve defined by $y - \delta^*(x) = 0$, y being the distance normal to the wall, defines the position as viewed from the external flow of an effective wall which accounts for the existence of a boundary layer of varying thickness.

We may extend this concept to turbulent, compressible flow by defining δ^* in a similar way, now with $\partial\delta^*/\partial t$ added to the right-hand side of (7-3). If the uniform stream is removed by Galilean transformation in x, t, then the linearized boundary condition on the acoustic field follows from Equation (2-11) and is

$$v_w = \left(\frac{\partial\phi}{\partial y}\right)_w = \frac{\partial\delta^*}{\partial t} \tag{7-4}$$

where ϕ is the velocity potential of the sound field. The right-hand side of (7-4) is now to be interpreted as the equivalent "piston velocity" at the wall.

To find δ^* from the boundary layer equations is the second part of the problem, and we shall limit our remarks here to the application of integral methods to the continuity equation, i.e., the integration of

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho u}{\partial x} + \frac{\partial\rho v}{\partial y} + \frac{\partial\rho w}{\partial z} = 0$$

with respect to y across the boundary layer. If this step is carried out, and the result separated into mean (barred) and fluctuating (primed) parts, and if turbulent fluctuations are of order ϵ, then

$$(\bar{\rho}v')_{y=\delta} = -\frac{\partial}{\partial t}\int_0^\delta \rho' \, dy - \frac{\partial}{\partial x}\int_0^\delta (\bar{\rho}u' + \bar{u}\rho' + \rho'u' - \overline{\rho'u'}) \, dy +$$

$$+ \frac{\partial}{\partial \tau}\int_0^\delta (\bar{\rho}w' + \rho'w' - \overline{\rho'w'}) \, dy + 0\left(\epsilon \frac{d\delta}{dx}\right) = \rho_\infty\left(\frac{\partial \delta^{*\prime}}{\partial t} + U_\infty \frac{\partial \delta^{*\prime}}{dx}\right) \quad (7\text{-}5)$$

In the last equation δ denotes the boundary layer thickness and is taken to be independent of t. It is seen from (7-5), viewed as a first-order partial differential equation for $\delta^{*\prime}$, that displacement thickness fluctuations relative to the laboratory frame can be found, in principle, from measurements of the fluctuating velocity and pressure, and their correlation through the boundary layer. One can, of course, seek to further simplify (7-5) on the basis of the numerical magnitude of the various terms. However, we shall not pursue this any further, and point out only that the measurement of the various terms on the right of (7-5) would provide the input for the "displacement-thickness" acoustical model proposed by Liepmann, that is, the right-hand side of (7-4)*.

In addition to the use of an equivalent displacement thickness, there are of course other possibilities. An attractive one in the case of boundary layer noise can be called the "wall-pressure" model. Here we seek to relate the boundary values of the sound pressure to the measured wall pressure. The chief difficulty with the wall-pressure model is that a rational theory that would give such a relationship seems to be lacking. However, a simple example, based upon an assumption of a linear relation, is discussed in Section 7.4.

7.3 Radiation from a Line

Simple examples which may be used to illustrate in more detail the preceding ideas are not difficult to find. We have (in Section 2.5) studied the noise produced by a sphere which expands and contracts radially, as a way of representing an isolated source. When one allows convection, however, the simplest element is not a source but rather a line distribution of sources, and we shall begin by first considering the noise produced by an infinite circular cylinder having random corrugations (Fig.3).

We might conceive of such a representation for the fluctuations induced by a fully developed turbulent jet, and the solution has been discussed from this point of view, in the case of low Mach number convection by Liepmann. We seek to solve

$$\phi_{tt} - \nabla^2\phi = 0, \qquad p = -\frac{\partial \phi}{\partial t} \quad (7\text{-}6)$$

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* Although to the knowledge of the authors the necessary experiments have not as yet been performed, it should be noted that only integrals through the boundary layer are involved, and this might allow a direct measurement. For a theoretical example illustrating the generation of a pressure wave by changes in shear at the boundary, the reader is referred to van Dyke, M., *Impulsive Motion of an Infinite Plate in a Viscous Compressible Fluid*, Zeitschrift für angewandte Mathematik und Physik, Vol.3, p.343, 1952.

with the boundary condition analogous to (7-4). Using y to denote the cylindrical radius, this condition is

$$2\pi\delta^* \left(\frac{\partial\phi}{\partial y}\right)_{y=\delta^*} = -2\pi\delta^* \frac{\partial\delta^*}{\partial t} = -\frac{\partial A}{\partial t} = -\dot{A} \qquad (7\text{-}7)$$

where A is the cross-sectional area of the jet. For convenience we have chosen units so that the wave speed and density are unity, and for a reference length we take a characteristic wavelength of the fluctuations $\dot{A}$. We will in fact take the mean value of δ^* to be small in these units, but still large compared to the fluctuations themselves*. Then (7-7) becomes, approximately

$$\lim_{y_0 \to 0} 2\pi y_0 \left(\frac{\partial\phi}{\partial y}\right)_{y=y_0} = -\dot{A}(x,t)$$

where y_0 is the mean value of δ^* .

If the function A defined in (7-8) is a random function of x and t , we may interpret the solution as noise radiated from a column of turbulence, the characteristic "eddy size" in the direction of the column (in accordance with the restriction on wave numbers) being large compared to the diameter of the column. It will be assumed that the statistical properties of $\dot{A}$ (and therefore of the radiated noise) are independent of the values assigned to x,t , that is, are stationary in both arguments. With this assumption, we may define the normalized correlation functions, corresponding to $\dot{A}$ and ϕ , by

$$\overline{\dot{A}^2}\, R_{\dot{A}}(\xi,\tau) = \overline{\dot{A}(x,t)\, \dot{A}(x+\xi, t+\tau)}\ ; \qquad R_{\dot{A}}(0,0) = 1 \qquad (7\text{-}8a)$$

$$\overline{\phi^2}(y)\, R_{\phi}(\xi,\tau;y) = \overline{\phi(x,t;y)\, \phi(x+\xi, t+\tau;y)}\ , \qquad R_{\phi}(0,0;y) = 1 \qquad (7\text{-}8b)$$

Note that y occurs in (7-8b) as a parameter. The energy spectrum functions are next defined by

$$R_{\dot{A}}(\xi,\tau) = \iint e^{i(\xi k+\tau w)}\, \Psi_{\dot{A}}(k,\omega)\, dk d\omega \qquad (7\text{-}9a)$$

$$R_{\phi}(\xi,\tau;y) = \iint e^{i(\xi k+\tau\omega)}\, \Psi_{\phi}(k,\omega;y)\, dk d\omega \qquad (7\text{-}9b)$$

where, unless the contrary is stated, integration is always over the entire two-dimensional space.

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* We might state this differently by saying that the cylinder is acoustically thin.

By standard procedures in the spectral analysis of solutions of linear equations with constant coefficients it may be shown that Ψ_ϕ and $\Psi_A^{\cdot}$ are related by*

$$\overline{\phi^2}(y)\,\Psi_\phi(k,\omega;y) \;=\; \overline{A^2}\,\left|s(k,\omega;y)\right|^2\,\Psi_A^{\cdot}(k,\omega) \qquad (7\text{-}10)$$

where s satisfies

$$s_{tt} - \nabla^2 s \;=\; 0 \qquad (7\text{-}11a)$$

$$\lim_{y\to 0}\, y_0\left(\frac{\partial s}{\partial y}\right)_{y=y_0} \;=\; -\frac{1}{2\pi}\,e^{i(kx+\omega t)} \qquad (7\text{-}11b)$$

The solution of (7-11) involves also a radiation condition of a familiar kind. As is evident from (7-11b), the solution for each (ω, k) will be a Bessel function of order zero, so that without going further into the analysis, it can be seen that

$$\left|s(k,\omega;y)\right|^2 \;=\; \frac{1}{16}\left[J_0^2(y\sqrt{\omega^2-k^2}) + Y_0^2\,(y\sqrt{\omega^2-k^2})\right], \qquad |\omega| > |k| \qquad (7\text{-}12a)$$

$$=\; \frac{1}{4\pi^2}\,K_0^2(y\sqrt{k^2-\omega^2}), \qquad |\omega| < |k| \qquad (7\text{-}12b)$$

The *phase speed*, with reference to the speed of sound, of the Fourier component s is defined by

$$v_p \;=\; -\,\omega/k \qquad (7\text{-}13)$$

According to (7-12), the contribution to the mean square fluctuation in ϕ, coming from those components of $\Psi_A^{\cdot}$ whose phase velocity is equal to v_p, is augmented by the factor $|s(k,-v_pk;y)|^2$. This factor is a function of $ky\sqrt{1-v_p^2}$ alone if $v_p < 1$ ("subsonic" phase speed) and of $ky\sqrt{v_p^2-1}$ if $v_p > 1$ ("supersonic" phase speed). Moreover, it follows from the behavior of K_0 and J_0 for large values of the argument (exponential decay for the former and slowly decaying oscillations for the latter) that the attentuation of the radiation is far greater where $v_p < 1$. The relation connecting the two energy spectra depends crucially upon this variation with phase speed, particularly at the higher wave numbers.

The relative weighting of subsonic and supersonic components depends also on the distance from the line, as is to be expected from the conventional division of the noise into its near and far fields. We can define the *far field* in the present problem by the condition

$$\left|v_p^2 - 1\right|^{1/2} y \gg 1 \qquad (7\text{-}14)$$

where we form v_p as given by (7-13) from a wavelength and frequency which characterize $\Psi_A^{\cdot}$. We shall at this point make the assumption (7-14) and pass to the mathematically

* See, e.g., Batchelor, G.K., *Homogeneous Turbulence*, Cambridge, 1959, Chapter IV.

much simpler representation of Ψ_ϕ in the far field. There is no real loss of generality in doing so, since it is in the far field that the model is sufficiently independent of the choice of $\Psi_A^{\cdot}$ to be useful anyway. In the following v_p need not (and will not) enter as a parameter, the far-field representation being obtained by expansion with respect to y. We may expect, in view of the asymptotic behavior of the Bessel functions noted earlier, that the principle contribution to the Fourier integral for $R_\phi(\xi,\tau;y)$ are likely to come only from the region $|\omega| > |k|$ of the phase space (that is, the (ω, k) plane). There is, however, a certain freedom in how we may pick the decay of $\psi(\omega, k)$ for large ω, k, which conceivably can give cases of relatively large contribution near $|\omega| = |k|$ (where $K_0(y\sqrt{k^2 - \omega^2})$ behaves like $\log y\sqrt{k^2 - \omega^2}$ approximately).

It turns out that if the decay of $\Psi_A^{\cdot}(k, -v_p k)$ with increasing k is fast enough, the asymptotic behavior of R_ϕ is what would be expected purely from the decay of the Bessel functions (i.e., from a discrete spectrum), and the error can be estimated to be close to the order of the neglected terms in the expansions of these functions. Our main result may then be expressed as follows: Suppose that $k^c\,\Psi_A^{\cdot}(k,\omega)$ is bounded uniformly over the phase plane, where c may be any number satisfying $0 \leqslant c < 1$, and suppose further that the function

$$R_\phi'(\xi,\tau) = \iint_{|\omega|>|k|} e^{i(\xi k+\tau\omega)} \frac{\Psi_A^{\cdot}(k,\omega)}{\sqrt{\omega^2 - k^2}}\, dk d\omega \tag{7-15}$$

exists for all ξ, τ. Then, as $y \to \infty$,

$$R_\phi(\xi,\tau;y) = \frac{R_\phi'(\xi,\tau)}{R_\phi'(0,0)} + \epsilon(y) \tag{7-16a}$$

$$\overline{\phi^2} = \frac{\overline{\dot{A}^2}}{8\pi y} R_\phi'(0,0) + \frac{\epsilon(y)}{y} \tag{7-16b}$$

where $y^\gamma \epsilon(y) \to 0$ as $y \to \infty$ for any $\gamma < 1$. It does not seem necessary to derive the estimates here, since (7-16) essentially bears out the observations made above. The main point is that contributions from the "transonic" phase speed regions near $|\omega| = |k|$ do not contribute a significant fraction of sound, provided that the decay of $\Psi_A^{\cdot}(k, -v_p k)$ as $k \to \infty$ is faster than k^{-1}.

Since we can now exclude the subsonic phase velocities, it is convenient, for the purpose of discussing the integral (7-15), to introduce a *phase angle* θ defined by

$$v_p = \frac{1}{\cos\theta}$$

This definition is illustrated in Figure 4, where it is seen that the discrete sources having a particular phase velocity are identified physically with sources moving with speed v_p relative to the medium. With the substitution (7-16), Equation (7-15) becomes

$$R'_{\phi}(\xi,\tau) = 2\int_0^{\infty}\int_0^{\pi} \Psi_{\dot{A}}(-\omega\cos\theta,\omega)\cos\omega(\tau-\xi\cos\theta)\,d\theta d\omega \qquad (7\text{-}17)$$

Let us return now to the idea that the noise of a jet can be represented by a line distribution of sources. For the intensity of the noise there is obtained

$$\overline{\phi_t^2} = \overline{p^2} = \frac{\overline{\dot{A}^2}}{4\pi y}\int_0^{\infty}\int_0^{\pi} \omega^2\Psi_{\dot{A}}(-\omega\cos\theta,\omega)\,d\theta d\omega \qquad (7\text{-}18)$$

$$= \frac{\overline{\dot{A}^2}}{4\pi y}\int_0^{\pi} D(\theta)\,d\theta$$

Since

$$\frac{d\theta}{y} = \frac{dx}{x^2+y^2}$$

the distribution function

$$D(\theta) = \int_0^{\infty} \omega^2\Psi_{\dot{A}}(-\omega\cos\theta,\omega)\,d\omega \qquad (7\text{-}19)$$

is equal to the intensity of the noise originating in a segment dx at the origin, as measured by an observer positioned on the line $\theta = \tan^{-1}(-y/x)$.

To take a specific example, let

$$\Psi_{\dot{A}}(k,\omega) = \frac{\tau l}{M_r} F\left[l^2k^2 + \frac{\tau^2}{M_r^2}(\omega + M_r k)^2\right]$$

where F is an arbitrary function. Then, if

$$C = \tau l \int_0^{\infty} \omega^2 F(\omega^2)\,d\omega$$

is finite,

$$D(\theta) = \frac{C}{M_r}\left[l^2\cos^2\theta + \frac{\tau^2}{M_r^2}(1 - M_r\cos\theta)^2\right]^{-3/2} \qquad (7\text{-}20)$$

Drawing upon the terminology of turbulence theory, the constants l and τ can be interpreted as the "eddy length" and "eddy lifetime", respectively. The constant M_r is seen to represent the Mach number of convection of the eddy pattern, relative to the free stream.

The two limiting forms of (7-20) which are of interest are (i) the case of low convection Mach number, $M_r \ll 1$, and (ii) the case of sharply directional (fully eddy Mach wave) radiation. The condition that the radiation be sharply directional is $\tau(M_r^2 - 1)^{1/2} \gg l$. This will be the case at high Mach number, in which case the sharp directionality occurs by virtue of the shallow angle of nearly all the eddy Mach waves, or at any $M_r > 1$ provided nearly all the eddies move with the same convective speed. For the example (7-19) the two limiting forms of the distribution of intensity with phase angle are

$$D(\theta) \sim \frac{CM_r^2}{\tau^3} \frac{1}{(1 - M_r \cos\theta)^3}$$

$$D(\theta) \sim \frac{CM_r^2}{l^2 + \tau^2(M_r^2 - 1)^{1/2}(\theta - \theta_r)^2} \qquad \text{(sharply directional about } \theta_r = \cos^{-1} 1/M_r)$$

It is further seen that $\overline{p^2}$ is proportional to $\overline{\dot{A}^2}\, M_r^2$ in the first case, and goes like $\overline{\dot{A}^2}\, M_r^2/\sqrt{M_r^2 - 1}$ in the second.

We may define an apparent convection speed for the noise in the far field by

$$M_r^f = \frac{1}{\cos\theta_r^f}, \qquad \text{where } [D'(\theta)]_{\theta=\theta_r^f} = 0$$

This definition of M_r^f is motivated by the fact that the maximum at θ_r^f should be observable in the laboratory, at least for the case of sharply directional noise. For our example, it is seen that

$$M_r^f = \frac{l^2 + \tau^2}{\tau^2} M_r > M_r \qquad (7\text{-}21)$$

According to (7-21) then, the sources will appear to be convected more rapidly in the far field. Relative to a frame moving to the right with speed $M_\infty > M_r$, therefore, the opposite is true and the eddies appear to move at a slower speed. The reason for this lies in the nature of eddy wave emission, which in the far field results in a shift of M_r in the direction of increased emission, that is, toward more rapid convection relative to the free stream*.

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* This result would seem to explain the values of convection speed measured by Laufer (see Chapter 8) above a turbulent boundary layer. However, in that experiment there is no question that, at least for $M_\infty > 3$, the noise was sharply directional, in which case the difficulty with (7-21) is that it will not account under these conditions for the substantial measured difference between M_r^f and M_r (taking here for the latter the value measured at the the wall), and certainly not for the observed variation of M_r^f/M_r with M_∞. It has been suggested by Laufer that in order to fully account for these results (as well as any equally precise statistical property of the noise) it is necessary to allow for the variation of M_r with k (or ω), i.e., to allow for a dispersion of phase velocity among the eddies of different lengths. We shall return to this point in Section 7.6.

7.4 Radiation from a Plane

If one attempts to represent a boundary layer over an *infinite* plane in terms of its displacement thickness and proceeds as before, there unfortunately occurs a divergent integral when integration is performed in the phase plane. The reason for this infinite result may be seen quite simply from the fact that the amplitude of the sound decreases as the inverse of the distance from the source and, when the sources are distributed over a large circle of radius L, the total intensity goes like $\log L$ as $L \to \infty$. However, we can deal with a *large* plate, and allow for convection in one direction (the direction of the x-axis, say) simply by integrating the line distribution of Section 7.2 along the plate; that is, by using the line sources as the fundamental solution in the construction of more complicated fields.

We seek to solve (7-6) with the condition

$$\left(\frac{\partial\phi}{\partial t}\right)_{t=0+} = -\frac{\partial\delta^*}{\partial t} = -\dot{\delta}^* \qquad (7\text{-}22)$$

on the strip $-\infty < x < +\infty$, $-L \leqslant y \leqslant +L$. The correlation functions are defined as before, but with the addition of a separation η in the z-coordinates and with $2\dot{\delta}^*$ replacing $\dot{A}$. The Fourier transforms are then defined by

$$R_{\dot{\delta}^*}(\xi,\eta,\tau) = \iiint e^{i(k_1\xi + k_2\eta + \omega\tau)}\,\Psi_{\dot{\delta}^*}(k_1, k_2,\omega)\,dk_1dk_2d\omega$$

etc. Also $s(k,\omega;y,z)$ is defined in a way analogous to $s(k,\omega;y)$ in one dimension. Then the theorem used earlier which states how $\psi(k,\omega)$ can be chosen in the far field has a simple corollary in the planar case. The derivation of R_ϕ' for the planar case is correspondingly quite similar to that for the full three-dimensional case, which we discuss in more detail in Section 7.5, and so we give now only the result for intensity. Suppose that z is very large compared to the statistical (correlation) lengths l_x, l_y for the distribution in the plane. Then for the intensity there results

$$\overline{p^2}(y,z) = 2\overline{\dot{\delta}^{*2}} \int_{\substack{-L\\ z\gg l_x,\, l_y}}^{+L}\int_0^\infty\int_0^\pi \left\{\frac{\omega^2\Psi_{\dot{\delta}^*}(-\omega\cos\theta,\omega\sin\theta\cos\lambda,\omega)}{\sqrt{(y-y_1)^2+z^2}}\right.\,d\theta d\omega dy_1\,, \qquad (7\text{-}23)$$

where λ is defined in Figure 5. The last equation may be compared to the result (7-17) for the line distribution.

It is seen from (7-23) that the growth of the integral for large L depends upon how the angle λ occurs in the integrand. For a source distribution over a plane (or for that matter for any distribution which can be obtained by differentiation of the source distribution with respect to time or in the plane) the dependence of the integrand upon λ is of no consequence near $\lambda = 0,\pi$, so the integral will diverge like $\log L$ as $L \to \infty$. On the other hand if we replace $\Psi_{\dot{\delta}^*}$ by $\omega^2\sin^2\theta\sin^2\lambda\,\Psi_{\dot{\delta}^*}$ which vanishes at the critical points, convergence is obtained. This disappearance of the divergent part clearly occurs because of cancellation, that is, because no noise is produced by a fluctuating dipole in the plane normal to its axis. On the basis of this result, we can assert that in the far field the mean square of the velocity perturbation normal to the plane, arising from a source distribution having spectrum $\Psi'_{\delta*}$, as well as the mean square pressure in the case of a distribution of *dipoles* with axis normal to the plane, having spectrum $\Psi'_{\delta*}$, both of these quantities will be finite for an infinite plane. This is because in both of these cases the factor $\sin^2\lambda$ is introduced into the integrand by a differentiation with respect to z .

We remark that the dipole distribution may be of interest as an alternative to the displacement thickness model, in cases where the wall pressure is assumed to be given. It can be shown from (7-23), or directly by Fourier analysis in two-dimensional (k_1, k_2) space, that the intensity of the radiated pressure is obtained by integrating the normalized spectrum function over the region $k_1^2 + k_2^2 < \omega^2$ and multiplying by the intensity of the wall pressure; in particular, the intensity is independent of distance from the plane. Such a model will then predict that at high convection speeds and Mach numbers the sound level in the free stream approaches that of the wall. Since observations in boundary layers show a drop in intensity of an order of magnitude between the wall and the free stream, the "wall pressure" in the model is not the actual wall pressure. However, it may be possible to separate the calculation of an effective wall value from the determination of the sound that is radiated, particularly if (as is usually the case) the dominant wavelengths are of the order of the boundary-layer thickness or longer. Then a simple connection, e.g., that the effective value is a certain function of the actual value, may be useful for the purpose of investigating the dependence of the radiated noise and its variation with convection Mach number upon the choice of spectrum function. To illustrate this possibility, take for the spectrum of the effective wall pressure

$$\frac{\kappa^2 \overline{p_w^2}\, l_1 l_2 \tau}{8\pi^{3/2} M_r} \exp\left\{-\frac{1}{4}\left[l_1^2 k_1^2 + l_2^2 k_2^2 + \frac{\tau^2}{M_r^2}(M_r k_1 + \omega)^2\right]\right\}$$

where $\overline{p_w^2}$ is the mean square wall pressure, and κ is now chosen to be a constant. To get the radiated intensity we integrate the last quantity over the region $k_1^2 + k_2^2 < \omega^2$ of the phase space, with the result

$$\overline{p^2} =$$

$$\frac{\kappa^2 \overline{p_w^2}\, l_1 l_2 \tau}{\pi M_r} \int_0^{\pi} \frac{\sin\theta \, d\theta}{\left[l_1^2 \cos^2\theta + \frac{\tau^2}{M_r^2}(1 + M_r \cos\theta)^2\right] \sqrt{l_1^2 \cos^2\theta + l_2^2 \sin^2\theta + \frac{\tau^2}{M_r^2}(1 - M_r \cos\theta)^2}}$$

If the radiated noise is sharply directional, so that the main contribution to the integral occurs near $\theta = \theta_r$, then the approximate form

$$\overline{p^2} = \frac{\kappa^2 \overline{p_w^2}}{\pi} \frac{M_r l_2}{l_1} \int_{-\infty}^{+\infty} \frac{dt}{(1 + t^2)\sqrt{1 + t^2 + \frac{(M_r^2 - 1)\, l_2^2}{l_1^2}}} \tag{7-24}$$

may be used. This similitude is of course an obvious consequence of the previous assumption, but it is interesting to recall that it will be valid not only in the case of rigid convection, but also when M_r is sufficiently large.

Certain consequences of (7-24) may be compared with the variation of the ratio of root mean square pressure with free stream Mach number given by the measurements of Laufer, and of Kistler and Chen (References 40 and 32; see Chapter 8). In these measurements the ratio M_r/M_∞ of the turbulent source field varied with M_∞ from a value of 0.2 in the subsonic range to about 0.4 at $M_\infty = 5$. If κ is taken to be independent of M_r, then one result of the model, that $\overline{p^2}/\overline{p_w^2}$ should tend to an upper limit when M_r is large, $M_r > 3$, for example, seems to be indicated by the data. It should be noted that, since a fairly sharp directional peak was observed for $M_\infty > 2.5$, even though M_r never exceeded about 2, the use of (7-24) must assume almost rigid convection of the dipole field. In the limited Mach number range of the data, (7-24) can be further simplified by assuming $l_2 \ll l_1$, i.e., that contours of constant correlation in the plane of the boundary layer are elongated in the streamwise direction. With this assumption (7-24) gives

$$\frac{\overline{p^2}}{\overline{p_w^2} M_r} = \text{constant}, \qquad 2 < M_\infty < 5 \tag{7-25}$$

If the values of M_r as a function of M_∞ given by Kistler and Chen are used, there is obtained the following table:

M_∞	M_r	Averaged $(\overline{p^2}/\tau_w)^{1/2}$	$(\overline{p_w^2}/\tau_w)^{1/2}$	$\left[\overline{p^2}/(\overline{p_w^2}M_r)\right]^{1/2}$
2	0.64	0.32	4.6	0.087
3	1.14	0.44	4.8	0.086
4	1.56	0.54	5.0	0.086
5	1.95	0.92	5.2	0.085

The fact that (7-25) seems to hold here is interesting because it shows that, for the assumed spectrum function, the Mach number variation which occurs solely because more sources are involved in the radiation of eddy Mach waves as M_∞ increases (the Mach number independence of the sources themselves corresponding to the assumed Mach number independence of κ) approximates very closely that of the far field wall pressure ratio, at least when $\tau|M_r^2 - 1|^{1/2} \gg l_1$ and $|M_r^2 - 1|^{1/2} l_2 \ll l_1$. Therefore the largely arbitrary assumptions of this model are in some way cancelling in their errors, as far as the Mach number variation is concerned. In interpreting this it should be remembered that, for the same reason as the displacement thickness model, the wall pressure model will fail to predict the observed far-field convection Mach number, and so cannot provide a very accurate form of the far-field spectrum.

7.5 The Radiation from an Elementary Eddy Volume

We pass now to the three-dimensional problem of determining the radiation from volume distribution of sources. Our reason for extending the results in this way are twofold. In the first place, the preceding cases, which reduce essentially to solving for a concentrated source distribution, can be obtained as limiting forms of a continuous distribution. In the second place, from the three-dimensional results we can most easily demonstrate the mathematical equivalence of the present analysis with that of Lighthill and Ffowcs Williams (Chapter 6).

We shall take the source distribution to be non-zero in an infinite region V given by $-\infty < x < +\infty$, y,z in Δ, where Δ is a bounded region in the y,z plane. The equation to be solved is

$$\phi_{tt} - \nabla^2\phi = \begin{cases} f(x,y,z,t) & \text{in } V \\ 0 & \text{outside } V \end{cases}$$

We shall assume that $f(x,y,z,t)$ is a stationary random function of all arguments. In order to connect our results to the case in which the same property holds over the entire space, we shall again require that V be large when measured in units equal to the various correlation lengths. In order to justify the far-field approximation used in Section 7.3, we must assume also that

$$|v_p^2 - 1|(y^2 + z^2) > C^2$$

where we stipulate that C is much larger than the smallest circle containing Δ. If the ϕ corresponding to the cylinder bounded by Δ is normalized by division by the area $a(\Delta)$ of Δ, we shall refer to the result as the radiation from an *elementary eddy volume*. We define in the usual way

$$\overline{\phi^2}R_\phi(\xi,\eta,\zeta,\tau;y,z) = \overline{\phi^2}\iint e^{i(\xi k + \tau\omega)}\ \Psi_\phi(k,\omega,\eta,\zeta;y,z)\ dk d\omega$$

Here we adopt the convention, e.g., that η in the place of k_2 indicates that Ψ_ϕ is a partial Fourier transform not involving η. A similar convention will apply to $\Psi_f(k_1,k_2,k_3,\omega)$ (the Fourier transform of the correlation function for the sources). We shall use k in place of k_1 when there is only one wave number involved. The expression for Ψ_ϕ in terms of Ψ_f is then found to be

$$\overline{\phi^2}\Psi_\phi(k,\omega,\eta,\zeta;y,z) =$$

$$\frac{\overline{f^2}}{a(\Delta)}\int_{\Delta^2} s^*(k,\omega,\eta_1,\zeta_1;y_1,z_1)\ s(k,\omega,\eta_2,\zeta_2;y_2,z_2)\ \Psi_f(k,\omega,\eta',\zeta')\ d\eta_1 d\zeta_1 d\eta_2 d\zeta_2$$

The functions $s(k,\omega,\eta_i,\zeta_i;y_i,z_i)$, $i = 1,2$, solve

$$s_{tt} - \nabla^2 s = \delta(y_i - \eta_i)\ \delta(z_i - \zeta_i)\ e^{i(kx + \omega t)}$$

with a condition that waves be outgoing. We have also defined $\eta = y_2 - y_1$, $\zeta = z_2 - z_1$, $\eta' = \eta_2 - \eta_1$, $\zeta' = \zeta_2 - \zeta_1$. In the far field we make use of the results of Section 7.3 (which apply here uniformly over the region by virtue of our definition of the far field in the present problem). In particular, (7-16) is again valid with f replacing $\dot{A}$ and

$$R'_\phi(\xi,\eta,\zeta,\tau;y,z) = \frac{1}{a(\Delta)} \iint_{|w|>|k|} e^{i(\xi k+\tau\omega)} \left[\int_{\Delta^2} e^{-\mathrm{sgn}\,\omega i\sqrt{\omega^2-k^2}\,J(y_i,z_i,\eta_i,\zeta_i)} \times \right.$$

$$\left. \times \frac{1}{\sqrt{\omega^2 - k^2}} \Psi_f(k,\omega,\eta',\zeta')\, d\eta_1 d\zeta_1 d\eta_2 d\zeta_2 \right] dk d\omega \qquad (7\text{-}26)$$

The function J appearing in the exponential factor is given by

$$J(y_i,z_i,\eta_i,\zeta_i) = \sqrt{(y_1 + \eta - \eta_2)^2 + (z_1 + \zeta - \zeta_2)^2} - \sqrt{(y_1 - \eta_1)^2 + (z_1 - \zeta_1)^2}$$

$$= \frac{y}{\sqrt{y^2 + z^2}} (\eta - \eta') + \frac{z}{\sqrt{y^2 + z^2}} (\zeta - \zeta') + 0(1/\sqrt{y^2 + z^2})$$

We can use this expansion in (7-24) without changing the error estimate. The integral may be partially evaluated as follows. We first introduce oblique coordinates $\eta', \zeta', \eta_1, \zeta_1$. The part of the integrand in (7-24) between the brackets can then be evaluated in terms of the full spectrum function $\psi(k_1,k_2,k_3,\omega)$ (it is assumed that the latter exists) provided that Δ is sufficiently large. Introducing the angles λ, θ, there results

$$R'_\phi(\xi,\eta,\zeta,\tau;\lambda) = 4\pi^2 \int_{-\infty}^{+\infty} \int_0^{\pi} \exp\, i\{-\xi\cos\theta - \eta\sin\theta\cos\lambda - \zeta\sin\theta\sin\lambda + \tau\}\omega \times$$

$$\times \Psi_f(-\omega\cos\theta,\ \omega\sin\theta\cos\lambda,\ \omega\sin\theta\sin\lambda,\omega)\ d\theta d\omega$$

In particular

$$\overline{p^2} = \pi \frac{\overline{f^2}}{\sqrt{y^2 + z^2}} \int_0^{\pi} \int_0^{\infty} \omega^2\ \Psi_f(-\omega\cos\theta,\ \omega\sin\theta\cos\lambda,\ \omega\sin\theta\sin\lambda,\omega)\ d\omega d\theta \qquad (7\text{-}27)$$

In (7-27) we have obtained the intensity in terms of averages in Fourier space over spheres of radius ω. The meaning of (7-27) is that only those components of the spectrum associated with phase speed in the direction of the observer (as fixed by θ and λ) contribute to the far-field noise level. That is, only those disturbances which approach an observer with sonic phase speed are contributing. The equivalent statement in terms of the correlation function appeared in Chapter 6, in Equation (6-18). That the approach we have taken here leads to the Lighthill result illustrates the ultimate mathematical equivalence of the two points of view. On the other hand, our starting with the spectrum function of a line distribution of sources has led to this result through a quite different sequence of steps.

7.6 Further Extensions

The particular examples given are of the very simplest kind, and in order to complete our discussion of acoustical models, we shall list briefly three extensions of the theory which appear to be needed on the basis of our intuitive knowledge of the problem, and which offer some hope of useful analytic results.

We first point out that the effect of dispersion in the sources, which has not appeared in the examples, is generally present and will affect any detailed study of the radiated spectrum. In this the medium remains non-dispersive, and only the form of the source spectrum function is changed. We may say that the sources are dispersive if the maximum of the distribution of intensity among wave angles θ occurs at a point θ_r which is not constant, but depends upon the frequency ω, or, what is essentially the same, on the wave number. For example, in the line-source model the general definition of θ_r,

$$\left[\frac{d}{d\theta}\int_0^\infty \omega^2 \Psi_A^{\cdot}(\omega\cos\theta,\omega)\ d\omega\right]_{\theta=\theta_r} = 0$$

can not, in the dispersive case, be reduced to a computation at one frequency. A qualitative discussion of the effect of dispersion upon the observed convection Mach number was given by Laufer[40], and similar conclusions obviously apply to the present models if the source spectrum behaves in the required way. Although a detailed analysis of a dispersive radiated spectrum is complicated by the integrals which must be performed, such a computation would be of great interest in cases where the wave number-frequency form of the source spectrum could be established with sufficient accuracy.

The next obvious generalization of the acoustical model would be to investigate, through a modification of the differential equation, the effects of mean and possibly also randomly distributed inhomogeneities in the gas. A first step in the direction has been taken, with only partial success, by Phillips (Chapter 5). A re-examination of the problem posed by Phillips is clearly needed, and it is certainly possible that limited information may be obtained without restricting the analytical problem to the asymptotic behavior of Fourier components well within the critical lines $k_1^2 + k_2^2 = \omega^2$, even though the Mach number is retained as the expansion parameter. A third limitation in the preceding examples is the assumption that the linear wave equation is relevant. It is evident that the generation of Mach waves by turbulence may be compared with the random flights of small projectiles about some mean speed. Although it may seem likely that in a turbulent field only "waves" and never "shocks" are so generated, such a conclusion cannot be valid for arbitrarily large Mach numbers unless the turbulence is damped to smaller fluctuations as the mean Mach number increases. We can therefore envisage a situation in which linear acoustic theory is insufficient to determine the structure of the "eddy shock" and so also a random distribution of such shocks. Although an entirely consistent non-linear statistical theory would be far too difficult to consider seriously, there is some hope of improving the linear theory by exploiting the projectile analogy directly, i.e., to make use of existing first-order wave theories from supersonic flow theory. Such an analysis might have an interesting application to the formation of shocks in stars, or in regions of the interstellar gas where turbulent fluctuations with large Mach numbers are found.

CHAPTER 8

THE EXPERIMENTAL APPROACH

8.1 Introduction

In the previous chapter some analytical methods applied to the boundary layer radiation problem have been described. The general conclusion one can make from these studies may be stated as follows: Using the Lighthill analogy, the radiation problem can be successfully formulated in a formal manner; at low speeds the intensity variation with Mach number can be predicted for the limiting cases of a large or small surface dimension compared to some typical wavelength; the absolute value of the intensity, however, cannot be evaluated because of its complicated integral form (Equation (5-10)); at high speeds the analogy is helpful to predict the changing character of the radiation ("eddy Mach waves") but gives only very limited information about its intensity. The "simplified model approach" of Chapter 7, on the other hand, is not far enough developed to provide more information than the acoustic analogy.

It is quite clear than an experimental approach is essential in order to further clarify the problem. Unfortunately, here too, one encounters some difficulties. The most obvious one is the setting up of an appropriate environment for the measurements. This in a sense is more difficult to do than in a jet noise experiment where a free field can be more easily approximated. In a wind tunnel the presence of the tunnel walls produces reflections that must be properly accounted for. In a ballistic range, on the other hand, the instrumentation problems related to the required small size sensing devices and to the time averaging method of the measurements cause complications. Finally, full scale tests involving planes or submarines are accompanied by the usual difficulties with extraneous noise producing effects. In addition to the environmental question, one has to provide for very high sensitivity transducers in order to be able to detect the relatively small sound intensities expected in the far field.

To the knowledge of the authors, no subsonic experiments have been performed on boundary layer noise. Perhaps the only relevent work is that of Wilson[83] who concerned himself with the noise generated from a turbulent region around a rotating cylinder, strictly speaking not a boundary layer problem. He found the radiation intensity to be very small indeed: near the noise level of the environment. A clear cut determination of the intensity variation with Mach number was, therefore, not possible, although the measurements tended to confirm a dipole rather than a quadrupole type of radiation, as expected from the discussion given in Chapter 5.

At supersonic speeds, the measurements of one of the authors (J.L.) gave more specific results concerning the intensity and nature of the radiation[40]. However, these experiments too, suffer from the fact that they were not made in a free field but rather in a wind tunnel, and therefore the absence of reflection effects has to be ascertained. This chapter will concern itself mainly with the description of these experiments.

8.2 Experimental Set-Up

The experiments were carried out in a supersonic wind tunnel having a cross section of 18 inches x 20 inches. The turbulence boundary layers on the four tunnel walls had thicknesses of one to approximately two inches in the test section, depending on the Mach number. The radiation from all four of these boundary layers was then measured. An arrangement of this type is far from being a "clean" set-up. The presence of the opposite walls might cause reflections that would result in erroneous conclusions from the intensity measurements. A short discussion on this point is therefore in order.

The reflections from the opposite walls, giving a larger intensity of radiation than one would detect in a free field, is, in general, difficult to isolate. Reference 40 describes a special experiment to investigate this point. The configuration of this test was such that, at the point of measurements, no reflection could have occurred. The intensity and spectrum measured this way were consistent with the rest of the results, as will be shown subsequently.

8.3 Method of Measurement

The pressures of the fluctuations were deduced with the application of the hot-wire technique. As is well known, the hot-wire responds to mass flow, m', and total temperature, T_t', fluctuation. However, in a sound field where the isentropic relations hold between pressure, temperature and density, both m' and T_t' may be expressed in terms of pressure and velocity fluctuations only. This is a fortunate circumstance, since a hot-wire having a very small size (0.0005 inch diameter and 0.012 inch length) is an ideal pressure pick-up with excellent "wave number response". In addition to the pressure intensity, the technique gives one additional information: Provided most of the sound energy is carried by plane waves moving approximately in the same direction, the direction can be calculated. This is so because, for a plane wave, the relation between the perturbations in velocity normal to the wave front and pressure is known, $u_n' = (1/M)\ (p'/\gamma\bar{p})$, and because a hot-wire responds to velocity perturbations in the direction of the flow, u', and not in the direction perpendicular to the wave front. Thus

$$\frac{u'}{\bar{U}} = \frac{p'}{\gamma\bar{p}}\frac{\cos\theta}{M_\infty} \quad \text{since} \quad u_n'\cos\theta = u' \qquad (8\text{-}1)$$

where θ is the angle between the flow direction and the normal to the wave front. However, if the sound intensity is distributed over waves having a broad angular distribution, the information available is insufficient to obtain the directional distribution.

8.4 The Intensity of Radiation

The hot-wire senses the pressure fluctuations emanating from all four boundary layers. In order to obtain the intensity due to one layer only, the measured mean square values of the fluctuations were divided by four under the assumption that the four boundary layers are identical and the sound fluctuations produced by them are independent. Figure 6 shows the root mean square pressure fluctuations calculated this way; they were normalized by the free-stream dynamic pressure. The effect of the Reynolds number is quite apparent in the figure. This suggests that the sound sources (T_{ij} in the Lighthill formulation) scale with τ_w, the wall shearing stress, as indeed the wall pressure fluctuations do. Figure 7 shows the wall and radiated pressure fluctuations normalized by the wall shear. It is seen that, within the experimental scatter, the Reynolds number effect disappears. The figure also indicates that the radiation intensity is at least two orders of magnitude smaller than the intensity measured on the wall[32]. Finally, the experimental point designated by a square was obtained with a configuration in which only one wall boundary layer was turbulent and no reflected waves could reach the point of measurements. It is seen that, within the expected accuracy, the result is consistent with the other measurements. (The accuracy at small Reynolds numbers is lower because the hot-wire calibration constants are not as well established under these conditions).

The interesting point to note is the rapid reduction in intensity as the supersonic free-stream Mach number approaches one. If the radiation is an "eddy Mach wave" (see Section 8.5) then this is understandable since no such radiation can take place at subsonic speeds.

8.5 The Directional Character of the Radiation

As mentioned already in Section 8.3 it is possible to obtain the direction of radiation from a single hot-wire sensor provided two conditions are satisfied: (1) the measurement is made in the far field of the source so that the assumption of plane waves is fulfilled; (2) the radiation is unidirectional. As will be shown from the spectrum measurements (Section 8.6), condition (1) is satisfied for the largest portion of the energy spectrum. With reference to the second condition, the measurements show[38] that the velocity and pressure fluctuations are perfectly anticorrelated. This is consistent with the conjecture that the field consists of backward facing waves with a directional preference. The angle of directionality is then given by Equation (8-1)

$$\cos\theta = \gamma M_\infty \frac{u'/U}{p'/p}$$

Once θ is known, the velocity of the "sources", U_s, producing the waves may be calculated, since

$$-\frac{1}{\cos\theta} = \frac{U_\infty - U_s}{a_\infty} = M_\infty\left(1 - \frac{U_s}{U_\infty}\right)$$

and therefore

$$\frac{U_s}{U_\infty} = 1 + \frac{1}{M\cos\theta}$$

8.6 The Statistical Nature of the Radiation

In the previous argument the source velocity was obtained in a rather indirect method. In order to investigate the streamwise component of the propagation velocities of the sound, a space-time correlation measurement was performed using two hot-wires displaced in the flow direction by an amount Δx . (A slight displacement in the perpendicular direction, Δy , was also necessary in order to avoid interference). Figure 8 shows a few typical correlation curves. It is seen that there exists a certain time delay $\Delta\tau$ for which the correlation between the pressure fluctuations at the two test stations is a maximum. This implies a propagation velocity, the x-component of which is $\Delta x/\Delta\tau \equiv U_s$. It is quite apparent, however, that, at the lower Mach numbers, the definition of a propagation velocity becomes rather obscure. The weak maximum of the correlation curve suggests that the convection occurs over a rather wide range of velocities. The implication of this result will be discussed subsequently.

Figure 9 shows the velocities U_s in the far field obtained from the space-time correlation measurements, together with the convection velocities, U_c , of the wall pressure fluctuation as measured by Kistler and Chen[32]. The interesting point to note is the large difference in these two quantities in the low Mach number range. If one would take the simple-minded point of view that the "eddy Mach wave" radiation is produced by a supersonically moving wavy wall (that carries a pressure field on its surface identical to that measured under a turbulent layer), the speed of the wall and the x-component of the propagation velocity in the far field would be equal. Obviously, this is not the case. As a matter of fact, at, say, $M_\infty = 2$ the wall pressure is convected subsonically with respect to the free stream $[M_\infty(1 - U_c/U_\infty) = 2(1 - 0.68) = 0.64]$ and therefore Mach wave radiation could not take place according to the wavy wall model. Nevertheless, the hot-wire measurements (and schlieren pictures taken in ballistic ranges) do indicate that some Mach wave radiation is taking place. At present the explanation may be conjectured only, because not enough information is known about certain statistical properties of the turbulent sources, specifically, about the wave number-frequency spectrum of the wall pressure fluctuations. However, it is well documented by now[81] that in a subsonic boundary layer (and there should be little doubt that in a supersonic layer also) the phase velocities of the pressure disturbances are frequency-dependent; they are dispersive. More specifically, pressure disturbances of large wave numbers have higher phase velocities relative to the free stream. Keeping this in mind, it is quite feasible that, although at Mach number 2 the energy containing perturbations move subsonically, the high wave number components have supersonic phase velocities with respect to the free stream and can produce Mach wavelets.

There are several important consequences of this conjecture which can be verified experimentally. First of all, at high enough Mach number where the phase velocities of most of the energy containing disturbances are supersonic, Mach wave radiation could take place throughout the wave number spectrum; therefore, the average phase velocities measured on the near and far fields could be expected to be the same. Indeed, at $M_\infty = 5$ one finds that $U_s \simeq U_c$. Secondly, at low Mach numbers the large wave number disturbances that radiate Mach waves contain a small amount of energy relative to the total disturbance energy; therefore, the intensity of radiation is expected to be small compared to that at high Mach numbers where presumably almost all of the disturbances take part in the radiation process. This is believed to be the main reason that at $M_\infty = 2$ the radiation intensity is found to be almost an order of magnitude smaller than at $M_\infty = 5$, even though the intensities of the wall pressure fluctuations and their spectrum distributions are about the same. (See Figures 7 and 10). Finally at the lower Mach numbers, one expects the normalized energy spectrum to contain relatively more energy in the large wave number region, compared to the low wave number regime (since presumably radiation takes place only here) than does the spectrum at the higher Mach number flows. Indeed, measurements made in the far field at $M_\infty = 4.5$ and $M_\infty = 2.0$ are consistent with this conclusion (Fig.11). The normalizing factor L_x was obtained from the wall spectrum measurements of Kistler and Chen shown in Figure 11. It is the integral scale defined by

$$L_x = \frac{2\pi E(0)}{4}$$

where $E(0)$ is the normalized wall pressure spectrum at zero wave number.

It should be emphasized that the foregoing arguments were made having had in mind the moving wavy wall or surface dipole model discussed in Section 7.4. The main justification for this is the fact that the model is simple and does contain the "eddy Mach wave" nature of this radiation. It is inadequate for predicting the intensity and energy spectrum of the radiation. One should mention at this point that in comparing the energy spectrum distributions measured at the wall and in the far field (Fig.12), it is quite apparent that the radiation field contains much less energy in the high wave number region, suggesting a strong filtering action within the boundary layer. Whether the filtering is due to some cancellation, scattering or reflection process is one of the important open questions of the problem.

8.7 The Question of Turbulence Damping

It has been pointed out in Chapter 1 that one of the most interesting new features of a compressible, turbulent boundary layer is the fact that it loses turbulent energy, not only by dissipative processes, but also by radiation. The question of whether under certain circumstances these losses could be strong enough to damp out turbulence, or at least seriously affect it, is an intriguing one and also from practical considerations a very important one.

The measurements described in this chapter give sufficient information to allow us to compare the rate of energy loss due to radiation, to the work done by the turbulent shearing stresses. To calculate the former, we shall assume that the boundary layer occupies the x,z plane (or at least a strip of this plane parallel to the free stream). Consider then the acoustic field, filling a region V bounded by the wall, a parallel plane above the boundary layer, and a cylinder normal to the planes whose cross-section S has a very large area. For sufficiently large area, the acoustic equations may be integrated to give the following expression for the energy radiated away through the upper plane, per unit time and unit area:

$$N = -\frac{1}{\text{area of } S} \int_S p'v' \, dxdz \tag{8-2}$$

In order to compute N, we immediately consider a sharply directional sound field, so that we may substitute

$$v' \simeq -\frac{p'}{\rho_\infty a_\infty} \sin\theta_r \ , \qquad \text{where} \quad \theta_r = \cos^{-1}\frac{1}{M_r}$$

in (8-2). This gives

$$\frac{N}{\rho_\infty a_\infty^3} = \frac{\overline{p'^2}}{\rho_\infty^2 a_\infty^4} \sin\theta_r$$

The rate at which energy is extracted from the mean flow (per unit area) is given simply by the product of the shearing stress and the wall speed relative to the medium

$$\frac{W}{\rho_\infty a_\infty^3} = \frac{\tau_w U_\infty}{\rho_\infty a_\infty^3} = \frac{C_f}{2} M_\infty^3$$

The ratio of N to W is then given by

$$\frac{N}{W} = \left(\frac{\overline{p'^2}}{\tau_w^2}\right) C_f M_\infty \sin\theta_r \qquad (8\text{-}3)$$

Since $\sin\theta_r \to 1$ as $M_r \to \infty$, the ultimate Mach number variation of N/W depends on the behavior of the quantities $\overline{p'}/\tau_w$ and $C_f M_\infty$. For the particular case M = 5 , the experimental results give

$$\frac{\overline{p'}}{\tau_w} = 0.6 \ , \qquad C_f = 6.6 \times 10^{-4} \ , \qquad M_r \simeq 2$$

so that N/W is of the order of 5×10^{-4} . Therefore, even if we allow for the fact that not all of the energy extracted from the mean flow actually goes into the kinetic energy of the turbulent fluctuations, the turbulence damping remains extremely weak, certainly less than one percent of the energy in the turbulence.

CHAPTER 9

CONCLUDING REMARKS

While a complete understanding of the sound producing mechanism due to a turbulent boundary layer is still lacking, important advances have been made in the last decade. In particular, three general approaches have been proposed that have either contributed substantially to our present knowledge or have the potentiality of furthering the present state of the art.

(i) The acoustic analogy formulates the problem in terms of a classical wave equation with a random forcing term; the forcing term is assumed to incorporate independently known incompressible flow behavior. The analogy predicts the inefficient nature of the generation mechanism, the directional behavior, and the strong dependence of the intensity on frequency and velocity. Unfortunately, in the formulation, a forcing term emerges which is difficult to obtain in detail, either by experimental or analytical means. The predictions of the theory are therefore limited in practice to information which is insensitive to the detailed structure of the forcing term or its spectrum.

(ii) This latter circumstance motivated the second line of inquiry: the acoustic model approach. Here the forcing function of the inhomogeneous wave equation is obtained, not in the formal manner of the analogy; rather an "equivalent forcing function" is sought that would fulfill the requirement of being directly measureable and of containing hopefully the main features of the "real" noise generating mechanism. An approach of this type cannot, of course, replace a formal solution but, as was shown in Chapter 7, it can bring out some of the important points of the problem with relatively simple examples. From this point of view, it is therefore a helpful undertaking.

(iii) The third approach seeks a solution of the generalized wave equation derived in Chapter 3. Recognizing that a general solution is not in sight, it searches for a valid expansion procedure by which the problem could be simplified. That Phillips has not succeeded in doing this does not detract from the promise of his method. It could turn out, of course, that a much better understanding of the general compressible turbulence problem will be necessary before this approach can be successful.

As far as the experimental line of inquiry is concerned, comparatively speaking, inadequate efforts have been made so far. It is true that the existing measurements described in Chapter 3 generated some valuable information, but much more could be done with well-conceived experimentation. One of the most interesting questions, however, concerning turbulence damping due to radiation losses has been answered: in supersonic flows apparently such damping is very weak indeed, and it is doubtful that even at hypersonic Mach numbers it can completely eliminate the turbulent fluctuations. The measurements do indicate that the radiation is more efficient in supersonic flows and that the notion of "eddy Mach wave" radiation is a correct one, but they are incomplete as yet as far as suggesting a generation mechanism. It seems, for instance, that the knowledge of the pressure field on the solid surface is inadequate for estimating the radiated field. The measurement of a more pertinent quantity, perhaps of the displacement thickness fluctuation - as suggested by Liepmann - might be a worthwhile undertaking to this effect.

Finally, it should be mentioned that the strongly nonlinear case, in which the generated sound field interacts with the turbulence itself, has not been touched upon at all. This is an area that might have some important applications in some astrophysical problem and therefore deserves serious experimental consideration.

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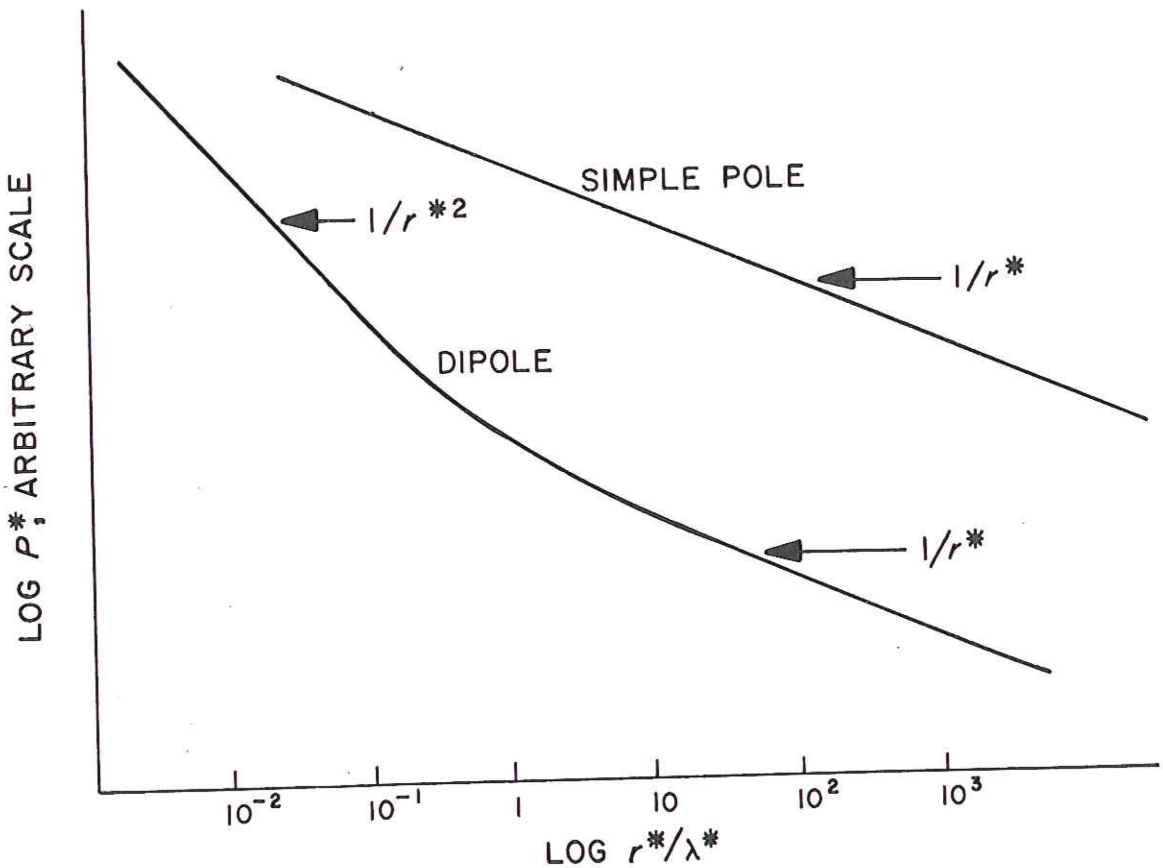

Fig.1 The Stokes effect

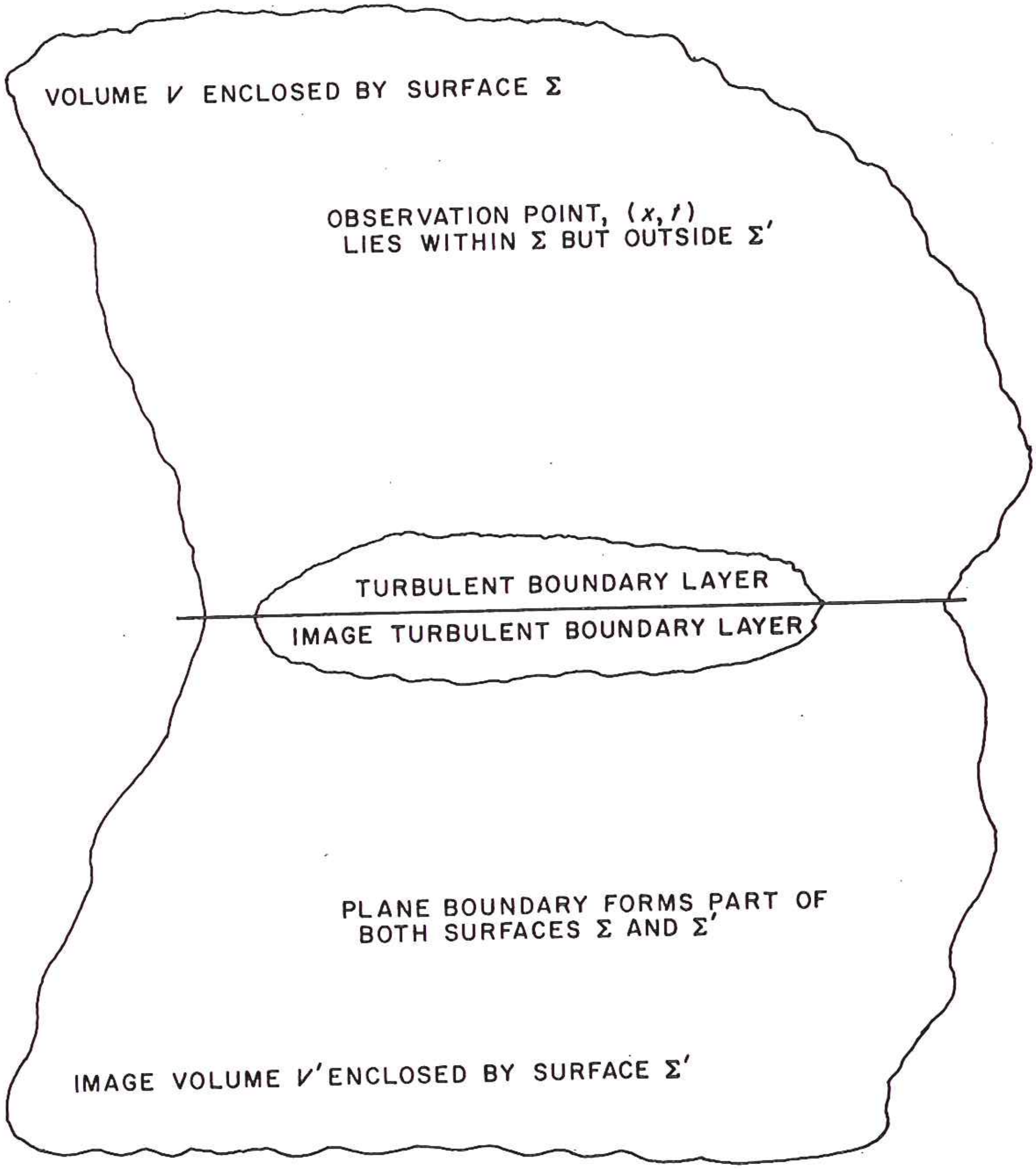

Fig. 2 Diagram illustration of the proof of acoustic reflection by a rigid boundary supporting a boundary layer

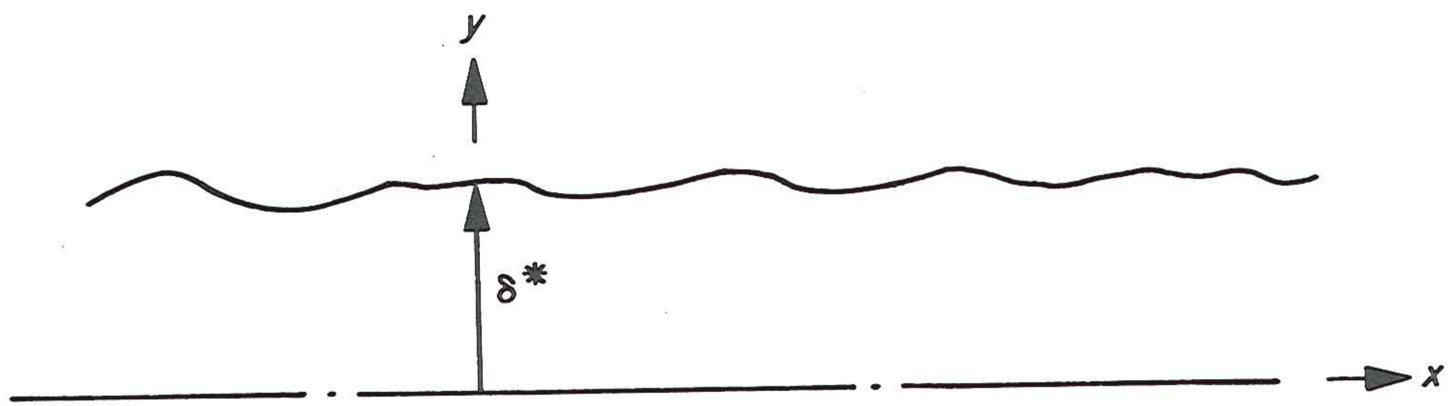

Fig.3 Flow over a randomly corrugated circular cylinder

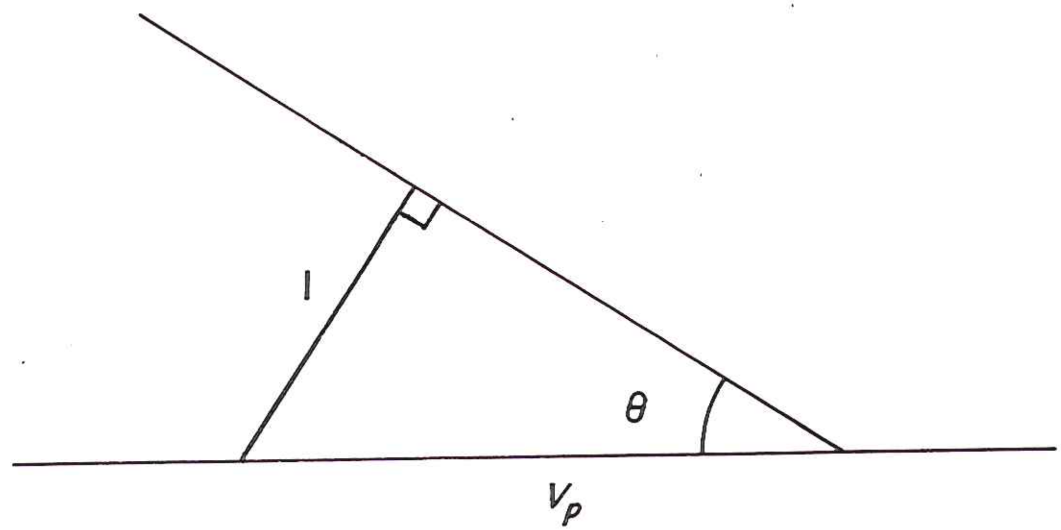

Fig.4 Definition of the phase angle θ

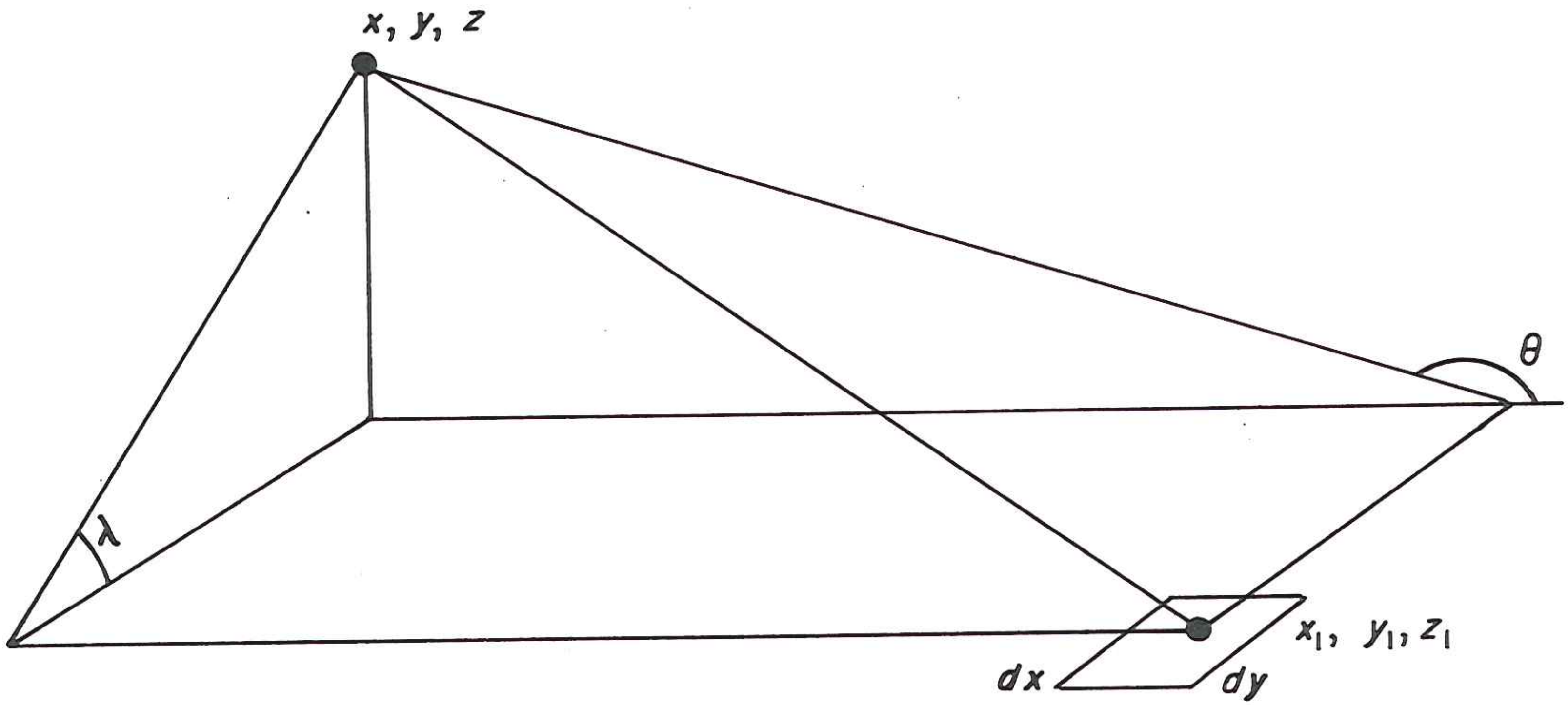

Fig.5 The coordinate system used in Equation (7-23)

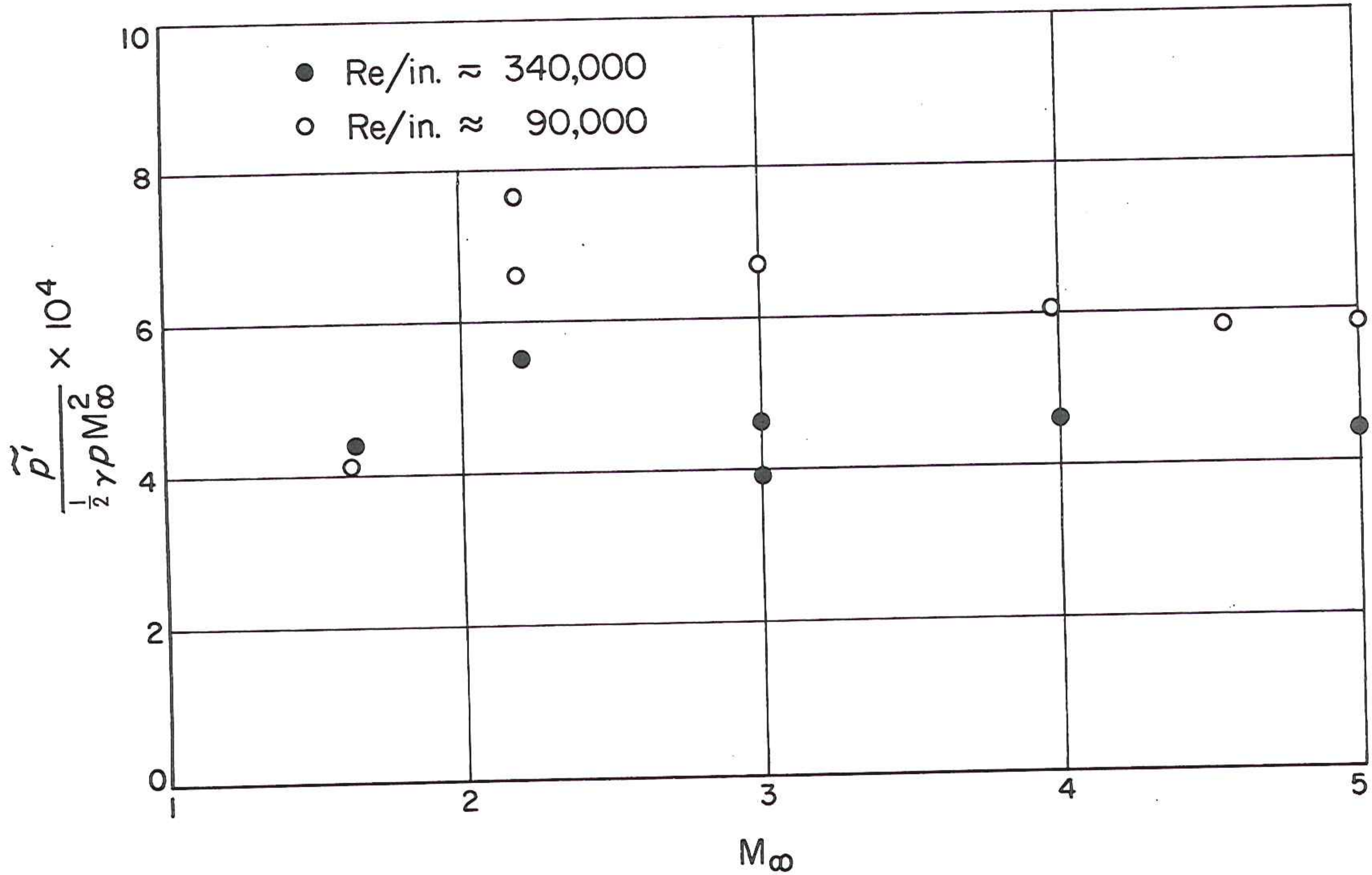

Fig.6 The root-mean-square radiated pressure variation with Mach number

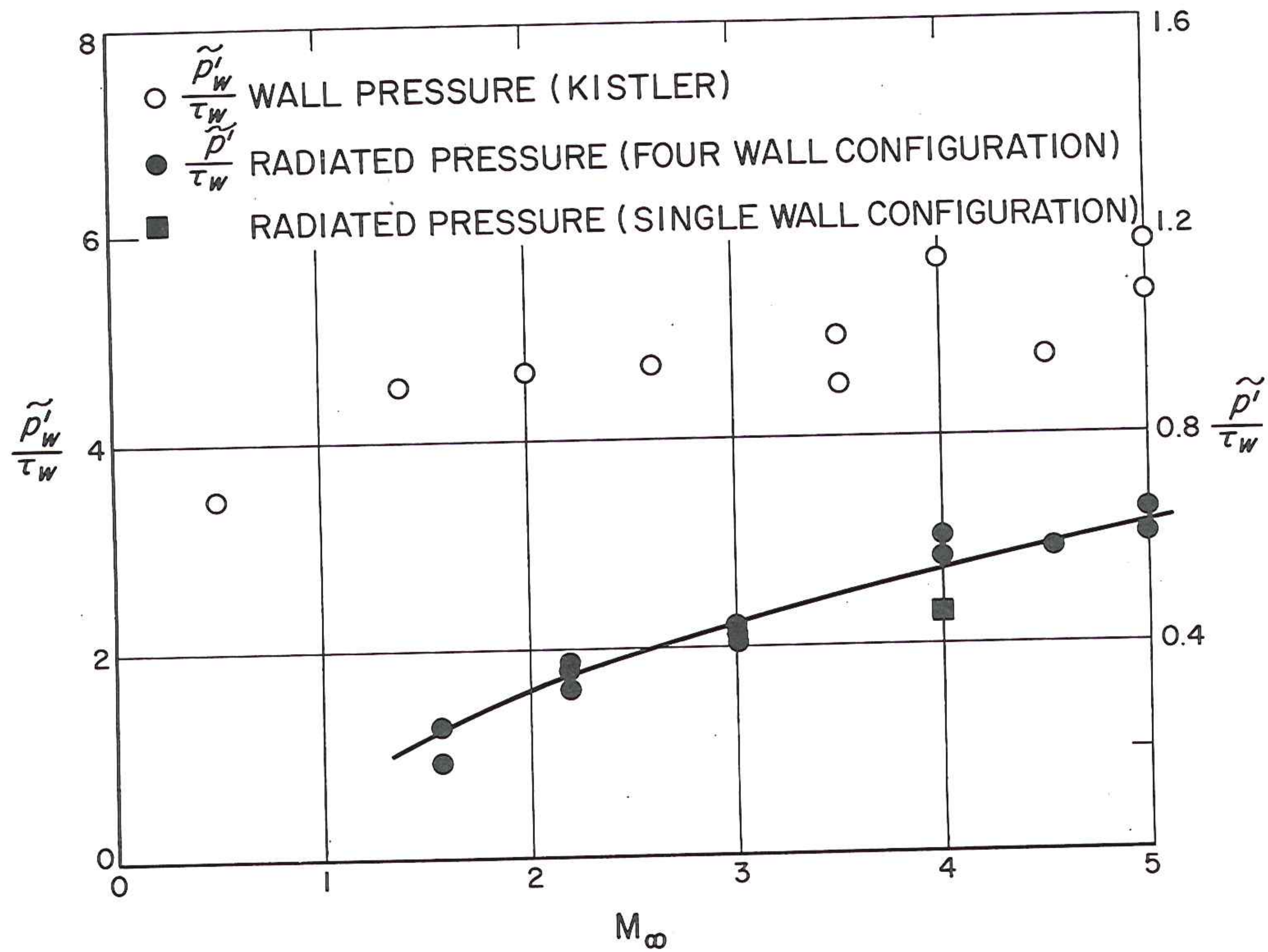

Fig.7 Comparison of the wall and radiated pressure fluctuations

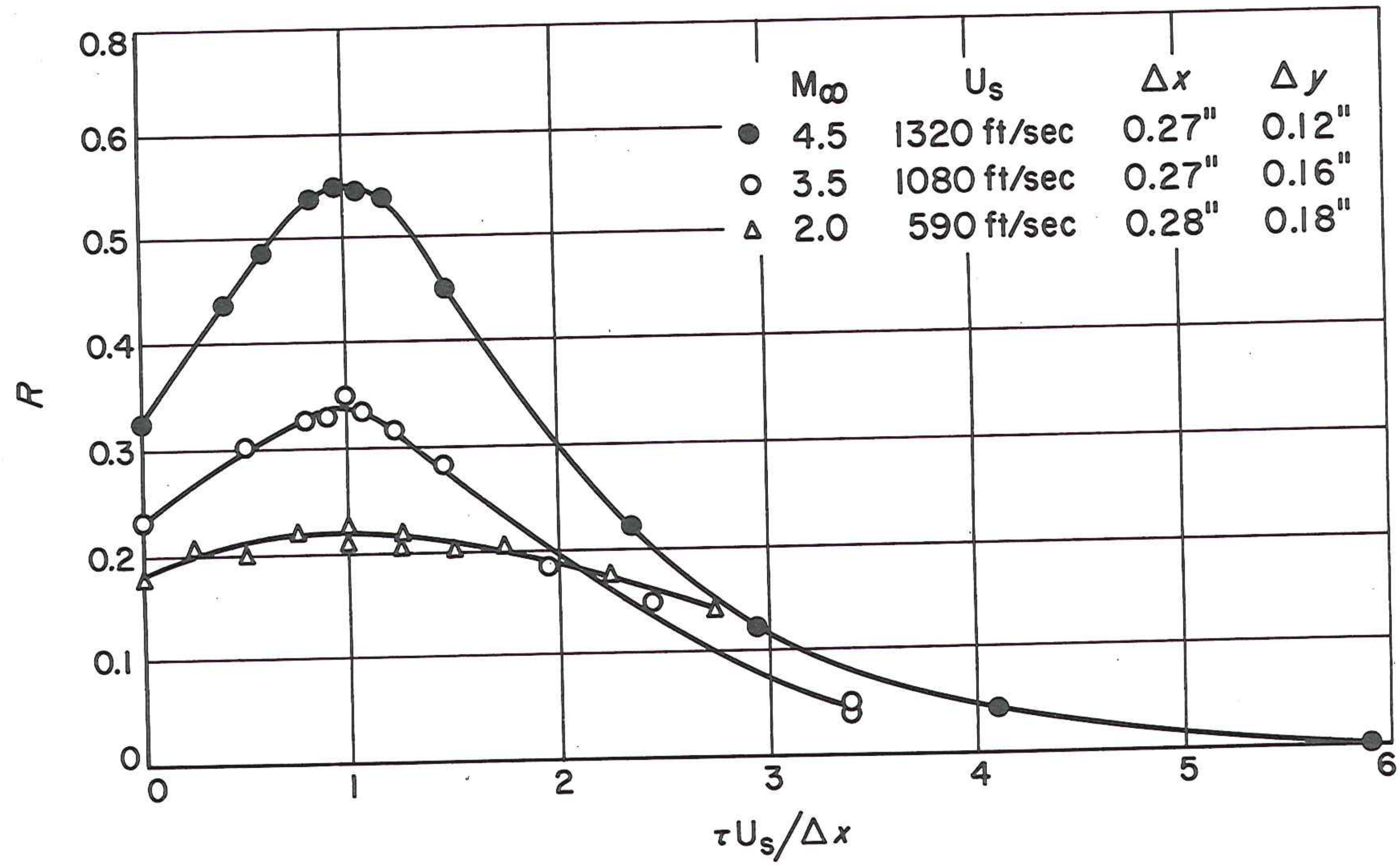

Fig.8 Space-time correlation

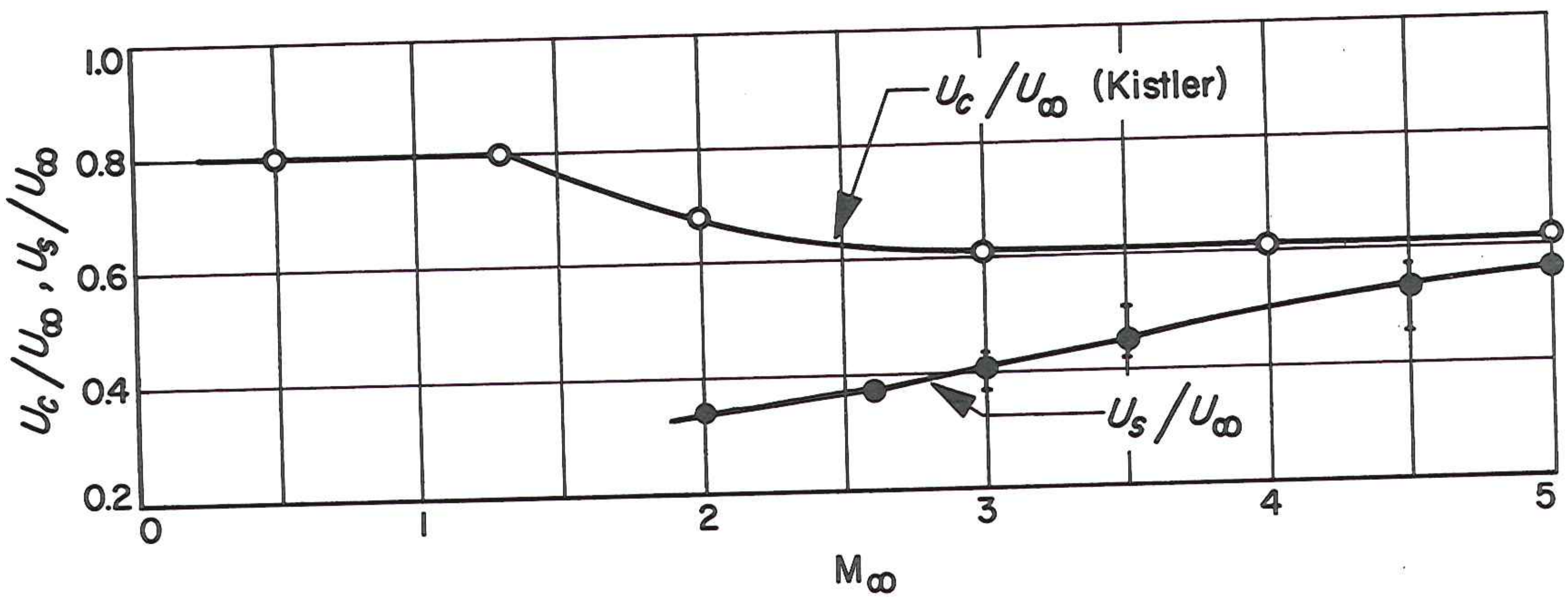

Fig.9 Convection and streamwise propagation speeds at the wall and in the far field, respectively

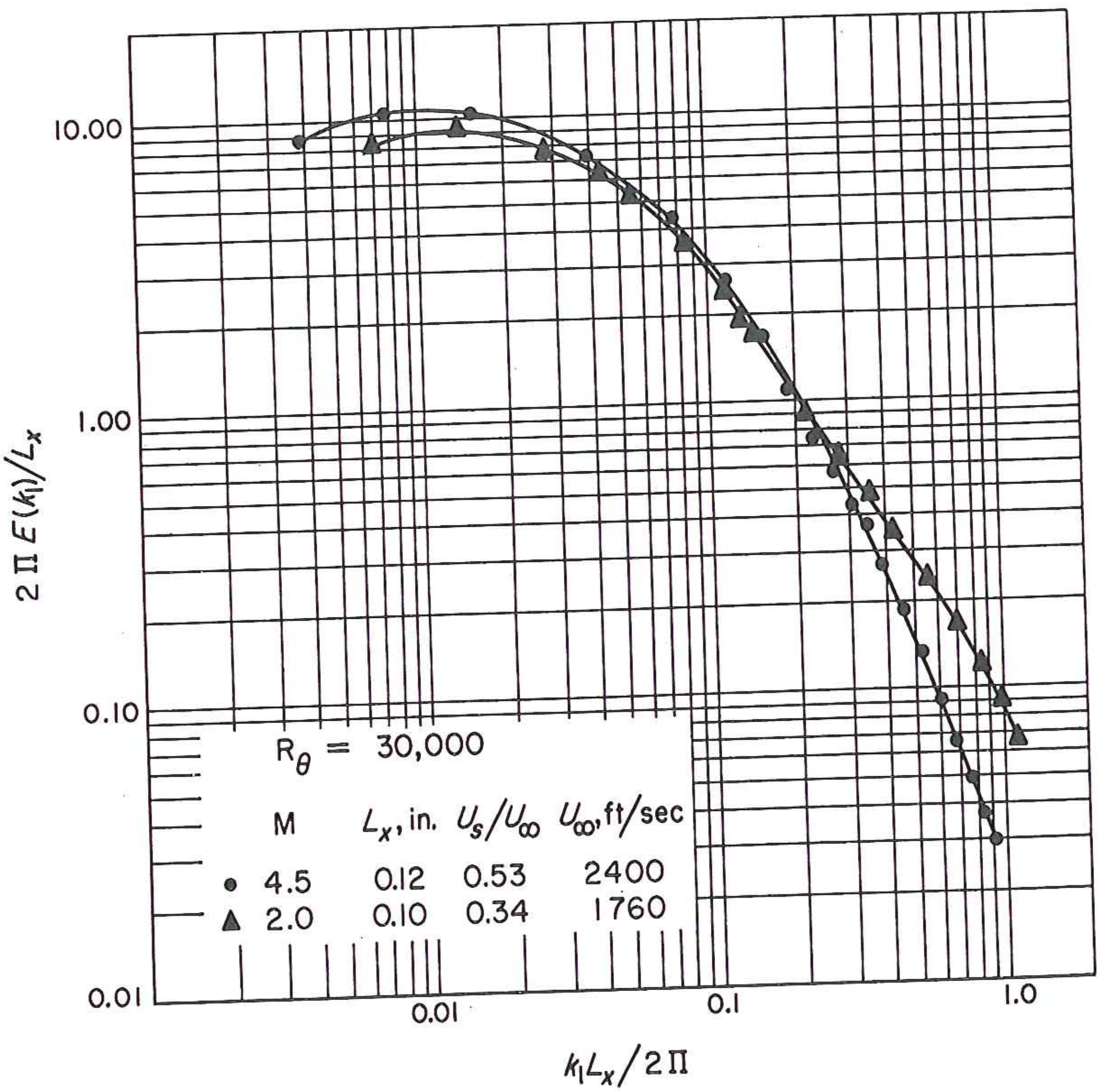

Fig.10 Pressure spectrum in the far field at M = 4.5 and M = 2.0

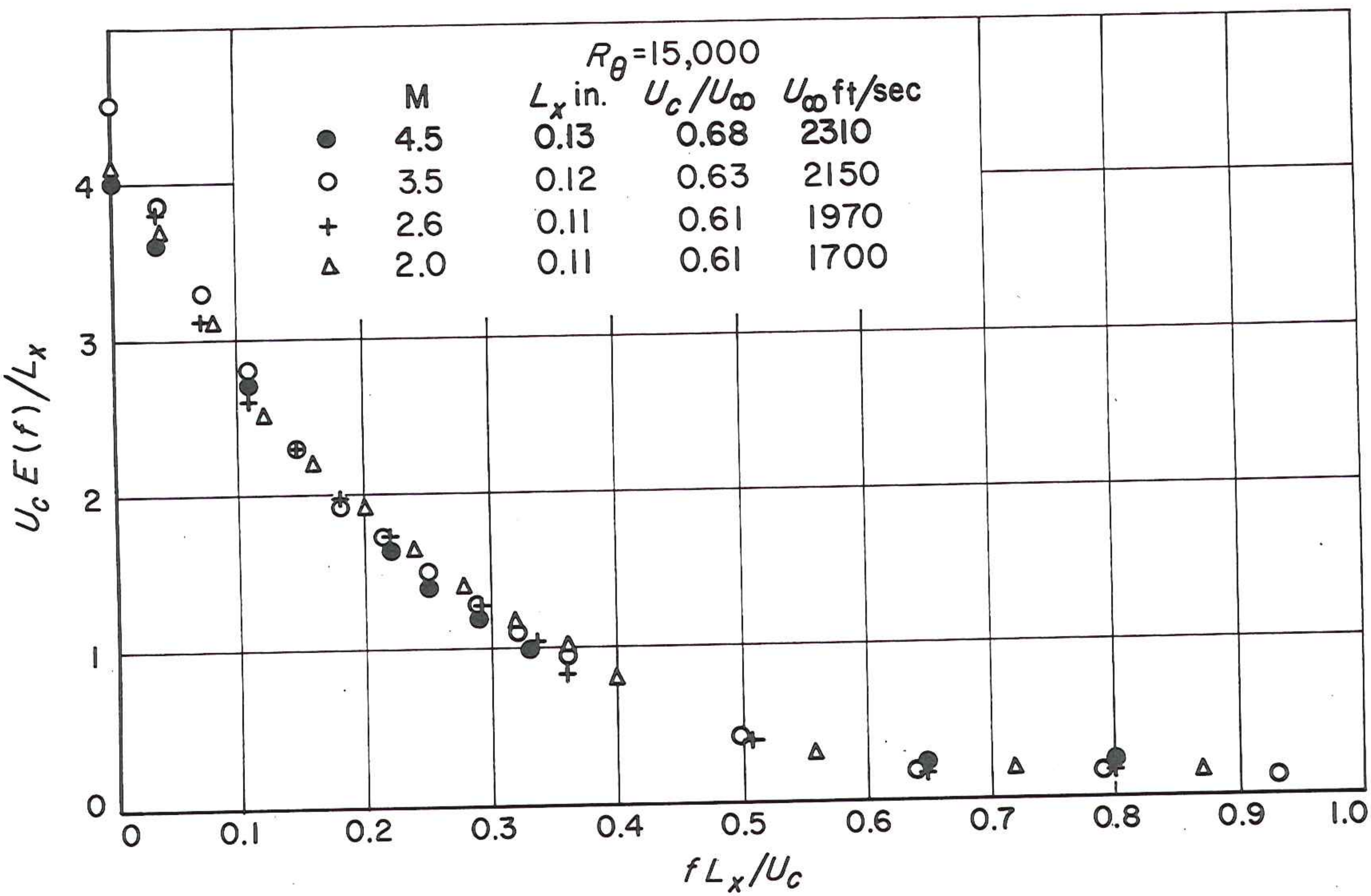

Fig.11 Pressure spectrum on the wall (Kistler and Chen, Ref.32)

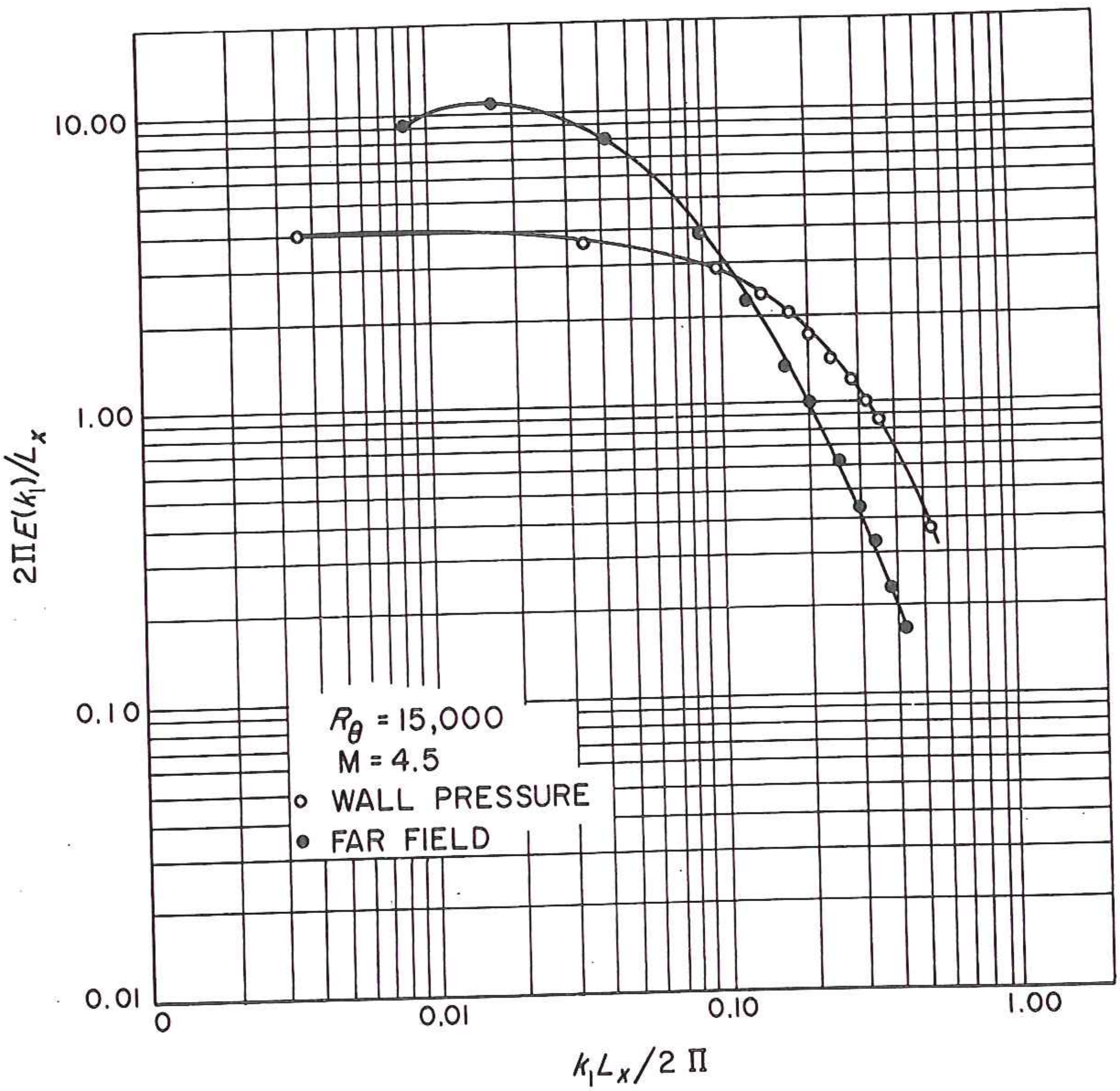

Fig.12 Far and near field pressure spectra

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