Indirect Observation Method for Nonlinear Problems

Select $u=n_0$ parameters, write one condition equation per observation of form

$$F_i(l_i, x) = l_i + G_i(x) = 0$$

(it's nonlinear in x)

write all n equations, one per obs.

$$\mathbf{F}(\mathbf{l},\mathbf{x}) = \mathbf{l} + \mathbf{G}(\mathbf{x}) = \mathbf{0}$$

make a taylor series approximation

$$\mathbf{F}(\mathbf{l},\mathbf{x}) \approx \mathbf{F}(\mathbf{l}^0,\mathbf{x}^0) + \frac{\partial \mathbf{F}}{\partial \mathbf{l}}\Delta\mathbf{l} + \frac{\partial \mathbf{F}}{\partial \mathbf{x}}\Delta\mathbf{x} = \mathbf{0}$$

$$\underset{(n,n)}{\frac{\partial \mathbf{F}}{\partial \mathbf{l}}} = \mathbf{I}_{\mathbf{n}}, \qquad \underset{(n,n_0)}{\frac{\partial \mathbf{F}}{\partial \mathbf{x}}} = \frac{\partial \mathbf{G}}{\partial \mathbf{x}} = \underset{(n,n_0)}{\mathbf{B}}$$

substitute and rearrange,

$$\Delta\mathbf{l} + \mathbf{B}\Delta\mathbf{x} = -\mathbf{F}(\mathbf{l}^0,\mathbf{x}^0) = -\mathbf{l}^0 - \mathbf{G}(\mathbf{x}^0)$$

but, $\mathbf{l} + \mathbf{v} = \mathbf{l}^0 + \Delta\mathbf{l}$,

$$\Delta\mathbf{l} = \mathbf{l} - \mathbf{l}^0 + \mathbf{v}$$

plugging this in above

$$\mathbf{v} + \mathbf{B}\Delta\mathbf{x} = -\mathbf{l} + \mathbf{l}^0 - \mathbf{l}^0 - \mathbf{G}(\mathbf{x}^0)$$

$$\mathbf{v} + \mathbf{B}\Delta = -\mathbf{l} - \mathbf{G}(\mathbf{x}^0) = -\mathbf{F}(\mathbf{l},\mathbf{x}^0) = \mathbf{f}$$

note the $\mathbf{l}^0$ terms disappear

$$\underset{(n,1)}{\mathbf{v}} + \underset{(n,u)}{\mathbf{B}}\,\underset{(u,1)}{\Delta} = \underset{(n,1)}{\mathbf{f}}$$

get approximations $\mathbf{x}^0$

get $\mathbf{W}$ from observation sigmas

evaluate $\mathbf{B} = \left.\frac{\partial \mathbf{G}}{\partial \mathbf{x}}\right|_{\mathbf{x}^0} = \left.\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\right|_{\mathbf{x}^0}$

evaluate $\mathbf{f} = -\mathbf{F}(\mathbf{l},\mathbf{x}^0)$

$$\Delta = (\mathbf{B}^T\mathbf{W}\mathbf{B})^{-1}\mathbf{B}^T\mathbf{W}\mathbf{f}$$

$$\mathbf{x}^0 = \mathbf{x}^0 + \Delta$$

check for small Δ, repeat if not

Convergence loop

$$\mathbf{v} = \mathbf{f} - \mathbf{B}\Delta$$

$$\hat{\mathbf{l}} = \mathbf{l} + \mathbf{v}$$

Left panel is the "derivation", right panel are the "cookbook" equations to use for solving a problem.

Observation Only Method for Nonlinear Problems

Write r condition equations which are nonlinear in the observations,

$$\mathbf{F}(\mathbf{l}) = \mathbf{0}$$

Now linearize by taylor series

$$\mathbf{F}(\mathbf{l}) \approx \mathbf{F}(\mathbf{l}^0) + \frac{\partial \mathbf{F}}{\partial \mathbf{l}} \Delta \mathbf{l} = \mathbf{0}$$

let, $\mathbf{A} = \left.\frac{\partial \mathbf{F}}{\partial \mathbf{l}}\right|_{\mathbf{l}^0}$, rearrange,

$$\mathbf{A}\,\Delta\mathbf{l} = -\mathbf{F}(\mathbf{l}^0)$$

relate $\Delta\mathbf{l}$ and $\mathbf{v}$,

$$\mathbf{l} + \mathbf{v} = \mathbf{l}^0 + \Delta\mathbf{l}, \quad \Delta\mathbf{l} = \mathbf{l} - \mathbf{l}^0 + \mathbf{v}$$

substitute,

$$\mathbf{A}(\mathbf{l} - \mathbf{l}^0 + \mathbf{v}) = -\mathbf{F}(\mathbf{l}^0)$$

$$\mathbf{A}\mathbf{v} = -\mathbf{F}(\mathbf{l}^0) - \mathbf{A}(\mathbf{l} - \mathbf{l}^0)$$

$$\mathbf{A}\mathbf{v} = \mathbf{f}$$

$$\underset{(r,n)}{\mathbf{A}}\ \underset{(n,1)}{\mathbf{v}} = \underset{(r,1)}{\mathbf{f}}$$

we already have $\mathbf{l}$ so we have first $\mathbf{l}^0$

get $\mathbf{W}$ from observation sigmas

evaluate $\mathbf{A} = \left.\frac{\partial \mathbf{F}}{\partial \mathbf{l}}\right|_{\mathbf{l}^0}$

evaluate $\mathbf{f} = -\mathbf{F}(\mathbf{l}^0) - \mathbf{A}(\mathbf{l} - \mathbf{l}^0)$

solve $\begin{bmatrix} -\mathbf{W} & \mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{k} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix}$

$$\mathbf{l}^0 = \mathbf{l} + \mathbf{v}$$

$$\Delta\mathbf{l} = \mathbf{l}^0_{\,new} - \mathbf{l}^0_{\,previous}$$

check for small $\Delta\mathbf{l}$, repeat if not

Convergence loop

$$\hat{\mathbf{l}} = \mathbf{l} + \mathbf{v}$$

Left panel is “derivation”, right panel are “cookbook” equations to use for solving a problem. Textbook ignores $\mathbf{l}^0$ and does not iterate.