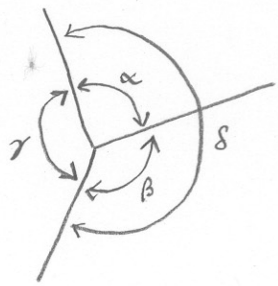
CE 506 – homework 4 solution
25 Sept. 2002

Linear Problems

4-6)



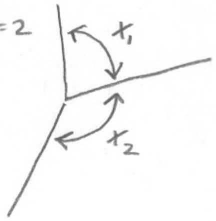
obs	value
α	110-15-20
β	130-40-08
γ	119-04-42
δ	240-55-43

$n = 4$
$n_0 = 2$
$r = 2$

$W = I$

Do it first by indirect observations, longhand method

Select $\mu = n_0 = 2$ parameters as shown



$$\left.\begin{aligned} \alpha + v_\alpha &= x_1 \\ \beta + v_\beta &= x_2 \\ \gamma + v_\gamma &= 360 - x_1 - x_2 \\ \delta + v_\delta &= x_1 + x_2 \end{aligned}\right\} \quad \begin{aligned} v_\alpha &= x_1 - \alpha \\ v_\beta &= x_2 - \beta \\ v_\gamma &= 360 - x_1 - x_2 - \gamma \\ v_\delta &= x_1 + x_2 - \delta \end{aligned}$$

$$\Phi = v_\alpha^2 + v_\beta^2 + v_\gamma^2 + v_\delta^2 \rightarrow \text{minimum}$$

$$\Phi = (x_1 - \alpha)^2 + (x_2 - \beta)^2 + (360 - x_1 - x_2 - \gamma)^2 + (x_1 + x_2 - \delta)^2$$

$$\frac{\partial \phi}{\partial x_1} = \cancel{2}(x_1 - \alpha) + \cancel{2}(360 - x_1 - x_2 - \gamma)\cdot(-1) + \cancel{2}(x_1 + x_2 - \delta) = 0$$

$$\frac{\partial \phi}{\partial x_2} = \cancel{2}(x_2 - \beta) + \cancel{2}(360 - x_1 - x_2 - \gamma)\cdot(-1) + \cancel{2}(x_1 + x_2 - \delta) = 0$$

$$3x_1 + 2x_2 = \alpha - \gamma + \delta + 360 = 592.105833$$
$$2x_1 + 3x_2 = \beta - \gamma + \delta + 360 = 612.519166$$

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 592.105833 \\ 612.519166 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 110.255833 \\ 130.669166 \end{bmatrix} = \begin{bmatrix} 110-15-21 \\ 130-40-09 \end{bmatrix}, \quad \hat{\ell} = \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\gamma} \\ \hat{\delta} \end{bmatrix} = \begin{bmatrix} 110-15-21 \\ 130-40-09 \\ 119-04-30 \\ 240-55-30 \end{bmatrix}, \quad \begin{bmatrix} v_\alpha \\ v_\beta \\ v_\gamma \\ v_\delta \end{bmatrix} = \begin{bmatrix} 1'' \\ 1'' \\ -12'' \\ -13'' \end{bmatrix}$$

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now, indirect observations by matrix approach

put into the form $v + B\Delta = f$, $W = I_4$

$$\left.\begin{aligned} v_\alpha - x_1 &= -\alpha \\ v_\beta - x_2 &= -\beta \\ v_\gamma + x_1 + x_2 &= 360 - \gamma \\ v_\delta - x_1 - x_2 &= -\delta \end{aligned}\right\} \quad \underset{v}{\begin{bmatrix} v_\alpha \\ v_\beta \\ v_\gamma \\ v_\delta \end{bmatrix}} + \underset{B}{\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}} \underset{\Delta}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{f}{\begin{bmatrix} -110.255555 \\ -130.668888 \\ 240.921666 \\ -240.928611 \end{bmatrix}}$$

$$\Delta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (B^T W B)^{-1} B^T W f = \begin{bmatrix} 110.^\circ 255833 \\ 130.^\circ 669166 \end{bmatrix}$$ agrees with prior result

$$v = f - B\Delta = \begin{bmatrix} .000277 \\ .000277 \\ -.003333 \\ -.003611 \end{bmatrix} \text{deg} = \begin{bmatrix} 1'' \\ 1'' \\ -12'' \\ -13'' \end{bmatrix}$$

now, 4-6 using observation only method – with matrix approach

$n=4$
$n_0=2$
$r=2$

$\Rightarrow$ need $r=2$ condition equations, put into form $Av=f$

$$1.\ \hat{\alpha}+\hat{\beta}+\hat{\gamma}=360^\circ \qquad 2.\ \hat{\alpha}+\hat{\beta}=\hat{\delta}$$

$$\hat{\alpha}+\hat{\beta}+\hat{\gamma}=360 \qquad \alpha+v_\alpha+\beta+v_\beta+\gamma+v_\gamma=360$$

$$\hat{\alpha}+\hat{\beta}-\hat{\delta}=0, \qquad \alpha+v_\alpha+\beta+v_\beta-\delta-v_\delta=0$$

$$v_\alpha+v_\beta+v_\gamma=360-\alpha-\beta-\gamma$$
$$v_\alpha+v_\beta-v_\delta=-\alpha-\beta+\delta$$

$$\begin{bmatrix}1&1&1&0\\1&1&0&-1\end{bmatrix}\begin{bmatrix}v_\alpha\\v_\beta\\v_\gamma\\v_\delta\end{bmatrix}=\begin{bmatrix}360\\0\end{bmatrix}-\begin{bmatrix}1&1&1&0\\1&1&0&-1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\\\gamma\\\delta\end{bmatrix}$$

$$A\quad v = \underbrace{d - A\ell}_{f}$$

$$\begin{bmatrix}1&1&1&0\\1&1&0&-1\end{bmatrix}\begin{bmatrix}v_\alpha\\v_\beta\\v_\gamma\\v_\delta\end{bmatrix}=\begin{bmatrix}-.002777\\.004166\end{bmatrix}$$

$$A\quad v = f$$

now solve simultaneously for the residuals and lagrange multipliers

$W=I_4$, full normal equations

$$\begin{bmatrix}-W & A^T\\ A & 0\end{bmatrix}\begin{bmatrix}v\\k\end{bmatrix}=\begin{bmatrix}0\\f\end{bmatrix},$$

solve using matlab

$$\begin{bmatrix}v_\alpha\\v_\beta\\v_\gamma\\v_\delta\\k_1\\k_2\end{bmatrix}=\begin{bmatrix}.000277\\.000277\\-.003333\\-.003611\\-.003333\\.003611\end{bmatrix}, \quad \begin{bmatrix}v_\alpha\\v_\beta\\v_\gamma\\v_\delta\end{bmatrix}=\begin{bmatrix}1''\\1''\\-12''\\-13''\end{bmatrix}$$

Same result as indirect observations ✓

Regression problem – Indirect Observations and Matrix method

x: constant
y: observation

$W=I_8$

x	y
2	3
5	4
9	6
11	7
15	8
17	9
18	10
23	12

$\hat{y}=mx+b$, $y+v_y=mx+b$, $\boxed{v_y-mx-b=-y}$ ← that is in form $v+B\Delta=f$

$$\begin{bmatrix}v_1\\v_2\\v_3\\v_4\\v_5\\v_6\\v_7\\v_8\end{bmatrix}+\begin{bmatrix}-2&-1\\-5&-1\\-9&-1\\-11&-1\\-15&-1\\-17&-1\\-18&-1\\-23&-1\end{bmatrix}\begin{bmatrix}m\\b\end{bmatrix}=\begin{bmatrix}-3\\-4\\-6\\-7\\-8\\-9\\-10\\-12\end{bmatrix}$$

$$v\ +\ B\ \Delta\ =\ f$$

solve via, matlab

$$\Delta=\begin{bmatrix}m\\b\end{bmatrix}=(B^TWB)^{-1}B^TWf$$

$$\Delta=\begin{bmatrix}.426724\\2.040948\end{bmatrix}$$

$$v=f-B\Delta=\begin{bmatrix}-.1056\\.1745\\-.1185\\-.2650\\.4418\\.2952\\-.2780\\-.1443\end{bmatrix}$$

$$\hat{\ell}=\ell+v=\begin{bmatrix}2.8943\\4.1745\\5.8814\\6.7349\\8.4418\\9.2952\\9.7219\\11.8556\end{bmatrix}$$