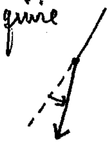
want to rotate about an arbitrary axis by ϕ = axis-angle rotation

(source: wolfram.com)

motivation

Z sys. errors will require



deflection angle

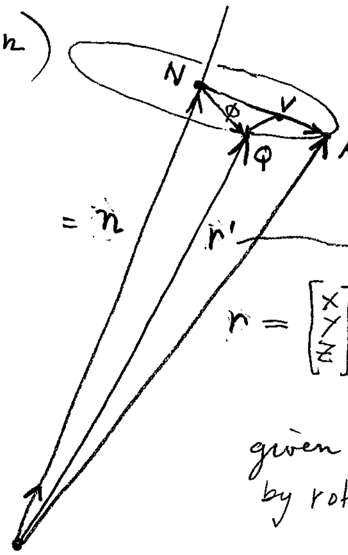
easy way to do

1. est. plane (N vect)
2. rotate by small ang

upper case = points

lower case = vectors (hat = unit vector)

(circle $\perp$ n)



$(r\cdot\hat{n})\hat{n} = n$

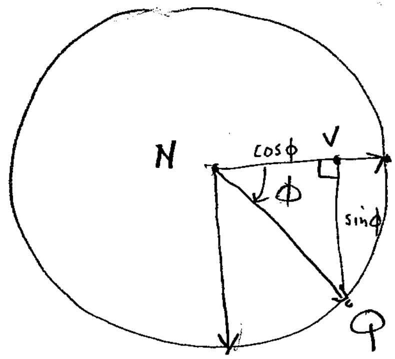
$r' \to \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$

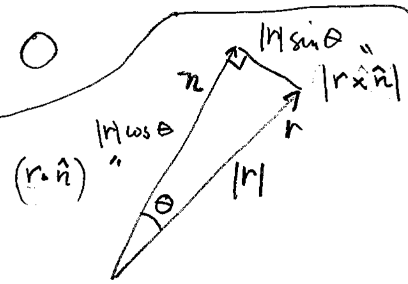
$r = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

given r, derive r' by rotation matrix mult.

$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \hat{n}$

O





$a\cdot b = |a||b|\cos\theta$

$a\times b = |a||b|\sin\theta\cdot\vec{v}_{RHR}$

$$r' = \overrightarrow{ON} + \overrightarrow{NV} + \overrightarrow{VQ}$$

$$r' = (r\cdot\hat{n})\hat{n} + \left[r - (r\cdot\hat{n})\hat{n}\right]\cos\phi + (r\times\hat{n})\sin\phi$$

$$r' = (r\cdot\hat{n})\hat{n} + r\cos\phi - (r\cdot\hat{n})\hat{n}\cos\phi + (r\times\hat{n})\sin\phi$$

$$r' = r\cos\phi + (1-\cos\phi)(r\cdot\hat{n})\hat{n} + (r\times\hat{n})\sin\phi$$

$r = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad \hat{n} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \cos\phi \begin{bmatrix} x \\ y \\ z \end{bmatrix} + (1-\cos\phi)(\alpha x + \beta y + \gamma z)\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} + \begin{vmatrix} i & j & k \\ x & y & z \\ \alpha & \beta & \gamma \end{vmatrix} \sin\phi$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \cos\phi\cdot I_3 \begin{bmatrix} x \\ y \\ z \end{bmatrix} + (1-\cos\phi)\begin{bmatrix} \alpha^2 x + \alpha\beta y + \alpha\gamma z \\ \alpha\beta x + \beta^2 y + \beta\gamma z \\ \alpha\gamma x + \beta\gamma y + \gamma^2 z \end{bmatrix} + \begin{bmatrix} y\gamma - z\beta \\ -(x\gamma - z\alpha) \\ x\beta - y\alpha \end{bmatrix}\sin\phi$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & 0 \\ 0 & \cos\phi & 0 \\ 0 & 0 & \cos\phi \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + (1-\cos\phi)\begin{bmatrix} \alpha^2 & \alpha\beta & \alpha\gamma \\ \alpha\beta & \beta^2 & \beta\gamma \\ \alpha\gamma & \beta\gamma & \gamma^2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \sin\phi\begin{bmatrix} 0 & \gamma & -\beta \\ -\gamma & 0 & \alpha \\ \beta & -\alpha & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \underbrace{\left[\begin{bmatrix} \cos\phi & 0 & 0 \\ 0 & \cos\phi & 0 \\ 0 & 0 & \cos\phi \end{bmatrix} + \begin{bmatrix} \alpha^2 & \alpha\beta & \alpha\gamma \\ \alpha\beta & \beta^2 & \beta\gamma \\ \alpha\gamma & \beta\gamma & \gamma^2 \end{bmatrix} (1-\cos\phi) + \begin{bmatrix} 0 & \gamma & -\beta \\ -\gamma & 0 & \alpha \\ \beta & -\alpha & 0 \end{bmatrix} \sin\phi \right]}_{\text{rotation matrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{bmatrix} \cos\phi + \alpha^2(1-\cos\phi) + 0 & 0 + \alpha\beta(1-\cos\phi) + \gamma\sin\phi & 0 + \alpha\gamma(1-\cos\phi) - \beta\sin\phi \\ 0 + \alpha\beta(1-\cos\phi) - \gamma\sin\phi & \cos\phi + \beta^2(1-\cos\phi) + 0 & 0 + \beta\gamma(1-\cos\phi) + \alpha\sin\phi \\ 0 + \alpha\gamma(1-\cos\phi) + \beta\sin\phi & 0 + \beta\gamma(1-\cos\phi) - \alpha\sin\phi & \cos\phi + \gamma^2(1-\cos\phi) + 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{bmatrix} \alpha^2(1-\cos\phi) + \cos\phi & \alpha\beta(1-\cos\phi) + \gamma\sin\phi & \alpha\gamma(1-\cos\phi) - \beta\sin\phi \\ \alpha\beta(1-\cos\phi) - \gamma\sin\phi & \beta^2(1-\cos\phi) + \cos\phi & \beta\gamma(1-\cos\phi) + \alpha\sin\phi \\ \alpha\gamma(1-\cos\phi) + \beta\sin\phi & \beta\gamma(1-\cos\phi) - \alpha\sin\phi & \gamma^2(1-\cos\phi) + \cos\phi \end{bmatrix}$$

go back, rotate object $-\phi$ by RHR
rotate coord. sys. $+\phi$ by RHR

---

if we want rotate object $+\phi$ by RHR
rotate coord sys. $-\phi$ by RHR
} all $\sin\phi$ become $-\sin\phi$
$\cos\phi$ <u>same</u>

$$\begin{bmatrix} \alpha^2(1-\cos\phi) + \cos\phi & \alpha\beta(1-\cos\phi) - \gamma\sin\phi & \alpha\gamma(1-\cos\phi) + \beta\sin\phi \\ \alpha\beta(1-\cos\phi) + \gamma\sin\phi & \beta^2(1-\cos\phi) + \cos\phi & \beta\gamma(1-\cos\phi) - \alpha\sin\phi \\ \alpha\gamma(1-\cos\phi) - \beta\sin\phi & \beta\gamma(1-\cos\phi) + \alpha\sin\phi & \gamma^2(1-\cos\phi) + \cos\phi \end{bmatrix}$$