Sequential LS from Sequential Formation of Normal Equations (EMM notation)

prior solution

$$N x = t, \quad x = N^{-1} t, \quad \Sigma_x = Q_x = N^{-1} \; (\sigma_0^2 = 1), \quad N = Q_x^{-1} \tag{1}$$

new observations

$$B x \approx f \quad (v + Bx = f), \quad W, Q \tag{2}$$

update using sequential formation of normal equations

$$x_n = (N + B^T W B)^{-1} (t + B^T W f) \tag{3}$$

substitute t and N from (1),

$$x_n = (Q_x^{-1} + B^T W B)^{-1} (N x + B^T W f) \tag{4}$$

$$x_n = (Q_x^{-1} + B^T W B)^{-1} (Q_x^{-1} x + B^T W f)$$

by inspection

$$Q_{xn} = (Q_x^{-1} + B^T W B)^{-1} \tag{5}$$

$$Q_{xn}^{-1} = Q_x^{-1} + B^T W B$$

Sherman, Morrison, Woodbury, Schur matrix inversion lemma

$$(A + UCV)^{-1} = A^{-1} - A^{-1} U (C^{-1} + V A^{-1} U)^{-1} V A^{-1} \tag{6}$$

expand equation (4) using this

$$x_n = \left[Q_x - \underbrace{Q_x B^T (Q + B Q_x B^T)^{-1}}_{K} B Q_x \right] (Q_x^{-1} x + B^T W f) \tag{7}$$

declare subexpression to be K

$$x_n = \left[Q_x - K B Q_x \right] (Q_x^{-1} x + B^T W f) \tag{8}$$

multiply out

$$x_n = x + Q_x B^T W f - K B x - K B Q_x B^T W f \tag{9}$$

from (8)

$$Q_{xn} = Q_x - K B Q_x = (I - KB) Q_x, \quad \boxed{Q_{xn} = (I - KB) Q_x} \tag{10}$$

expression for K

$$\boxed{K = Q_x B^T (Q + B Q_x B^T)^{-1}} \tag{11}$$

insert $Q_{xn} Q_{xn}^{-1}$ and WQ, (both $= I$)

$$K = Q_{xn} Q_{xn}^{-1} Q_x B^T W \underbrace{Q (Q + B Q_x B^T)^{-1}}_{} \tag{12}$$

look at subexpression and reorganize using $(AB)^{-1} = B^{-1}A^{-1}$

$$Q(Q + BQ_xB^T)^{-1} = [(Q + BQ_xB^T)W]^{-1} = (I + BQ_xB^TW)^{-1} \quad (13)$$

plug this back into (12)

$$K = Q_{xn}Q_{xn}^{-1}Q_xB^TW(I + BQ_xB^TW)^{-1} \quad (14)$$

plug expression for Q_{xn}^{-1} from (5)

$$K = Q_{xn}(Q_x^{-1} + B^TWB)Q_xB^TW(I + BQ_xB^TW)^{-1} \quad (15)$$

multiply middle factor to the left

$$K = Q_{xn}(B^TW + B^TWBQ_xB^TW)(I + BQ_xB^TW)^{-1} \quad (16)$$

factor left

$$K = Q_{xn}B^TW\underbrace{(I + BQ_xB^TW)(I + BQ_xB^TW)^{-1}}_{= I} \quad (17)$$

inverses cancel, product = I

$$\boxed{K = Q_{xn}B^TW} \quad (18)$$

rearrange (9)

$$X_n = Q_xB^TWf - KBQ_xB^TWf + x - KBx \quad (19)$$

factor

$$X_n = (I - KB)Q_xB^TWf + x - KBx \quad (20)$$

recall from (10), $Q_{xn} = (I - KB)Q_x$

$$X_n = Q_{xn}B^TWf + x - KBx \quad (21)$$

replace subexpression from (18)

$$X_n = Kf + x - KBx$$

$$\boxed{X_n = x + K(f - Bx)} \quad (22)$$

recall (10)

$$\boxed{Q_{xn} = (I - KB)Q_x} \quad (23)$$

compare w/ Brown + Hwang (KF) Figure 5.8

(KF equations derived from LS)

$$\boxed{\begin{array}{l} K = P^-H^T(HP^-H^T + R)^{-1} \\ X = X^- + K(z - HX^-) \\ P = (I - KH)P^- \\ X_{k+1}^- = \Phi X_k \\ P_{k+1}^- = Q_k + \Phi P_k \Phi^T \end{array}} \begin{array}{l} (11) \\ (22) \\ (10,23) \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{Dyn. Model} \end{array} \quad (24)$$