Error Propagation Law $\vec{x}, \Sigma_{xx}$ 21-1

$\underline{\Sigma_{yy} = ?}$ $\exists\ \vec{y} = A\vec{x}$ A: matrix of constants

$$\underbrace{E(y)}_{\to \mu_y} = E(Ax) = A\underbrace{E(x)}_{} = A\mu_x$$

$$\Sigma_{yy} = E\{(\underset{\substack{\uparrow\\ Ax}}{y} - \underset{\substack{\uparrow\\ A\mu_x}}{\mu_y})(y-\mu_y)^T\}$$ defn. cov. mx.

Oct 12-9:24 AM

21-2

$$\Sigma_{yy} = E\{(Ax - A\mu_x)(Ax - A\mu_x)^T\}$$

$(x^T A^T - \mu_x^T A^T)$

$$\Sigma_{yy} = E\{A(x-\mu_x)\underline{(x^T - \mu_x^T)}A^T\}$$

$(x-\mu_x)^T$

$$\Sigma_{yy} = A\,\underbrace{E\{(x-\mu_x)(x-\mu_x)^T\}}\,A^T$$

$$\boxed{\Sigma_{yy} = A\Sigma_{xx}A^T}$$ $y = Ax,\ \Sigma_{xx}$

Oct 12-9:24 AM

examples 21-3

$$y = a_1 X_1 + a_2 X_2 \qquad \Sigma_{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} \sigma_{x_1}^2 & 0 \\ 0 & \sigma_{x_2}^2 \end{pmatrix}$$

$$y = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\sigma_y^2 = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{pmatrix} \sigma_{x_1}^2 & 0 \\ 0 & \sigma_{x_2}^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\sigma_y^2 = a_1^2 \sigma_{x_1}^2 + a_2^2 \sigma_{x_2}^2 \checkmark$$

Oct 12-9:24 AM

$$\boxed{y = a_1 x_1 + a_2 x_2} \qquad \Sigma_{xx} = \begin{pmatrix} \sigma_{x_1}^2 & \sigma_{x_1 x_2} \\ \sigma_{x_2 x_1} & \sigma_{x_2}^2 \end{pmatrix} \qquad 21\text{-}4$$

$$y = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\sigma_y^2 = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} \sigma_{x_1}^2 & \sigma_{x_1 x_2} \\ \sigma_{x_2 x_1} & \sigma_{x_2}^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\sigma_y^2 = a_1^2 \sigma_{x_1}^2 + \underline{2 a_1 a_2 \sigma_{x_1 x_2}} + a_2^2 \sigma_{x_2}^2$$

Oct 12-9:24 AM

21-5

$y_1 = x_1 + 2x_2$

$y_2 = 2x_1 + x_2$

$\Sigma_{xx} = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$

$z \leftarrow y \leftarrow x$

$z = \underline{y_1} + 2\,\underline{y_2}$

$z \leftarrow x$

$z = x_1 + 2x_2 + 2(2x_1 + x_2)$ 1-step

$z = 5x_1 + 4x_2$

$z = \begin{bmatrix} 5 & 4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\sigma_z^2 = \begin{bmatrix} 5 & 4 \end{bmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 204$

$\sigma_z = \sqrt{204}$

Oct 12-9:24 AM

21-6

$y_1 = x_1 + 2x_2$

$y_2 = 2x_1 + x_2$

$\Sigma_{xx} = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ $\Sigma_{yy} = A\Sigma_{xx}A^T$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\Sigma_{yy} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 24 & 21 \\ 21 & 24 \end{pmatrix}}}$

Two-Step

$z = y_1 + 2y_2$

$z = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\Sigma_{zz} = \sigma_z^2 = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{pmatrix} 24 & 21 \\ 21 & 24 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 204$

$\sigma_z^2 = 204$, $\sigma_z = \sqrt{204}$ 1 step & 2 step yield SAME RESULT

Oct 12-9:24 AM

EI for I/O LS method $\Sigma_{\delta\delta} = ?$ 21-7

$$\underline{\delta = (B^T W B)^{-1} B^T W f} \Longrightarrow \delta = \square\, l,\ \Sigma_{ll}$$

$$f = d - l = d - I l$$

$$\boxed{\Sigma = \sigma_0^2 Q}$$

$$f = -\underline{I l} + d$$

$$Q_{ff} = (-I) Q_{ll} (-I)^T \qquad y = Ax,\ \underline{y = Ax + b}$$

$$Q_{ff} = Q_{ll} \quad (Q = Q_{ll}) \qquad \underline{\delta} = \underline{\square} \cdot \underline{f}$$

Oct 12-9:24 AM

$$\delta = \underbrace{(B^T W B)^{-1} B^T W}\, f \qquad \underline{Q_{ff} = Q} \qquad WQ = I \quad 21\text{-}8$$

$$Q_{\delta\delta} = (B^T W B)^{-1} B^T \underline{W \cdot Q} \cdot \left[(B^T W B)^{-1} B^T W\right]^T$$

$$\underbrace{\qquad WB}\underbrace{(B^T W B)^{-1}}_{I}$$

$$Q_{\delta\delta} = (B^T W B)^{-1} = N^{-1}$$

$$N\delta = t, \quad \delta = N^{-1} t$$

if $\sigma_0^2 = \underline{1}$

$$\boxed{Q_{\delta\delta} = N^{-1}}, \quad \boxed{\Sigma_{\delta\delta} = \sigma_0^2 Q_{\delta\delta}}$$

then $\Sigma_{\delta\delta} = Q_{\delta\delta}$

Oct 12-9:24 AM

21-9

$y = Ax + b$ $\qquad \Sigma_{yy} = A \Sigma_{xx} A^T$

similar for linear & nonlinear models

$y = F(x)$

$y = F^0 + \frac{dF}{dx}\delta x$

$\underline{y = y^0 + J\delta x}$ $\qquad \Sigma_{yy} = J \Sigma_{xx} J^T$

$\Sigma = \sigma_0^2 Q$ $\qquad \Sigma_{ww} = \sigma_0^2 Q_{ww}$

$\Sigma_{xx} = \sigma_0^2 Q_{xx}$ $\qquad \Sigma_{\hat{\ell}\hat{\ell}} = \sigma_0^2 Q_{\hat{\ell}\hat{\ell}}$

$\Sigma_{vv} = \sigma_0^2 Q_{vv}$

Oct 12-9:24 AM

21-10

σ_0^2 $\qquad \sigma_0^2 / \sigma_i^2 = w_i$ $\qquad$ prior post

$\boxed{\hat{\sigma}_0^2 = \frac{v^T W v}{r}}$

$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$ $\qquad \frac{r \cdot \hat{\sigma}_0^2}{\sigma_0^2} \sim \chi^2_r$

σ_0^2: a priori ref. var.

$\hat{\sigma}_0^2$: a posteriori ref var.

$\frac{r \cdot \frac{v^T W v}{r}}{\sigma_0^2} = \boxed{\frac{v^T W v}{\sigma_0^2} \sim \chi^2_r}$

Oct 12-9:24 AM

21-11

when you choose a value for σ_0^2, think of it as choosing the observation (and variance) that will be assigned $w = \underline{1}$. That choice is arbitrary. ~~After~~ After adjustment, the correction to that observation might be much <u>less</u> or much <u>greater</u> than σ_0. That indicates that our results are not consistent with assumptions. We will test this formally by comparing $\hat{\sigma}_0^2$ with σ_0^2 (post vs. prior): $r \cdot \hat{\sigma}_0^2 / \sigma_0^2 \sim \chi_r^2$

$$\hat{\sigma}_0^2 = \frac{v^T W v}{r}, \quad \boxed{\frac{v^T W v}{\sigma_0^2} \sim \chi_r^2}$$

test statistic and distribution

Oct 12-9:24 AM