HW1 Solution, Sep. 2015

1. $n = 4$ indirect observations, choose parameters a_0, a_1, a_2

$n_0 = 3$ model : $\hat{y} = a_0 + a_1 x + a_2 x^2$

$r = 1$

(Scalar Method)

$$y + v_y = a_0 + a_1 x + a_2 x^2$$

$$v_y = a_0 + a_1 x + a_2 x^2 - y_1$$

write 4 condition equations :

$$v_1 = a_0 + a_1(0.75) + a_2(0.75)^2 - 0.7$$
$$v_2 = a_0 + a_1(1.75) + a_2(1.75)^2 - 3.8$$
$$v_3 = a_0 + a_1(3.00) + a_2(3.00)^2 - 4.3$$
$$v_4 = a_0 + a_1(4.00) + a_2(4.00)^2 - 1.2$$

subsititute into objective function ;

$$\phi = v_1^2 + v_2^2 + v_3^2 + v_4^2 = \left[a_0 + a_1(0.75) + a_2(0.75)^2 - 0.7\right]^2 + \left[a_0 + a_1(1.75) + a_2(1.75)^2 - 3.8\right]^2 + \left[a_0 + a_1(3.00) + a_2(3.00)^2 - 4.3\right]^2 + \left[a_0 + a_1(4.00) + a_2(4.00)^2 - 1.2\right]^2$$

$$\frac{\partial\phi}{\partial a_0} = \cancel{2}\left[a_0 + a_1(0.75) + a_2(0.75)^2 - 0.7\right] + \cancel{2}\left[a_0 + a_1(1.75) + a_2(1.75)^2 - 3.8\right] + \cancel{2}\left[a_0 + a_1(3) + a_2(3)^2 - 4.3\right] + \cancel{2}\left[a_0 + a_1(4) + a_2(4)^2 - 1.2\right] = 0$$

$$\frac{\partial\phi}{\partial a_1} = \cancel{2}\left[a_0 + a_1(.75) + a_2(.75)^2 - .7\right](.75) + \cancel{2}\left[a_0 + a_1(1.75) + a_2(1.75)^2 - 3.8\right](1.75) + \cancel{2}\left[a_0 + a_1(3) + a_2(3)^2 - 4.3\right](3) + \cancel{2}\left[a_0 + a_1(4) + a_2(4)^2 - 1.2\right](4) = 0$$

$$\frac{\partial\phi}{\partial a_2} = \cancel{2}\left[a_0 + a_1(.75) + a_2(.75)^2 - .7\right](.75)^2 + \cancel{2}\left[a_0 + a_1(1.75) + a_2(1.75)^2 - 3.8\right](1.75)^2 + \cancel{2}\left[a_0 + a_1(3) + a_2(3)^2 - 4.3\right](3)^2 + \cancel{2}\left[a_0 + a_1(4) + a_1(4)^2 - 1.2\right](4)^2 = 0$$

normal equations, collect like terms :

$$a_0 + a_0 + a_0 + a_0 + (.75 + 1.75 + 3 + 4)\,a_1 + \left[(.75)^2 + (1.75)^2 + (3)^2 + (4)^2\right] a_2 = .7 + 3.8 + 4.3 + 1.2$$

$$(.75 + 1.75 + 3 + 4)\,a_0 + \left[(.75)^2 + (1.75)^2 + (3)^2 + (4)^2\right] a_1 + \left[(.75)^3 + (1.75)^3 + (3)^3 + (4)^3\right] a_2 = (.7)(.75) + (3.8)(1.75) + (4.3)(3) + (1.2)(4)$$

$$\left[(.75)^2 + (1.75)^2 + (3)^2 + (4)^2\right] a_0 + \left[(.75)^3 + (1.75)^3 + (3)^3 + (4)^3\right] a_1 + \left[(.75)^4 + (1.75)^4 + (3)^4 + (4)^4\right] a_2 = (.7)(.75)^2 + (3.8)(1.75)^2 + (4.3)(3)^2 + (1.2)(4)^2$$

$$4a_0 + 9.5a_1 + 28.625a_2 = 10.0$$
$$9.5a_0 + 28.625a_1 + 96.78125a_2 = 24.875$$
$$28.625a_0 + 96.78125a_1 + 346.6953125a_2 = 69.93125$$

$$\Rightarrow \begin{bmatrix} 4 & 9.5 & 28.625 \\ 9.5 & 28.625 & 96.78125 \\ 28.625 & 96.78125 & 346.6953125 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 10.0 \\ 24.875 \\ 69.93125 \end{bmatrix}$$

solve in matlab ;

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -3.6241 \\ 6.7300 \\ -1.3778 \end{bmatrix} \qquad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} -0.0515 \\ 0.1340 \\ -0.1340 \\ 0.0515 \end{bmatrix} \qquad \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \hat{y}_4 \end{bmatrix} = \begin{bmatrix} 0.6485 \\ 3.9340 \\ 4.1660 \\ 1.2515 \end{bmatrix}$$

no prior info. on uncertainty of observations

substitute into earlier equations for v_i

get adjusted observations

$$\hat{y}_i = y_i + v_i$$

2. $n = 9$ observations only
$n_0 = 4$ $\quad c = r = 5$
$\overline{r = 5}$

$$\hat{l}_1 + \hat{l}_2 + \hat{l}_3 = \hat{l}_9$$
$$\hat{l}_4 + \hat{l}_5 = \hat{l}_9$$
$$\hat{l}_1 = \hat{l}_6$$
$$\hat{l}_7 + \hat{l}_3 = \hat{l}_5$$
$$\hat{l}_4 + \hat{l}_7 = \hat{l}_8$$

$$l = \begin{bmatrix} 20 \\ 85 \\ 24 \\ 52 \\ 76 \\ 19 \\ 52 \\ 106 \\ 129 \end{bmatrix}$$

$$v_1 + v_2 + v_3 - v_9 = -(l_1 + l_2 + l_3 - l_9) = 0$$
$$v_4 + v_5 - v_9 = -(l_4 + l_5 - l_9) = 1$$
$$v_1 - v_6 = -(l_1 - l_6) = -1$$
$$v_7 + v_3 - v_5 = -(l_7 + l_3 - l_5) = 0$$
$$v_4 + v_7 - v_8 = -(l_4 + l_7 - l_8) = 2$$

note coefficients same for v's and l's, except neg. sign.

make augmented objective function using lagrange multipliers

$$\phi' = v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2 + v_6^2 + v_7^2 + v_8^2 + v_9^2 - 2k_1(v_1 + v_2 + v_3 - v_9) - 2k_2(v_4 + v_5 - v_9 - 1)$$
$$- 2k_3(v_1 - v_6 + 1) - 2k_4(v_7 + v_3 - v_5) - 2k_5(v_4 + v_7 - v_8 - 2)$$

differentiate with respect to all v's and k's and set equal to zero:

$$\frac{\partial\phi'}{\partial v_1} = \cancel{2}v_1 - \cancel{2}k_1 - \cancel{2}k_3 = 0$$
$$\frac{\partial\phi'}{\partial v_2} = \cancel{2}v_2 - \cancel{2}k_1 = 0$$
$$\frac{\partial\phi'}{\partial v_3} = \cancel{2}v_3 - \cancel{2}k_1 - \cancel{2}k_4 = 0$$
$$\frac{\partial\phi'}{\partial v_4} = \cancel{2}v_4 - \cancel{2}k_2 - \cancel{2}k_5 = 0$$
$$\frac{\partial\phi'}{\partial v_5} = \cancel{2}v_5 - \cancel{2}k_2 + \cancel{2}k_4 = 0$$
$$\frac{\partial\phi'}{\partial v_6} = \cancel{2}v_6 + \cancel{2}k_3 = 0$$
$$\frac{\partial\phi'}{\partial v_7} = \cancel{2}v_7 - \cancel{2}k_4 - \cancel{2}k_5 = 0$$
$$\frac{\partial\phi'}{\partial v_8} = \cancel{2}v_8 + \cancel{2}k_5 = 0$$
$$\frac{\partial\phi'}{\partial v_9} = \cancel{2}v_9 + \cancel{2}k_1 + \cancel{2}k_2 = 0$$

$$\frac{\partial\phi'}{\partial k_1} = -\cancel{2}v_1 - \cancel{2}v_2 - \cancel{2}v_3 + \cancel{2}v_9 = 0$$
$$\frac{\partial\phi'}{\partial k_2} = -\cancel{2}v_4 - \cancel{2}v_5 + \cancel{2}v_9 + \cancel{2}^{1} = 0$$
$$\frac{\partial\phi'}{\partial k_3} = -\cancel{2}v_1 + \cancel{2}v_6 - \cancel{2}^{1} = 0$$
$$\frac{\partial\phi'}{\partial k_4} = -\cancel{2}v_7 - \cancel{2}v_3 + \cancel{2}v_5 = 0$$
$$\frac{\partial\phi'}{\partial k_5} = -\cancel{2}v_4 - \cancel{2}v_7 + \cancel{2}v_8 + \cancel{4}^{2} = 0$$

collect coefficients into big normal equation matrix

$$\left[\begin{array}{ccccccccc|ccccc}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
\hline
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ \hline k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{array}\right]
=
\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 1 \\ -1 \\ 0 \\ 2 \end{array}\right]$$

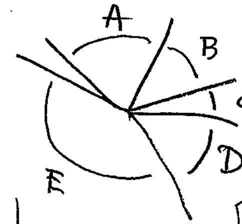
$$\left[\begin{array}{c|c} -I & A^T \\ \hline A & Z_5 \end{array}\right]\left[\begin{array}{c} V \\ \hline K \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline f \end{array}\right]$$, since $W = I$, $$\begin{bmatrix} -W & A^T \\ A & Z_5 \end{bmatrix}\begin{bmatrix} v \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

solving with matlab,

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{bmatrix} = \begin{bmatrix} -.3125 \\ .3750 \\ -.1562 \\ .6563 \\ .2500 \\ .6875 \\ .4063 \\ -.9375 \\ -.0938 \end{bmatrix} \qquad \begin{bmatrix} \hat{l}_1 \\ \hat{l}_2 \\ \hat{l}_3 \\ \hat{l}_4 \\ \hat{l}_5 \\ \hat{l}_6 \\ \hat{l}_7 \\ \hat{l}_8 \\ \hat{l}_9 \end{bmatrix} = \begin{bmatrix} 19.6875 \\ 85.3750 \\ 23.8438 \\ 52.6563 \\ 76.2500 \\ 19.6875 \\ 52.4063 \\ 105.0625 \\ 128.9063 \end{bmatrix}$$

$$\hat{l} = l + v$$

3. $n = 10$ indirect observations via matrix methods
$\underline{n_0 = 5}$ $u = n_0 = 5$ parameters A, B, C, D, E $\longrightarrow$
$r = 5$



$\hat{a}_1 = A$	$v_1 - A = -a_1$	
$\hat{a}_2 = B + C$	$v_2 - B - C = -a_2$	
$\hat{a}_3 = D$	$v_3 - D = -a_3$	
$\hat{a}_4 = E$	$v_4 - E = -a_4$	
$\hat{a}_5 = B$	$v_5 - B = -a_5$	
$\hat{a}_6 = C$	$v_6 - C = -a_6$	
$\hat{a}_7 = 360 - A - B - C$	$v_7 + A + B + C = 360 - a_7$	
$\hat{a}_8 = 360 - B - C - D - E$	$v_8 + B + C + D + E = 360 - a_8$	
$\hat{a}_9 = C + D$	$v_9 - C - D = -a_9$	
$\hat{a}_{10} = 360 - A - B - C - D - E$	$v_{10} + A + B + C + D + E = 360 - a_{10}$	

$$\Rightarrow \quad f = \begin{bmatrix} -68 \\ -67 \\ -72 \\ -142 \\ -44 \\ -24 \\ 135 \\ 279 \\ -95 \\ 348 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} -68 \\ -67 \\ -72 \\ -142 \\ -44 \\ -24 \\ 135 \\ 279 \\ -95 \\ 348 \end{bmatrix}, \qquad \sigma_0^2 = 1, \quad w_i = \sigma_0^2 / \sigma_i^2 \quad \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \\ w_{10} \end{bmatrix} = \begin{bmatrix} 1/1 \\ 1/1 \\ 1/1 \\ 1/4 \\ 1/4 \\ 1/1 \\ 1/1 \\ 1/1 \\ 1/4 \\ 1/9 \end{bmatrix}$$

$$v \; + \; B \cdot \Delta \; = \; f$$, solve via matlab

$$\Delta = (B^T W B)^{-1} B^T W f = \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 68.0737 \\ 43.2000 \\ 23.7158 \\ 71.6211 \\ 140.8211 \end{bmatrix}$$

$$v = f - B\Delta$$

$$\hat{a} = a + v$$

$$v = \begin{bmatrix} .0737 \\ -.0842 \\ -.3789 \\ -1.1789 \\ -.8000 \\ -.2842 \\ .0105 \\ -.3579 \\ .3368 \\ .5684 \end{bmatrix} \qquad \hat{a} = \begin{bmatrix} 68.0737 \\ 66.9158 \\ 71.6211 \\ 140.8211 \\ 43.2000 \\ 23.7158 \\ 225.0105 \\ 80.6421 \\ 95.3368 \\ 12.5684 \end{bmatrix}$$

residuals a little small compared to σ's.