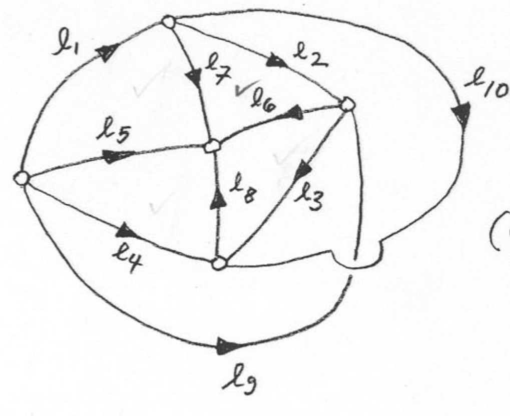
(1 hour) CE 506 Exam I 16 October 2003 ______________ Name

(1 page of notes is allowed)

1. For the following level network, give the elements of the model: n, n_0, r, and write the condition equations for least squares adjustment in the form $Av = f$.



(arrows point uphill)

2. We make 2 observations (l_1, l_2) of a single quantity. The standard deviations for these 2 observations are, respectively, (σ_1, σ_2). Show that the "weighted mean" $\bar{l} = \dfrac{w_1 l_1 + w_2 l_2}{w_1 + w_2}$ is equivalent to the least squares solution.

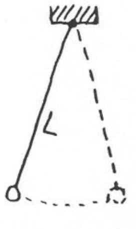
3. The period, T, of a pendulum is related to its length, L, and the acceleration, g, due to gravity by the following equation:

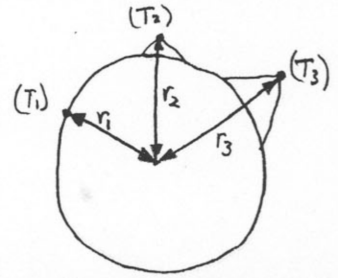
$$T = 2\pi\sqrt{\frac{L}{g}}$$

g is related to the earth's gravitational constant, GMe, and the distance, r, from the center of the earth by:

$$g = \frac{GMe}{r^2}$$

At 3 different elevations, we <u>observe</u> r and T. We also make a single <u>observation</u> of L. ($n = 7$: $r_1, T_1, r_2, T_2, r_3, T_3, L$)





(a) what are n_0 and r (redundancy)? write the condition equations for least squares adjustment by observations only

(b) write the condition equations in the form $Av = f$

(Hint: this problem contains <u>spatial</u> and <u>time</u> observations. To obtain n_0 you must be able to reconstruct all of the given observations.)

CE506 Exam I Solution, 20 October 2003

1. $n = 10$, $n_0 = 4$, $r = 6$

for observations only, we need $r = c = 6$ condition equations

1. $\hat{l}_1 + \hat{l}_7 - \hat{l}_5 = 0$
2. $\hat{l}_2 + \hat{l}_6 - \hat{l}_7 = 0$
3. $\hat{l}_3 + \hat{l}_8 - \hat{l}_6 = 0$
4. $\hat{l}_4 + \hat{l}_8 - \hat{l}_5 = 0$
5. $\hat{l}_9 + \hat{l}_6 - \hat{l}_5 = 0$
6. $\hat{l}_{10} + \hat{l}_8 - \hat{l}_7 = 0$

$$\begin{array}{c}
\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
\underbrace{\begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1
\end{bmatrix}}_{A}
\end{array}
\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{10} \end{bmatrix}}_{v}
= -\mathbf{A}\begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ \vdots \\ l_{10} \end{bmatrix}$$

$$v = \underbrace{-A\,l}_{f}$$

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2. $n = 2$, $n_0 = 1$, $r = 1$ — solve by indirect observations, choose parameter: x

$$w_1 = \frac{1}{\sigma_1^2} \qquad W = \begin{bmatrix} w_1 & \\ & w_2 \end{bmatrix} \qquad w_2 = \frac{1}{\sigma_2^2}$$

$\hat{l}_1 = x$

$\hat{l}_2 = x$

$\hat{l}_1 - x = 0$

$\hat{l}_2 - x = 0$

$$\left.\begin{array}{l} v_1 - x = -l_1 \\ v_2 - x = -l_2 \end{array}\right\} \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix}[x] = \begin{bmatrix} -l_1 \\ -l_2 \end{bmatrix}$$

$$v + B\Delta = f$$

$$\Delta = (B^TWB)^{-1}B^TWf$$

$$\Delta = \left[\begin{bmatrix} -1 & -1 \end{bmatrix}\begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right]^{-1}\begin{bmatrix} -1 & -1 \end{bmatrix}\begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix}\begin{bmatrix} -l_1 \\ -l_2 \end{bmatrix}$$

$$\Delta = \left[\begin{bmatrix} -w_1 & -w_2 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right]^{-1}\begin{bmatrix} -w_1 & -w_2 \end{bmatrix}\begin{bmatrix} -l_1 \\ -l_2 \end{bmatrix}$$

$$\Delta = (w_1 + w_2)^{-1}\, w_1 l_1 + w_2 l_2$$

$$\boxed{\Delta = \frac{w_1 l_1 + w_2 l_2}{w_1 + w_2}}$$

(note: see end for a solution by observations only)

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3. $n = 7$, $n_0 = 4$, $r = 3$

$(\underbrace{r_1, r_2, r_3, L}_{4}$ allow us to compute $T_1, T_2, T_3)$

for $Av = f$, $c = r = 3$

$$\left.\begin{array}{l} T_1 = 2\pi\sqrt{\dfrac{L r_1^2}{GMe}} \\ T_2 = 2\pi\sqrt{\dfrac{L r_2^2}{GMe}} \\ T_3 = 2\pi\sqrt{\dfrac{L r_3^2}{GMe}} \end{array}\right\}$$

they are nonlinear in the observations

(notation: Let $G = GMe$)

$F_1 = T_1 - 2\pi\left[L r_1^2/G\right]^{1/2} = 0$

$F_2 = T_2 - 2\pi\left[L r_2^2/G\right]^{1/2} = 0$

$F_3 = T_3 - 2\pi\left[L r_3^2/G\right]^{1/2} = 0$

$\dfrac{\partial F_1}{\partial T_1} = 1$, $\dfrac{\partial F_1}{\partial L} = -\cancel{\tfrac{1}{2}}\cdot\cancel{2}\pi[\;]^{-1/2}(r_1^2/G)$, $\dfrac{\partial F_1}{\partial r_1} = -\cancel{\tfrac{1}{2}}\cdot\cancel{2}\pi[\;]^{-1/2}\cdot 2Lr_1/G$

$\dfrac{\partial F_2}{\partial T_2} = 1$, $\dfrac{\partial F_2}{\partial L} = -\cancel{\tfrac{1}{2}}\cdot\cancel{2}\pi[\;]^{-1/2}(r_2^2/G)$, $\dfrac{\partial F_2}{\partial r_2} = -\cancel{\tfrac{1}{2}}\cdot\cancel{2}\pi[\;]^{-1/2}\cdot 2Lr_2/G$

$\dfrac{\partial F_3}{\partial T_3} = 1$, $\dfrac{\partial F_3}{\partial L} = -\cancel{\tfrac{1}{2}}\cdot\cancel{2}\pi[\;]^{-1/2}(r_3^2/G)$, $\dfrac{\partial F_3}{\partial r_3} = -\cancel{\tfrac{1}{2}}\cdot\cancel{2}\pi[\;]^{-1/2}\cdot 2Lr_3/G$

$$\begin{array}{c}
\begin{array}{ccccccc} (T_1) & (T_2) & (T_3) & (L) & (r_1) & (r_2) & (r_3) \end{array} \\
\underbrace{\begin{bmatrix}
1 & 0 & 0 & \dfrac{-\pi r_1^2}{[L r_1^2 G]^{1/2}} & \dfrac{-2\pi r_1 L}{[L r_1^2 G]^{1/2}} & 0 & 0 \\
0 & 1 & 0 & \dfrac{-\pi r_2^2}{[L r_2^2 G]^{1/2}} & 0 & \dfrac{-2\pi r_2 L}{[L r_2^2 G]^{1/2}} & 0 \\
0 & 0 & 1 & \dfrac{-\pi r_3^2}{[L r_3^2 G]^{1/2}} & 0 & 0 & \dfrac{-2\pi r_3 L}{[L r_3^2 G]^{1/2}}
\end{bmatrix}}_{(A)}
\end{array}
\underbrace{\begin{bmatrix} v_{T_1} \\ v_{T_2} \\ v_{T_3} \\ v_L \\ v_{r_1} \\ v_{r_2} \\ v_{r_3} \end{bmatrix}}_{(v)}
= \underbrace{\begin{bmatrix} -F_1 \\ -F_2 \\ -F_3 \end{bmatrix}}_{f} = A(l - l^0)$$

2. alternate solution by observations only

$$\hat{\ell}_1 - \hat{\ell}_2 = 0$$

$$v_1 - v_2 = -\ell_1 + \ell_2$$

$$v_1 = -\ell_1 + \ell_2 + v_2$$

solve LS by substitution

$$\phi = w_1 v_1^2 + w_2 v_2^2$$

$$\phi = w_1(-\ell_1 + \ell_2 + v_2)^2 + w_2 v_2$$

$$\frac{\partial \phi}{\partial v_2} = \cancel{2} w_1(-\ell_1 + \ell_2 + v_2) + \cancel{2} w_2 v_2 = 0$$

$$-w_1 \ell_1 + w_1 \ell_2 + w_1 v_2 + w_2 v_2 = 0$$

$$v_2 (w_1 + w_2) = w_1 \ell_1 - w_1 \ell_2$$

$$v_2 = \frac{w_1 \ell_1 - w_1 \ell_2}{w_1 + w_2}$$

$$\hat{\ell}_2 = \ell_2 + v_2$$

$$\hat{\ell}_2 = \underbrace{\frac{(w_1 + w_2)\ell_2}{(w_1 + w_2)}}_{''\ \ell_2} + \underbrace{\frac{w_1 \ell_1 - w_1 \ell_2}{w_1 + w_2}}_{''\ v_2}$$

$$\boxed{\hat{\ell}_2 = \frac{w_1 \ell_1 + w_2 \ell_2}{w_1 + w_2}}$$