

## CONTENTS (Continued)

## Discussions (Continued):

Relation Between Rail and Waterway Transportation: A Symposium. By MESSRS. R. B. KITTRIDGE, HENRY A. PALMER, and J. PARKER SNOW...	1123
General Specifications for Steel Railway Bridges: Prepared by the Conference Committees of the American Society of Civil Engineers and the American Rail- way Engineering Association. By MESSRS. G. G. THOMAS, HENRY S. JACOBY, GLENN B. WOODRUFF, HAROLD C. BIRD, W. A. DUFF, CHARLES STRATTON DAVIS, C. D. PURDON, G. E. TEB- BETTS, J. R. WORCESTER, A. W. CARPENTER, D. B. STEINMAN, W. CHASE THOMSON, JONATHAN JONES, and P. G. LANG, JR.....	1131
Experimental Research on Vibration Dampers and Insulators. By MESSRS. W. M. DAWLEY and STEPHEN E. SLOCUM.....	1177

## Memoirs:

WILLIAM BROWN, M. AM. SOC. C. E.....	1197
CHARLES HALL CARPENTER, M. AM. SOC. C. E.....	1198
ERNEST LESTER JONES, M. AM. SOC. C. E.....	1200
EDWARD STANLEY SAFFORD, M. AM. SOC. C. E.....	1202
IRVING MASON WOLVERTON, M. AM. SOC. C. E.....	1204
ADOLPHUS JAMES EDDY, ASSOC. M. AM. SOC. C. E.....	1204
JOHN LUCEY, ASSOC. M. AM. SOC. C. E.....	1206
CHARLES HENRY REYNOLDS, ASSOC. M. AM. SOC. C. E.....	1208
BOYD FREEZE WALKER, ASSOC. M. AM. SOC. C. E.....	1209

For Index to all Papers, the discussion of which is current in *Proceedings*,  
see the second page of the cover .

## AMERICAN SOCIETY OF CIVIL ENGINEERS

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## PAPERS AND DISCUSSIONS

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ANALYSIS OF CONTINUOUS FRAMES  
BY DISTRIBUTING FIXED-END MOMENTS

By HARDY CROSS,\* M. AM. SOC. C. E.

## SYNOPSIS

The purpose of this paper is to explain briefly a method which has been found useful in analyzing frames which are statistically indeterminate. The essential idea which the writer wishes to present involves no mathematical relations except the simplest arithmetic. It is true that in order to apply the method it is necessary to determine certain constants mathematically, but the means to be used in determining these constants are not discussed in the paper, nor are they a part of the method. These constants have been derived by so many writers and in so many slightly different ways that there is little occasion to repeat here the whole procedure.

The reactions in beams, bents, and arches which are immovably fixed at their ends have been extensively discussed. They can be found comparatively readily by methods which are more or less standard. The method of analysis herein presented enables one to derive from these the moments, shears, and thrusts required in the design of complicated continuous frames.

## DEFINITIONS

For convenience of reference, definitions of three terms will be introduced at once. These terms are "fixed-end moment", "stiffness", and "carry-over factor".

By "fixed-end moment" in a member is meant the moment which would exist at the ends of the member if its ends were fixed against rotation.

"Stiffness", as herein used, is the moment at one end of a member (which is on unyielding supports at both ends) necessary to produce unit rotation of that end when the other end is fixed.

If one end of a member which is on unyielding supports at both ends is rotated while the other end is held fixed the ratio of the moment at the fixed

NOTE.—Written discussion on this paper will be closed in September, 1930, *Proceedings*.

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end to the moment producing rotation at the rotating end is herein called the "carry-over factor."

#### EFFECT OF JOINT ROTATION

Imagine any joint in a structure, the members of which are being deformed by loads, or in some other way, to be first held against rotation and then released. Call the algebraic sum of the fixed-end moments at the joint the "unbalanced fixed-end moment". Before the joint is released this unbalanced fixed-end moment will not usually be zero; after the joint is released, the sum of the end moments at the joint must be zero. The total change in end moments, then, must equal the unbalanced fixed-end moment. This may be stated in another way by saying that the unbalanced fixed-end moment has been "distributed to" the connecting members in some ratio.

When the joint is released all connecting members rotate through the same angle and this rotation at the end is accompanied by a change in end moment. The change in end moments is proportional to the "stiffness" of the members.

Hence, it may be said that when the joint is released the unbalanced fixed-end moment is distributed among the connecting members in proportion to their stiffness.

The rotation of the joint to produce equilibrium induces moments at the other ends of the connecting members. These are equal in each member to the moments distributed at the rotating joint multiplied by the carry-over factor at the rotating end of the member. This follows from the definition of "carry-over factor".

#### MOMENT DISTRIBUTION

The method of moment distribution is this: (a) Imagine all joints in the structure held so that they cannot rotate. Compute the moments at the ends of the members for this condition; (b) at each joint distribute the unbalanced fixed-end moment among the connecting members in proportion to the constant for each member defined as "stiffness"; (c) multiply the moment distributed to each member at a joint by the carry-over factor at that end of the member and set this product at the other end of the member; (d) distribute these moments just "carried over"; (e) repeat the process until the moments to be carried over are small enough to be neglected; and (f) add all moments—fixed-end moments, distributed moments, moments carried over—at each end of each member to obtain the true moment at the end.

To the mathematically inclined the method will appear as one of solving a series of normal simultaneous equations by successive approximation. From an engineering viewpoint it seems simpler and more useful to think of the solution as if it were a physical occurrence. The beams are loaded or otherwise distorted while the joints are held against rotation; one joint is then allowed to rotate with accompanying distribution of the unbalanced moment at that joint and the resulting moments are carried over to the adjacent joints; then another joint is allowed to rotate while the others are held against rotation; and the process is repeated until all the joints are "eased down" into equilibrium.

#### BEAM CONSTANTS

This method of analysis is dependent on the solution of three problems in the mechanics of materials. These are the determination of the fixed-end moments, of the stiffness at each end, and of the carry-over factor at each end for each member of the frame under consideration. The determination of these values is not a part of the method of moment distribution and is not discussed in this paper.

The stiffness of a beam of constant section is proportional to the moment of inertia divided by the span length, and the carry-over factor is  $-\frac{1}{2}$ .

The proof or derivation of these two statements and the derivation of formulas for fixed-end moments is left to the reader. They can be deduced by the use of the calculus; by the theorems of area-moments; from relations stated in *Bulletin 108* of the Engineering Experiment Station of the University of Illinois (the Slope-Deflection *Bulletin*); from the theorem of three moments; by what is known to some as the column analogy method;\* or by any of the other corollaries of geometry as applied to a bent member. Formulas for fixed-end moments in beams of uniform section may be found in any structural handbook.

#### SIGNS OF THE BENDING MOMENTS

It has seemed to the writer very important to maintain the usual and familiar conventions for signs of bending moments, since these are the conventions used in design.

For girders the usual convention is used, positive moment being such as sags the beam. For vertical members the same convention is applicable as for girders if the sheet is turned to read from the right as vertical members on a drawing are usually read. The usual conventions for bending moments are, then, applicable to both girders and columns if they are looked at as a drawing is usually lettered and read.

Moments at the top of a column, as the column stands in the structure, should be written above the column and those at the bottom of the column, as the column stands in the structure, should be written below the column when the sheet is in position to read the columns. This is necessary because positive moment at the right end of a beam and at the top of a column both represent tendencies to rotate the connected joint in the clockwise direction.

It makes no difference whether girder moments are written above or below the girder. Either arrangement may be convenient. Confusion will be avoided by writing column moments parallel to the column and girder moments parallel to the girders.

When any joint is balanced the total moment to the right and to the left of the support is the same, both in absolute value and in sign. The unbalanced moment is the algebraic difference of the moments on the two sides of the joint.

#### LIMITATION OF METHOD

From the fact that the terms, "stiffness" and "carry-over factor", have been defined for beams resting on unyielding supports, it follows that direct

\* "The Column Analogy," by Hardy Cross, M. Am. Soc. C. E., *Bulletin* (203) Eng. Experiment Station, Univ. of Illinois, Urbana, Ill. (In press.)

application of the method is restricted to those cases where the joints do not move during the process of moment distribution. The method, however, can be applied in an indirect way to cases in which the joints are displaced during the moment distribution, as indicated later.

As the method has been stated, it is restricted only by this condition that the joints are not displaced. If this condition is satisfied it makes no difference whether the members are of constant or of varying section, curved or straight, provided the constants, (a) fixed-end moments at each end, (b) stiffness at each end, and (c) carry-over factor at each end, are known or can be determined. Such values can be derived by standard methods and may be tabulated for different types of members and conditions of loading.

It will be found that in most cases accuracy is needed only in the fixed-end moments. It does not ordinarily make very much difference how, within reason, the unbalanced moments are distributed, nor, within reason, how much of the distributed moments are carried over.

In the illustration which follows it has been assumed that the members are straight and of uniform section. The stiffnesses, then, are proportional to the moments of inertia, ( $I$ ), divided by the lengths, ( $L$ ), but the relative values given for  $\frac{I}{L}$  in this problem might quite as well be the relative stiffness of a series of beams of varying section. In this latter case, however, the carry-over factors for the beams would not be  $-\frac{1}{2}$ .

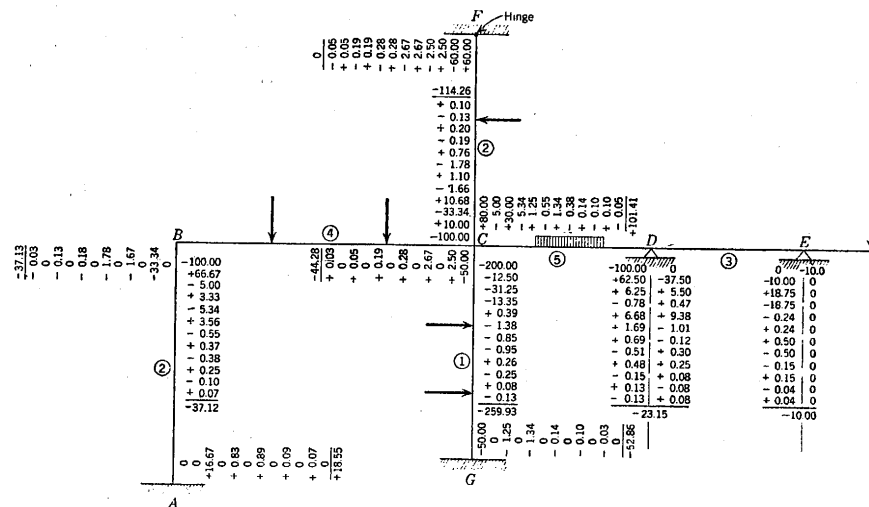


FIG. 1.

ILLUSTRATION

The illustration given (Fig. 1) is entirely academic. It is not intended to represent any particular type of structure nor any probable condition of loading. It has the advantage for the purpose of this paper that it involves all the conditions that can occur in a frame which is made up of straight members and in which the joints are not displaced.

The loads on the frame are supposed to be as indicated. The relative values of  $\frac{I}{L}$  for the different members are indicated in circles.

The fixed-end moments in all members are first written. In this problem they are arbitrarily assumed to be as shown, as follows: at A, 0; at B, in BA, 0, and in BC,  $-100$ ; at C, in CB,  $-100$ , in CF,  $+80$ , in CD,  $-200$ , and in CG,  $-50$ ; at F,  $+60$ ; at G,  $-50$ ; at D, in DC,  $-100$ , and in DE, 0; at E, in ED, 0, and in the cantilever,  $-10$ .

Before proceeding to a solution of the problem, attention may be called to the arrangement of the computations. The moments in the girders are written parallel to the girders; those in the columns, parallel to the columns. The original fixed-end moments are written next to the members in which they occur, the successive moments distributed or carried over being written above or below these, but farther from the member.

The arrangement of the moments in the columns in positions above the columns, when the paper is turned into a position to write these moments, for the top of the columns (at B, F, and C), and in positions below the columns for the bottom of the columns (at A, C, and G), is an essential part of the sign convention adopted.

The moment at C in the girder, BC, is written above the girder in order to get it out of the way. Otherwise, it makes no difference whether the moments are written above or below the girder.

The signs of the fixed-end moments are determined by observing the direction of flexure at the ends of the members due to the loads. In order to apply to the columns the ordinary conventions for signs of bending moments it is necessary to turn the drawing of the structure.

The reader should realize that the solution is built up step by step. It is always the last figures showing that are to be operated on—distributed or carried over—so that in ordinary framework there is little chance for confusion as to what step should be taken next.

Distribute at each joint the unbalanced moment, as follows:

- 1.—At A there is no moment.
- 2.—At B there is an unbalanced moment of  $-100$  on one side of the joint.

This moment is distributed to BA and to BC in the ratio, 2 : 4, so that the distributed moment to BA is  $\frac{2}{2+4} 100 = 33.33$  and to BC,  $\frac{4}{2+4} 100 = 66.67$ .

The signs are written in the only way possible to balance the joint by giving the same total moment ( $-33.33$ ) both to left and right of the joint.

3.—At C, the unbalanced moments are, in CB,  $-100$ , and in CG,  $-50$ , giving a total of  $-150$  on the left of the joint; in CF,  $+80$ , and in CD,  $-200$ , giving a total of  $-120$  on the right of the joint. The total unbalanced moment at the joint, which is the difference between the total moment on the left and on the right of the joint, is 30. This is now distributed in the respective proportions, as follows:

To  $CB$ ,

$$\frac{4}{4 + 2 + 5 + 1} 30 = 10$$

to  $CF$ .

$$\frac{2}{4 + 2 + 5 + 1} 30 = 5$$

to  $CD$ ,

$$\frac{5}{4 + 2 + 5 + 1} 30 = 12.5$$

and, to  $CG$ ,

$$\frac{1}{4 + 2 + 5 + 1} 30 = 2.5$$

There is only one way to place the signs of the distributed moments so that the total is the same on both sides of the joint. This is done by reducing the excess negative moment on the left and increasing the negative moment on the right.

4.—At  $F$ , the unbalanced moment is  $+60$ . The hinge has no stiffness. The moment, then, is distributed between the member,  $FC$ , and the hinge in the ratio,  $2 : 0$ ; all of it goes to the member. The total balanced moment is  $+60 - 60 = 0$ , as it must be at a free end.

5.—At  $G$ , the abutment is infinitely stiff and the unbalanced moment,  $-50$ , is distributed between the member,  $GC$ , and the abutment in the ratio,  $1 : \infty$ . The member gets none of it; the end stays fixed.

6.—At  $D$ , the unbalanced moment,  $-100$ , is distributed to  $DC$  and to  $DE$  in the ratio of  $5 : 3$ .

7.—At  $E$ , the unbalanced moment is  $-10$  in the cantilever. Since the cantilever has no stiffness, this unbalanced moment is distributed between the beam,  $ED$ , and the cantilever in the ratio,  $3 : 0$ . This means that all of it goes to  $ED$ .

All joints have now been balanced. Next, carry over from each end of each member one-half the distributed moment just written, reverse the sign, and write it at the other end of the member. Thus, carry over, successively, in  $AB$ ,  $0$  from  $A$  to  $B$  and  $+16.67$  from  $B$  to  $A$ ; in  $BC$ ,  $-33.34$  from  $B$  to  $C$  and  $-5.0$  from  $C$  to  $B$ ; in  $CF$ ,  $+2.5$  from  $C$  to  $F$  and  $+30$  from  $F$  to  $C$ ; in  $CG$ ,  $0$  from  $G$  to  $C$  and  $-1.25$  from  $C$  to  $G$ ; in  $CD$ ,  $+6.25$  from  $C$  to  $D$  and  $-31.25$  from  $D$  to  $C$ ; and in  $DE$ ,  $+18.75$  from  $D$  to  $E$  and  $+5.00$  from  $E$  to  $D$ .

Distribute the moments just carried over exactly as the original fixed-end moments were distributed. Thus, at  $A$ ,  $+16.67$  is distributed  $0$  to  $AB$  (fixed-ended); at  $B$ ,  $-5.0$  is distributed as  $-1.67$  and  $+3.32$ ; at  $C$ , the unbalanced moment is  $(-33.34 + 0) - (+30.00 - 31.25) = -32.09$  which is distributed as  $+2.67$ ,  $+10.68$ ,  $-5.34$ , and  $-13.35$ ; at  $F$ ,  $+2.50$  is distributed as  $-2.5$  to the member;  $G$  is fixed-ended; at  $D$ ,  $+1.25$  is distributed as  $-0.78$  and  $+0.47$ ; at  $E$ , the unbalanced  $+18.75$  is distributed to the member as  $-18.75$ .

The moments distributed are now carried over as before and then re-distributed; and the process is repeated as often as desired. The procedure should

be stopped after each distribution, however, and a check made to see that statics ( $\Sigma M = 0$ ) is satisfied.

When it is felt that the process has gone far enough, all moments at each end of each member are added to give the total moment at the joint. After the moments at the joints have been determined, all other quantities, such as moments and shears, may be obtained by applying the laws of statics.

#### CONVERGENCE OF RESULTS

The distribution herein has been carried out with more precision than is ordinarily necessary, in order to show the convergence of the results. To show the rate of convergence, the successive values of the moments at the joints after successive distributions are given in Table 1.

TABLE 1.—CONVERGENCE OF RESULTS.

Successive values of bending moment at joint.	After one distribution (two rows of figures).	After two distributions (four rows of figures).	After three distributions (six rows of figures).	After four distributions (eight rows of figures).	After five distributions (ten rows of figures).	After six distributions (twelve rows of figures).
$A$	.....	0	+ 16.67	+ 17.50	+ 18.89	+ 18.88
$B$	.....	- 33.34	- 35.01	- 36.79	- 36.97	- 37.10
$C$	In $CB$	- 90.00	- 112.86	- 113.22	- 114.24	- 114.23
	" $CF$	+ 75.00	+ 99.86	+ 100.36	+ 101.82	+ 101.86
	" $CD$	- 212.50	- 257.10	- 258.09	- 259.89	- 259.88
$D$	" $CG$	- 47.50	- 44.88	- 44.55	- 44.86	- 44.81
	.....	- 37.50	- 32.03	- 29.66	- 28.46	- 28.15
$E$	.....	- 10.00	- 10.00	- 10.00	- 10.00	- 10.00
$F$	.....	0	0	0	0	0
$G$	.....	- 50.00	- 51.25	- 52.59	- 52.73	- 52.88

For most purposes the computations might as well have been stopped after the second distribution. Had this been done, the solution would have appeared as shown in Fig. 2.

For any practical purpose the computation might in this case have been stopped after the third distribution. In general, two or three distributions are sufficient. This is not true in all instances, but in any case the exactness of the solution at any stage will be indicated by the magnitude of the moments carried over in the members.

#### VARIATIONS OF THE METHOD

The writer has developed and used at different times several variations of the method shown, but the original method is itself so simple and so easy to remember that he finds himself inclined to discard the variants.

One variant is perhaps worth recording. It is rather tedious to carry moments out to the end of a member which is free to rotate and then balance the moment and carry it back again. This may be avoided by releasing the free end once for all and leaving it free. In this case, for beams of constant section, the stiffness of the beam is to be taken three-fourths as great\* as

\* The moment needed to produce a given rotation at one end of a beam when the other end is free is three-fourths as great as if the other end is fixed.

the relative  $\frac{I}{L}$ -value would indicate. After the end of the beam is once released, no moments are carried over to it.

CORRECTING FOR SIDE-SWAY

Single square or trapezoidal frames, portals, L-frames, box culverts, and similar structures act as simple continuous beams if there is no transverse deflection. If they are symmetrical as to form and loading, they will not deflect sidewise and if they are restrained against sidewise movement, they cannot so deflect.

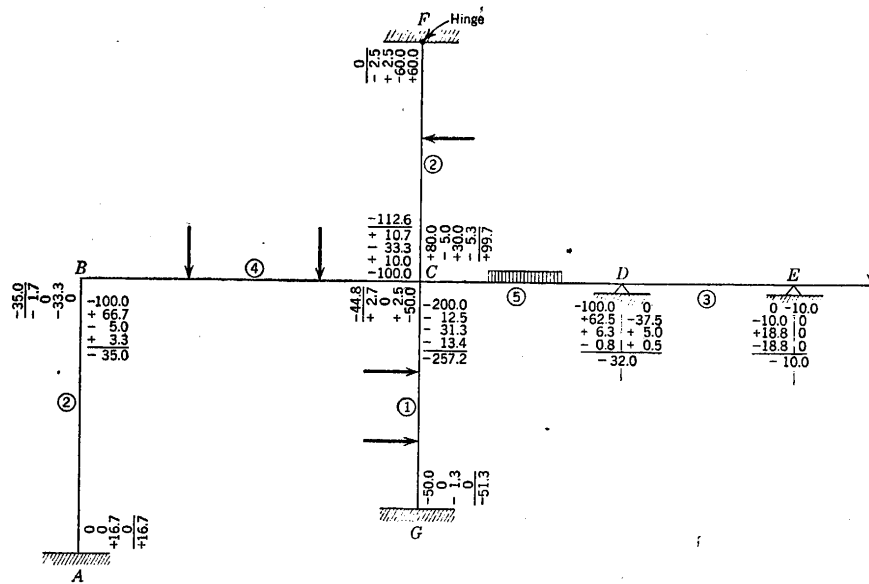


FIG. 2.

Side-sway of frames due to dissymmetry of the frame is rarely an important factor in design. Correction for side-sway may be made by a method which may be applied also in cases of transverse loading on bents. The method is to consider that the bent does not sway sidewise and analyze it as a series of continuous beams. The total shear in the legs will not now, except by accident, equal the shear which is known to exist. The difference must be a force which prevents side-sway.

bent

Now, assume all joints held against rotation, but the top of the beam moved sidewise. Assume any series of fixed-end moments in the legs such that all legs have the same deflection. In this case for members of uniform section fixed-end moments in columns vary as  $\frac{I}{L^2}$ . Distribute these fixed-end moments and find the total shear in the legs. The changes in moments due to side-sway will then be to the moments just computed in the same algebraic ratio as the total unbalanced horizontal shear in the legs due to side-sway, when the frame is analyzed as a continuous girder, is to the shear just computed.

MULTI-STORIED BENTS

Bents of more than one story, subject to side-sway, either as a result of unbalanced loading or due to horizontal forces, may be solved by this method. It is understood that exact solution of such problems is not commonly of great interest. It is the approximate effect that is desired rather than exact analysis.

To analyze by this method a two-story bent it will be necessary to make two configurations—one for each story. From the assumed shear in each story (producing, of course, shears in the other stories), a set of moment values may be obtained. These may be combined to obtain the true shears, and from the true shears the true moments follow.

GENERAL APPLICATION OF THE METHOD

The method herein indicated of distributing unbalanced moments may be extended to include unbalanced joint forces. As thus extended it has very wide application. Horizontal or vertical reactions may be distributed and carried over and thus a quick estimate made of the effect of many complicating elements in design. The writer has used it in studying such problems as continuous arch series, the effect of the deflection of supporting girders, and other phenomena.

An obvious application of moment distribution occurs in the computation of secondary stresses in trusses. Many other applications will doubtless suggest themselves, but it has been thought best to restrict this paper chiefly to continuous frames in which the joints do not move.

CONCLUSION

The paper has been confined to a method of analysis, because it has seemed wiser to so restrict it. It is not then an oversight that it does not deal with: (1) Methods of constructing curves of maximum moments; (2) methods of constructing curves of maximum shears; (3) the importance of analyses for continuity in the design of concrete girders; (4) flexural stresses in concrete columns; (5) methods of constructing influence lines; (6) the degree to which continuity exists in ordinary steel frames; (7) continuity in welded steel frames; (8) plastic deformation beyond the yield point as an element in interpreting secondary stress computations; (9) the effect of time yield on moments and shears in continuous concrete frames; (10) plastic flow of concrete as a factor in the design of continuous concrete frames; (11) whether in concrete frames it is better to guess at the moments, to take results from studies made by Winkler fifty years ago, or to compute them; (12) the effect of torsion of connecting members; (13) the relative economy of continuous structures; (14) the relative flexibility of continuous structures; (15) the application of methods of continuous frame analysis to the design of flat slabs; (16) probability of loading and reversal of stress as factors in the design of continuous frames; (17) the relation of precision in the determination of shears and moments to precision in the determination of fiber stresses and a dozen other considerations bearing on the design of continuous frames.

The writer has discussed several of these questions elsewhere. He hopes that readers will discuss some of them now.

A method of analysis has value if it is ultimately useful to the designer; not otherwise. There are apparently three schools of thought as to the value of analyses of continuous frames. Some say, "Since these problems cannot be solved with exactness because of physical uncertainties, why try to solve them at all?" Others say, "The values of the moments and shears cannot be found exactly; do not try to find them exactly; use a method of analysis which will combine reasonable precision with speed." Still others say, "It is best to be absolutely exact in the analysis and to introduce all elements of judgment after making the analysis."

The writer belongs to the second school; he respects but finds difficulty in understanding the viewpoint of the other two. Those who agree with his viewpoint will find the method herein explained a useful guide to judgment in design.

Members of the last named school of thought should note that the method here presented is absolutely exact if absolute exactness is desired. It is a method of successive approximations; not an approximate method.

May, 1930.]

929

## AMERICAN SOCIETY OF CIVIL ENGINEERS

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### PAPERS AND DISCUSSIONS

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#### THE TRAINING WALL ACROSS THE LIAO BAR IN MANCHURIA

By P. N. FAWCETT,\* M. AM. SOC. C. E.

##### SYNOPSIS

This paper describes in detail the problems in harbor work encountered at the Port of Niuchuang in Southern Manchuria. The discharge of the Liao River has been such as to form a bar at its mouth that continually hampers free navigation. The paper presents a description of the movement of this bar before the construction of the single training wall as well as during its construction. It also contains supplementary data concerning tides, winds, rainfall, velocities of river flow, and other information important to a study of this kind.

##### LOCATION

The Liao River is the most important waterway in Manchuria. Its length is 600 miles (of which 350 are navigable for junk traffic) and its drainage area is 80 000 sq. miles. It empties into the northern extremity of the shallow Gulf of Liaotung and the Gulf of Pechihli. The Port of Niuchuang, for which the improvements described in this paper were made, is shown in Fig. 1, at the northern end of the Gulf of Liaotung.

##### TIDAL DATA

Fig. 2 shows the tidal curves for this river. Spring tides range between 10 and 13 ft. and neap tides between 7 and 8 ft.; the diurnal inequality during the spring tides is sometimes as much as 4 ft., the variation being confined to the high waters, with practically none in the low waters. Owing to the peculiar formation of the entrance to the Gulf of Pechihli the tides are greatly influenced by winds. A northerly wind drives the water out of the Gulf, thus reducing the depths along the coasts, while a southerly wind forces the water into it, between Korea and the Shantung Peninsula, thereby aug-

NOTE.—Written discussion on this paper will be closed in August, 1930, *Proceedings*.  
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