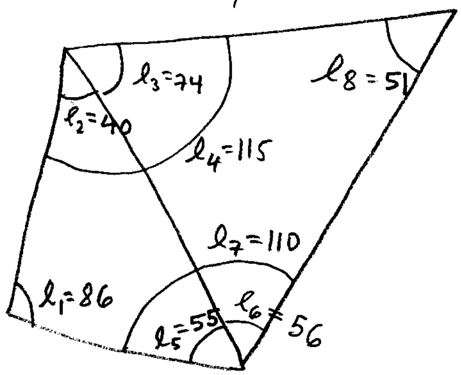
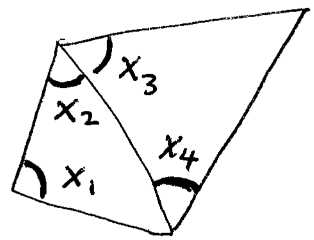
Data 1, Fall 2009 Homework 1 - Solution

1.



$n = 8$
$n_0 = 4$
$r = 4$

choose $n_0 = 4$ parameters for the solution by <u>indirect observations</u>:



$$\begin{aligned}
l_1+v_1 &= x_1 \\
l_2+v_2 &= x_2 \\
l_3+v_3 &= x_3 \\
l_4+v_4 &= x_2+x_3 \\
l_5+v_5 &= 180-x_1-x_2 \\
l_6+v_6 &= x_4 \\
l_7+v_7 &= 180-x_1-x_2+x_4 \\
l_8+v_8 &= 180-x_3-x_4
\end{aligned}$$

$\circledast$

$$\begin{aligned}
v_1 &= x_1-86 \\
v_2 &= x_2-40 \\
v_3 &= x_3-74 \\
v_4 &= x_2+x_3-115 \\
v_5 &= 180-x_1-x_2-55 = 125-x_1-x_2 \\
v_6 &= x_4-56 \\
v_7 &= 180-x_1-x_2+x_4-110 = 70-x_1-x_2+x_4 \\
v_8 &= 180-x_3-x_4-51 = 129-x_3-x_4
\end{aligned}$$

$$\Phi = (x_1-86)^2 + (x_2-40)^2 + (x_3-74)^2 + (x_2+x_3-115)^2 + (125-x_1-x_2)^2 + (x_4-56)^2 + (70-x_1-x_2+x_4)^2 + (129-x_3-x_4)^2$$

$$\partial\phi/\partial x_1 = 2(x_1-86) + 2(125-x_1-x_2)(-1) + 2(70-x_1-x_2+x_4)(-1) = 0$$

$$\partial\phi/\partial x_2 = 2(x_2-40) + 2(x_2+x_3-115) + 2(125-x_1-x_2)(-1) + 2(70-x_1-x_2+x_4)(-1) = 0$$

$$\partial\phi/\partial x_3 = 2(x_3-74) + 2(x_2+x_3-115) + 2(129-x_3-x_4)(-1) = 0$$

$$\partial\phi/\partial x_4 = 2(x_4-56) + 2(70-x_1-x_2+x_4) + 2(129-x_3-x_4)(-1) = 0$$

$$\begin{aligned}
x_1+x_1+x_2+x_1+x_2-x_4 &= 86+125+70 \\
x_2+x_2+x_3+x_1+x_2+x_1+x_2-x_4 &= 40+115+125+70 \\
x_3+x_2+x_3+x_3+x_4 &= 74+115+129 \\
x_4-x_1-x_2+x_4+x_3+x_4 &= 56-70+129
\end{aligned}$$

$$\begin{bmatrix} 3 & 2 & 0 & -1 \\ 2 & 4 & 1 & -1 \\ 0 & 1 & 3 & 1 \\ -1 & -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 281 \\ 350 \\ 318 \\ 115 \end{bmatrix}$$

Solve by Matlab ...

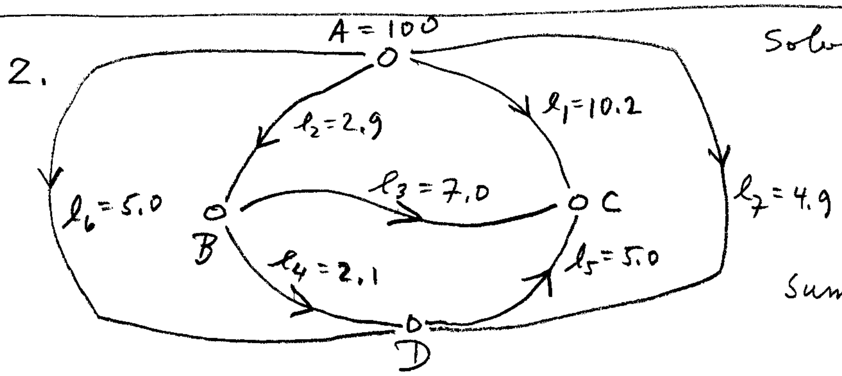
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 85.4000 \\ 40.1333 \\ 74.1333 \\ 55.4667 \end{bmatrix}$$

plug values for x into ⊛ to obtain residuals

$\hat{\ell} = \ell + v$:

$v_1 = -0.6000$	$\hat{\ell}_1 = 85.4$
$v_2 = 0.1333$	$\hat{\ell}_2 = 40.1333$
$v_3 = 0.1333$	$\hat{\ell}_3 = 74.1333$
$v_4 = -0.7333$	$\hat{\ell}_4 = 114.2667$
$v_5 = -0.5333$	$\hat{\ell}_5 = 54.4667$
$v_6 = -0.5333$	$\hat{\ell}_6 = 55.4667$
$v_7 = -0.0667$	$\hat{\ell}_7 = 109.9333$
$v_8 = -0.6000$	$\hat{\ell}_8 = 50.4$

2.



Solve by <u>observations only</u>

$n = 7$
$n_0 = 3$
$r = 4 \implies c = 4$

Sum around 4 independent loops:

$\hat{\ell}_1 - \hat{\ell}_3 - \hat{\ell}_2 = 0$	$v_1 - v_3 - v_2 = -\ell_1 + \ell_3 + \ell_2$	$= -10.2 + 7.0 + 2.9 = -0.3$
$\hat{\ell}_3 - \hat{\ell}_5 - \hat{\ell}_4 = 0$	$v_3 - v_5 - v_4 = -\ell_3 + \ell_5 + \ell_4$	$= -7.0 + 5.0 + 2.1 = 0.1$
$\hat{\ell}_7 + \hat{\ell}_5 - \hat{\ell}_1 = 0$	$v_7 + v_5 - v_1 = -\ell_7 - \ell_5 + \ell_1$	$= -4.9 - 5.0 + 10.2 = 0.3$
$\hat{\ell}_2 + \hat{\ell}_4 - \hat{\ell}_6 = 0$	$v_2 + v_4 - v_6 = -\ell_2 - \ell_4 + \ell_6$	$= -2.9 - 2.1 + 5.0 = 0$

Solve first by <u>substitution</u>: Keep $\boxed{1, 2, 4}$, express $\boxed{3, 5, 6, 7}$ in terms of 1, 2, 4 via the condition equations:

$v_3 = v_1 - v_2 + 0.3$

$v_5 = v_3 - v_4 - 0.1$ (subs again for v_3)

$\quad = v_1 - v_2 + 0.3 - v_4 - 0.1$

$\quad = v_1 - v_2 - v_4 + 0.2$

$v_6 = v_2 + v_4$

$v_7 = v_1 - v_5 + 0.3$

$\quad = v_1 - v_1 + v_2 + v_4 - 0.2 + 0.3$

$\quad = v_2 + v_4 + 0.1$

$$\left.\begin{aligned} v_3 &= v_1 - v_2 + 0.3 \\ v_5 &= v_1 - v_2 - v_4 + 0.2 \\ v_6 &= v_2 + v_4 \\ v_7 &= v_2 + v_4 + 0.1 \end{aligned}\right\} \quad ⊛$$

now express objective function in terms of $n_0 = 3$ of the v's:

$$\phi = V_1^2 + V_2^2 + V_3^2 + V_4^2 + V_5^2 + V_6^2 + V_7^2$$

$$\phi = V_1^2 + V_2^2 + (V_1 - V_2 + 0.3)^2 + V_4^2 + (V_1 - V_2 - V_4 + 0.2)^2 + (V_2 + V_4)^2 + (V_2 + V_4 + 0.1)^2$$

now differentiate with respect to V_1, V_2, V_3 & set $= 0$:

$$\frac{\partial\phi}{\partial V_1} = \cancel{2}V_1 + \cancel{2}(V_1 - V_2 + 0.3) + \cancel{2}(V_1 - V_2 - V_4 + 0.2) = 0$$

$$\frac{\partial\phi}{\partial V_2} = \cancel{2}V_2 + \cancel{2}(V_1 - V_2 + 0.3)(-1) + \cancel{2}(V_1 - V_2 - V_4 + 0.2)(-1) + \cancel{2}(V_2 + V_4) + \cancel{2}(V_2 + V_4 + 0.1) = 0$$

$$\frac{\partial\phi}{\partial V_4} = \cancel{2}V_4 + \cancel{2}(V_1 - V_2 - V_4 + 0.2)(-1) + \cancel{2}(V_2 + V_4) + \cancel{2}(V_2 + V_4 + 0.1) = 0$$

collect terms:

$$\begin{aligned} 3V_1 - 2V_2 - V_4 &= -0.5 \\ -2V_1 + 5V_2 + 3V_4 &= 0.4 \\ -V_1 + 3V_2 + 4V_4 &= 0.1 \end{aligned} \Rightarrow \begin{bmatrix} 3 & -2 & -1 \\ -2 & 5 & 3 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_4 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.4 \\ 0.1 \end{bmatrix}$$

solve linear equations by Matlab:

$$\begin{bmatrix} V_1 \\ V_2 \\ V_4 \end{bmatrix} = \begin{bmatrix} -0.15 \\ 0.05 \\ -0.05 \end{bmatrix}$$

use equations ⊛ to solve for the other V's $\Rightarrow$

$$V = \begin{bmatrix} -0.15 \\ 0.05 \\ 0.10 \\ -0.05 \\ 0.05 \\ 0 \\ 0.1 \end{bmatrix} \qquad \hat{\ell} = \begin{bmatrix} 10.05 \\ 2.95 \\ 7.10 \\ 2.05 \\ 5.05 \\ 5.00 \\ 5.00 \end{bmatrix}$$

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now solve by Lagrange Multipliers:

$$\phi' = V_1^2 + V_2^2 + V_3^2 + V_4^2 + V_5^2 + V_6^2 + V_7^2 + 2\lambda_1(V_1 - V_3 - V_2 + 0.3) + 2\lambda_2(V_3 - V_5 - V_4 - 0.1) + 2\lambda_3(V_7 + V_5 - V_1 - 0.3) + 2\lambda_4(V_2 + V_4 - V_6)$$

$$\begin{aligned}
\partial\phi'/\partial V_1 &= 2V_1 + 2\lambda_1 - 2\lambda_3 = 0 \\
\partial\phi'/\partial V_2 &= 2V_2 - 2\lambda_1 + 2\lambda_4 = 0 \\
\partial\phi'/\partial V_3 &= 2V_3 - 2\lambda_1 + 2\lambda_2 = 0 \\
\partial\phi'/\partial V_4 &= 2V_4 - 2\lambda_2 + 2\lambda_4 = 0 \\
\partial\phi'/\partial V_5 &= 2V_5 - 2\lambda_2 + 2\lambda_3 = 0 \\
\partial\phi'/\partial V_6 &= 2V_6 - 2\lambda_4 = 0 \\
\partial\phi'/\partial V_7 &= 2V_7 + 2\lambda_3 = 0
\end{aligned}$$

$$\begin{aligned}
\partial\phi'/\partial\lambda_1 &: \quad V_1 - V_3 - V_2 = -0.3 \\
\partial\phi'/\partial\lambda_2 &: \quad V_3 - V_5 - V_4 = 0.1 \\
\partial\phi'/\partial\lambda_3 &: \quad V_7 + V_5 - V_1 = 0.3 \\
\partial\phi'/\partial\lambda_4 &: \quad V_2 + V_4 - V_6 = 0
\end{aligned}$$

now extract coefficients for matrix representation:

full normal equations:

$$
\begin{array}{c}
\begin{array}{ccccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{array} \\
\left[\begin{array}{ccccccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
\left[\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ \hline \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{array}\right]
=
\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline -0.3 \\ 0.1 \\ 0.3 \\ 0 \end{array}\right]
$$

Solution by Matlab: $\big[\underbrace{-.15;\ .05;\ .10;\ -.05;\ .05;\ 0;\ .10}_{v\text{'s}}\ \vdots\ \underbrace{.05;\ -.05;\ -.10;\ 0}_{\lambda\text{'s}}\big]$

same as before! ✓

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solve separately for k & v: (partitioned, block elimination)

⊛ $v_1 = -\lambda_1 + \lambda_3$ $v_2 = \lambda_1 - \lambda_4$ $v_3 = \lambda_1 - \lambda_2$ $v_4 = \lambda_2 - \lambda_4$ $v_5 = \lambda_2 - \lambda_3$ $v_6 = \lambda_4$ $v_7 = -\lambda_3$	subs. these expressions into the condition equations	$-\lambda_1+\lambda_3-\lambda_1+\lambda_2-\lambda_1+\lambda_4 = -0.3$ $\lambda_1-\lambda_2-\lambda_2+\lambda_3-\lambda_2+\lambda_4 = 0.1$ $-\lambda_3+\lambda_2-\lambda_3+\lambda_1-\lambda_3 = 0.3$ $\lambda_1-\lambda_4+\lambda_2-\lambda_4-\lambda_4 = 0$	$-3\lambda_1+\lambda_2+\lambda_3+\lambda_4 = -0.3$ $\lambda_1-3\lambda_2+\lambda_3+\lambda_4 = 0.1$ $\lambda_1+\lambda_2-3\lambda_3 = 0.3$ $\lambda_1+\lambda_2-3\lambda_4 = 0$

$$
\begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 0 \\ 1 & 1 & 0 & -3 \end{bmatrix}
\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix}
=
\begin{bmatrix} -0.3 \\ 0.1 \\ 0.3 \\ 0 \end{bmatrix}, \quad
\lambda = \begin{bmatrix} 0.05 \\ -0.05 \\ -0.10 \\ 0 \end{bmatrix}
$$

plug these values into ⊛ and obtain:

$$V = [-.15;\ .05;\ .10;\ -.05;\ .05;\ 0;\ .10]$$

all solutions agree! ✓