Example: Least Squares Line Fit or Regression using Method of Indirect Observations & Longhand or Brute Force Approach

CE 506
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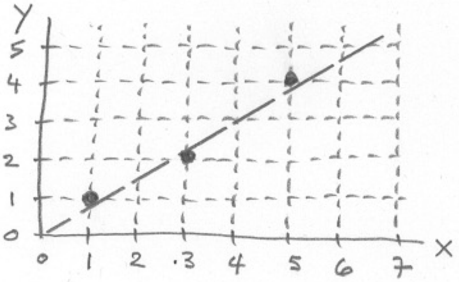

X	Y
1	1
3	2
5	4

X: constant
Y: observation

$n = 3$ total observations
$-\ n_0 = 2$ minimum observations to fix model
$r = 1$ redundancy, extra observations

For the method of indirect observations you must carry n_0 (=2) unknowns, and you must write n (=3) condition equations.

Our equations will be of the form $y = mx + b$ (3 of them), and we must add a correction (residual) to each observation,

$$\begin{aligned} y_1 + v_1 &= m x_1 + b \\ y_2 + v_2 &= m x_2 + b \\ y_3 + v_3 &= m x_3 + b \end{aligned} \Rightarrow \begin{aligned} v_1 &= m x_1 + b - y_1 \\ v_2 &= m x_2 + b - y_2 \\ v_3 &= m x_3 + b - y_3 \end{aligned} \Rightarrow \overset{(*)}{\begin{aligned} v_1 &= m \cdot 1 + b - 1 \\ v_2 &= m \cdot 3 + b - 2 \\ v_3 &= m \cdot 5 + b - 4 \end{aligned}}$$

We must minimize the Least Squares Objective Function $\Phi = \sum_{i=1}^{n} v_i^2$

In this case, $\Phi = v_1^2 + v_2^2 + v_3^2$. Plug in expressions for v_i, differentiate with respect to m, b the two unknowns, set equal to zero, and solve the normal equations.

$$\Phi = v_1^2 + v_2^2 + v_3^2 = (m + b - 1)^2 + (3m + b - 2)^2 + (5m + b - 4)^2 \rightarrow \text{minimum}$$

$$\frac{\partial \Phi}{\partial m} = \cancel{2}(m + b - 1) + \cancel{2}(3m + b - 2)3 + \cancel{2}(5m + b - 4)5 = 0$$

$$\frac{\partial \Phi}{\partial b} = \cancel{2}(m + b - 1) + \cancel{2}(3m + b - 2) + \cancel{2}(5m + b - 4) = 0$$

cancel the 2's &, collect terms,

$$\begin{aligned} 35m + 9b &= 27 \\ 9m + 3b &= 7 \end{aligned} \Rightarrow \begin{bmatrix} 35 & 9 \\ 9 & 3 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 27 \\ 7 \end{bmatrix} \Rightarrow$$

Normal Equations (Symmetric)

Solve with matlab
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N = [35 9; 9 3];
t = [27; 7];
Δ = inv(N) * t
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$$\begin{bmatrix} \hat{m} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} 35 & 9 \\ 9 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 27 \\ 7 \end{bmatrix} = \begin{bmatrix} 0.7500 \\ 0.0833 \end{bmatrix}$$

Plug into (*) to determine the v's

$$\begin{aligned} v_1 &= .7500 \times 1 + .0833 - 1 = -.1667 \\ v_2 &= .7500 \times 3 + .0833 - 2 = .3333 \\ v_3 &= .7500 \times 5 + .0833 - 4 = -.1667 \end{aligned}$$

$\Rightarrow$ corrected observations

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \end{bmatrix} = \begin{bmatrix} y_1 + v_1 \\ y_2 + v_2 \\ y_3 + v_3 \end{bmatrix} = \begin{bmatrix} 0.8333 \\ 2.3333 \\ 3.8333 \end{bmatrix}$$