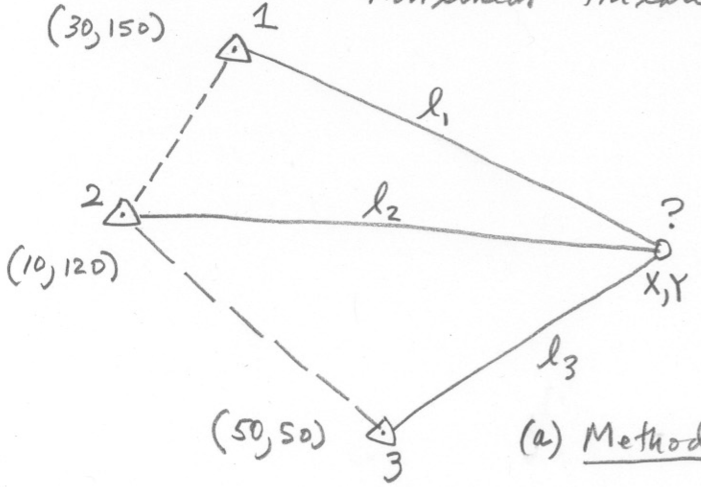
Homework 5 Solutions
Nonlinear Trilateration Problem

1-Oct-02



#	l	σ
1	125.0	0.5
2	133.5	0.2
3	98.6	0.2

choose $\sigma_0 = 0.5$

$w_1 = 1$

$w_2 = w_3 = 6.25$

$w_i = \sigma_0^2 / \sigma_i^2$

$n = 3$

$n_0 = 2$

$r = 1$

(a) Method of Indirect Observations

Select $n_0 = u = 2$ parameters : X,Y of unknown point

write one equation per observation in terms of those parameters

$$l_i = \left[(x-x_i)^2 + (y-y_i)^2\right]^{1/2} \quad , \text{ or } \quad F_i = l_i - \left[(x-x_i)^2 + (y-y_i)^2\right]^{1/2} = 0$$

get initial approximations graphically (intersection of 3 circles) $x^0, y^0 = (140, 90)$

evaluate

$$B = \begin{bmatrix} \partial F_1/\partial x & \partial F_1/\partial y \\ \partial F_2/\partial x & \partial F_2/\partial y \\ \partial F_3/\partial x & \partial F_3/\partial y \end{bmatrix}_{(x^0, y^0)} , \quad f = \begin{bmatrix} -F_1 \\ -F_2 \\ -F_3 \end{bmatrix}_{(x^0, y^0)}$$

for first iteration,

$$B = \begin{bmatrix} -.8779 & .4789 \\ -.9744 & .2249 \\ -.9138 & -.4061 \end{bmatrix} , \quad f = \begin{bmatrix} .2996 \\ -.0834 \\ -.1114 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} .0662 \\ .1739 \end{bmatrix} , \quad \begin{bmatrix} x^0 \\ y^0 \end{bmatrix}_{updated} = \begin{bmatrix} 140.0662 \\ 90.1739 \end{bmatrix}$$

do again ....

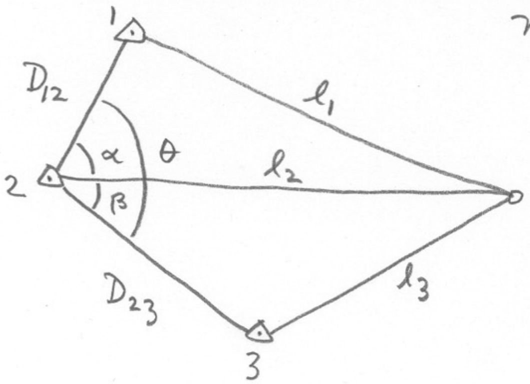
after 3 iterations $\Delta = \begin{bmatrix} .0669 \\ -.1682 \end{bmatrix} \times 10^{-7}$, $\dfrac{v^TWv_{new} - v^TWv_{old}}{v^TWv_{old}} < 10^{-6}$

final result

$$\begin{bmatrix} \hat{X} \\ \hat{Y} \end{bmatrix} = \begin{bmatrix} 140.0660 \\ 90.1739 \end{bmatrix} , \quad v = \begin{bmatrix} .2745 \\ -.0580 \\ .0197 \end{bmatrix} , \quad \hat{l} = \begin{bmatrix} 125.2745 \\ 133.4420 \\ 98.6197 \end{bmatrix}$$

See files trilati.m, trilati.lst for matlab code + results.

(b) method of observations only



note: analysis of probem is the same
we need $r = c = 1$ condition equation among the observations. D_{12}, D_{23}, θ are computed from the known control points.

use law of cosines to write one equation per triangle,

1. $l_1^2 = l_2^2 + D_{12}^2 - 2 l_2 D_{12} \cos\alpha$
2. $l_3^2 = l_2^2 + D_{23}^2 - 2 l_2 D_{23} \cos\beta$

but $\beta = \theta - \alpha$, so rewrite the second equation

3. $l_3^2 = l_2^2 + D_{23}^2 - 2 l_2 D_{23} \cos(\theta - \alpha)$

now, solve equation 1 for α, substitute into equation 3.

$$l_1^2 - l_2^2 - D_{12}^2 = -2 l_2 D_{12} \cos\alpha$$

$$\cos\alpha = \frac{-l_1^2 + l_2^2 + D_{12}^2}{2 l_2 D_{12}}$$

4. $\alpha = \cos^{-1}\left[\dfrac{-l_1^2 + l_2^2 + D_{12}^2}{2 l_2 D_{12}}\right]$, now substitute into equation 3

5. $l_3^2 = l_2^2 + D_{23}^2 - 2 l_2 D_{23} \cos\left[\theta - \cos^{-1}\left(\dfrac{-l_1^2 + l_2^2 + D_{12}^2}{2 l_2 D_{12}}\right)\right]$, or rewriting it

6. $\boxed{F(l) = l_3^2 - l_2^2 - D_{23}^2 + 2 l_2 D_{23} \cos\left[\theta - \cos^{-1}\left(\dfrac{l_1^2 + l_2^2 + D_{12}^2}{2 l_2 D_{12}}\right)\right] = 0}$

that is the (nonlinear) equation that we need, now $\partial F / \partial l_i$

Use chain rule : $\dfrac{d}{dx} f(g(h(k(u)))) = \dfrac{df}{dg} \cdot \dfrac{dg}{dh} \cdot \dfrac{dh}{dk} \cdot \dfrac{dk}{du} \cdot \dfrac{du}{dx}$, etc.

parse the equation into components to apply chain rule,

7. $F(l) = \underbrace{l_3^2 - l_2^2 - D_{23}^2}_{f_1} + \underbrace{2 l_2 D_{23}}_{f_2} \cos\left[\theta - \cos^{-1}\left(\dfrac{l_1^2 + l_2^2 + D_{12}^2}{2 l_2 D_{12}}\right)\right]$

f_3, f_4, f_5, f_6, f_7

8. $F(\ell) = f_1 + f_2 \cdot f_3$

9. $f_3 = \cos(f_4)$

10. $f_4 = \theta - \cos^{-1}(f_5)$

11. $f_5 = f_6 / f_7$

12. $f_1 = \ell_3^2 - \ell_2^2 - D_{23}^2$

13. $f_2 = 2\,\ell_2 D_{23}$

14. $f_6 = \ell_1^2 + \ell_2^2 + D_{12}^2$

15. $f_7 = 2\,\ell_2 D_{12}$

Start at the top level and work down,

16. $$\frac{\partial F}{\partial \ell_i} = \frac{\partial f_1}{\partial \ell_i} + f_2 \frac{df_3}{\partial \ell_i} + \frac{\partial f_2}{\partial \ell_i} \cdot f_3$$

17. $$\frac{\partial f_3}{\partial \ell_i} = \frac{\partial f_3}{\partial f_4} \cdot \frac{\partial f_4}{\partial f_5} \cdot \frac{\partial f_5}{\partial \ell_i}$$

18. $$\frac{\partial f_3}{\partial f_4} = -\sin(f_4)$$

19. $$\frac{\partial f_4}{\partial f_5} = \frac{1}{\sqrt{1-(f_5)^2}}$$, inverse cosine

20. $$\frac{\partial f_5}{\partial \ell_i} = \frac{f_7 \cdot \partial f_6 / \partial \ell_i - f_6 \cdot \partial f_7 / \partial \ell_i}{(f_7)^2}$$, quotient rule

$\partial f_1 / \partial \ell_1 = 0$

$\partial f_1 / \partial \ell_2 = -2\ell_2$

$\partial f_1 / \partial \ell_3 = 2\ell_3$

$\partial f_6 / \partial \ell_1 = 2\ell_1$

$\partial f_6 / \partial \ell_2 = 2\ell_2$

$\partial f_6 / \partial \ell_3 = 0$

$\partial f_2 / \partial \ell_1 = 0$

$\partial f_2 / \partial \ell_2 = 2D_{23}$

$\partial f_2 / \partial \ell_3 = 0$

$\partial f_7 / \partial \ell_1 = 0$

$\partial f_7 / \partial \ell_2 = 2D_{12}$

$\partial f_7 / \partial \ell_3 = 0$

There, now we have all raw materials to evaluate equation 16 $\partial F / \partial \ell_i$ for each of the 3 observations:

$$A = \left[\frac{\partial F}{\partial \ell_1} \quad \frac{\partial F}{\partial \ell_2} \quad \frac{\partial F}{\partial \ell_3} \right] \Bigg|_{\ell^0}$$

for iteration 1, $\ell^\circ = \ell$

evaluate $f = -F(\ell)\big|_{\ell^\circ} - A\big|_{\ell^\circ}(\ell - \ell^\circ)$

for iteration 1,

$$A = \begin{bmatrix} 444.9608 & -587.4202 & 197.2000 \end{bmatrix}$$

$$f = 159.3191$$

solve $\begin{bmatrix} -W & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$ (full normal equations)

OR

$K = (AQA^T)^{-1} f$

$V = QA^T K$

where $Q = W^{-1}$

(reduced normal equations)

$$l_{o\,(new)} = l + v$$

do again, linearizing + re-evaluating A & f at $l_{o\,(new)}$

until $\dfrac{v^TWv_{new} - v^TWv_{old}}{v^TWv_{old}}$ is small $(< 10^{-6})$

we converge after 3 iterations,

$$V = \begin{bmatrix} .2745 \\ -.0580 \\ .0197 \end{bmatrix}, \quad \hat{\ell} = \begin{bmatrix} 125.2745 \\ 133.4420 \\ 98.6197 \end{bmatrix}$$

(same as indirect observation solution!)

See files trilat0.m, trilat0.lst for matlab code + results.