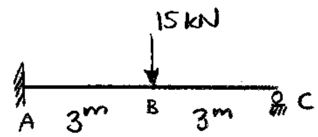
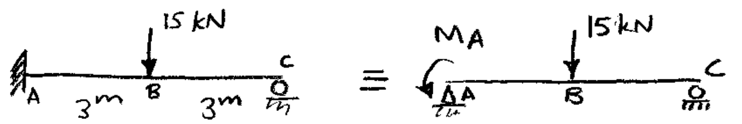
Application of Principle of Virtual Work to Find Displacement in Statically Indeterminate Structures

<u>Example</u>: Find Δ_B, the vertical displacement at *B*. Consider flexural response only; assume *EI* is constant.

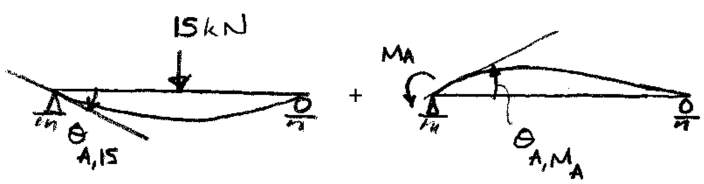


First of all, we need to find the curvature distribution in this statically indeterminate to 1st degree propped-cantilever structure.

Let's treat the moment reaction at *A* as the redundant reaction and use *method of consistent deformations* (also known as *compatibility method* or *flexibility method*) to solve the system.



w/ $\theta_A = 0$ displacement compatibility condition

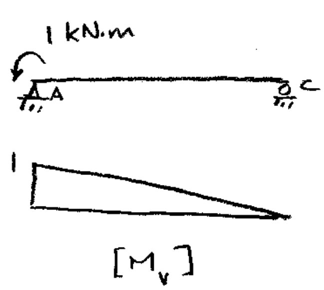


$\theta_{A,15} + \theta_{A,M_A} = 0$

We can use the virtual force method to find $\theta_{A,15}$ and θ_{A,M_A}. That is, we first apply a tracer virtual unit moment at *A* and then convolve the resulting virtual bending moment distribution with the curvature distributions in the two simply supported beams loaded with real external force 15 kN and support reaction M_A, respectively, to find the corresponding internal virtual strain energy results. Equating these to the respective external virtual work in each case we can find $\theta_{A,15}$ and θ_{A,M_A}.

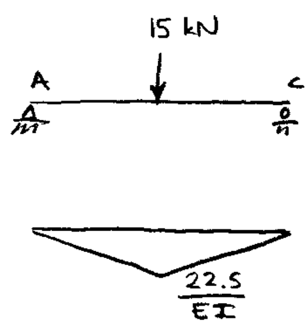
<u>for $\theta_{A,15}$:</u>

virtual system



$[M_v]$

real system



$[M/EI]$ ($=\varphi$: curvature)

Equating external virtual work to internal virtual strain energy

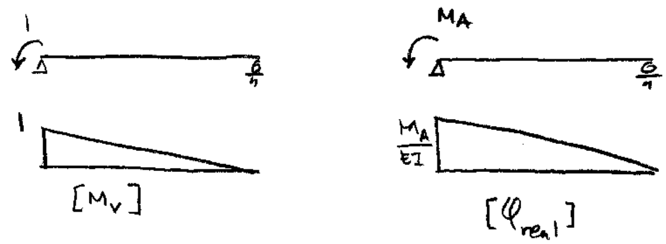
$$W_v = U_v$$

$$W_v = 1\ \text{kN·m} \times \theta_{A,15}$$

$$U_v = \int_0^6 M_v(x)\,\frac{M(x)}{EI(x)}\,dx = -\frac{1}{4}(1)\left(\frac{22.5}{EI}\right)6 = -\frac{33.75}{EI}$$

$$\Rightarrow\quad 1 \times \theta_{A,15} = -\frac{33.75}{EI} \quad\Rightarrow\quad \theta_{A,15} = -\frac{33.75}{EI}$$

for θ_{A,M_A}



$$W_v = U_v \qquad 1 \times \theta_{A,M_A} \qquad U_v = \frac{1}{3}(1)\left(\frac{M_A}{EI}\right)6 = 2\frac{M_A}{EI}$$

$$\Rightarrow\quad \theta_{A,M_A} = \frac{2M_A}{EI}$$

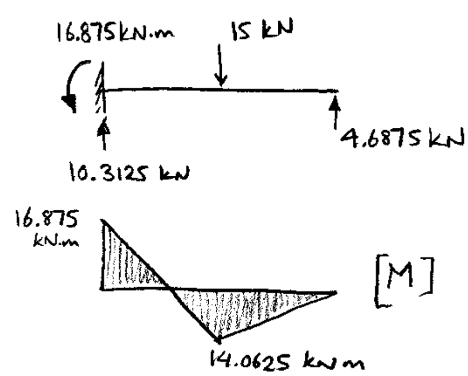
Now requiring the compatibility condition that net slope change at *A* should be zero, we can find M_A.

$$\theta_{A,15} + \theta_{A,M_A} = 0$$

$$-\frac{33.75}{EI} + \frac{2M_A}{EI} = 0$$

$$M_A = 16.875\ \text{kN m}$$

$\Rightarrow$

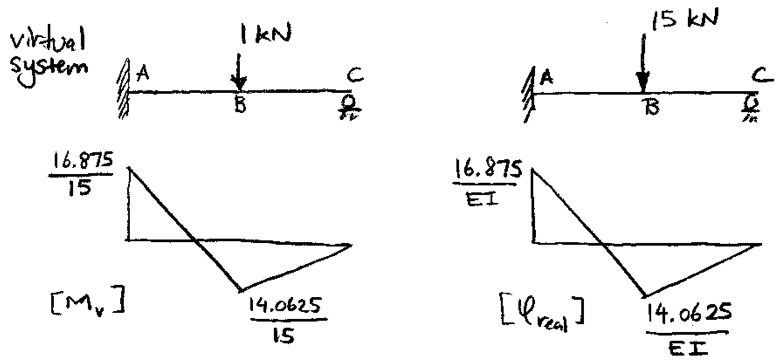


Now that we have found the moment distribution in the propped cantilever when it loaded by 15 kN downward force at midspan point *B*, we can now find vertical displacement at *B*, Δ_B. We will do so using virtual force approach –aside: this method is also known as "dummy load" method or "unit load" method. We will find Δ_B using three different "virtual systems". In each case, we will apply a virtual unit vertical load at *B*, of course, and find the corresponding virtual bending moment distribution. Then using principle of virtual work we will find Δ_B.

The three distinct structures in the virtual systems we will consider are:

a) a propped-cantilever identical to the real structure (statically indeterminate to 1st degree).
b) a simplified structure with moment constraint at *A* released. This is a simply supported beam.
c) a simplified structure with roller support at *C* released. This is a simple cantilever beam.

a) virtual system is a propped-cantilever with a unit vertical load applied at B.



$$W_v = U_v$$

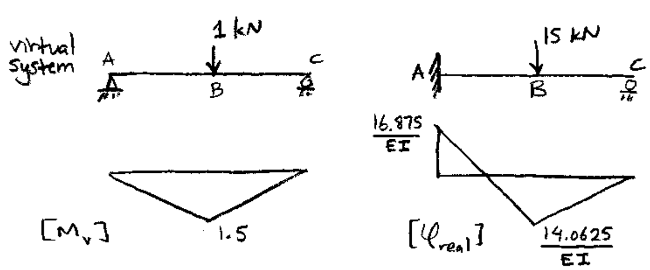
$$1 \times \Delta_B = \int_0^6 M_v(x) \left(\frac{M_{real}(x)}{EI(x)}\right) dx \qquad \varphi_{real}(x)$$

$$U_v = \frac{1}{6}\left[\frac{16.875}{15}\left(2\times 16.875 - 14.0625\right) - \frac{14.0625}{15}\left(16.875 - 2\times 14.0625\right)\right]\cdot\frac{3}{EI}$$

$$+\frac{1}{3}\,\frac{14.0625}{15}\,\frac{14.0625}{EI}\cdot 3$$

$$U_v = 29.53125\,\frac{1}{EI} \quad \Rightarrow \quad \left[\Delta_B = \frac{29.53125}{EI}\ \downarrow\right]$$

b) Virtual system is a simply supported beam with a unit vertical load applied at B.



$$W_v = U_v$$

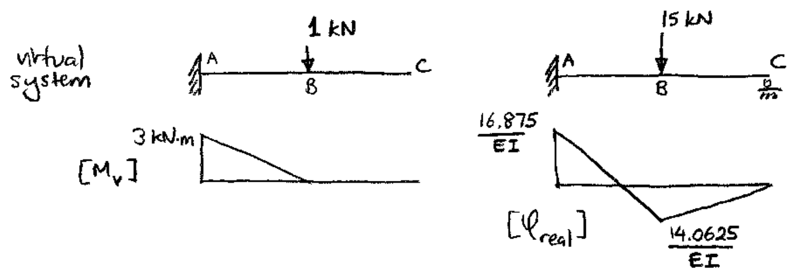
$$1\times\Delta_B \qquad U_v = -\frac{1}{6}(1.5)\left(16.875 - 2\times 14.0625\right)\frac{3}{EI} + \frac{1}{3}(1.5)(14.0625)\frac{3}{EI}$$

$$U_v = \frac{29.53125}{EI}$$

$$\Rightarrow \left[\Delta_B = \frac{29.53125}{EI}\ \downarrow\right]$$

Note that if we had added a $\frac{16.875}{15}$ kNm moment (CCW) at support *A* to the virtual system structure, we would have gone back to the original, statically indeterminate propped-cantilever for our virtual structure. The virtual strain energy contribution to this $\frac{16.875}{15}$ kNm virtual moment would be equal to the virtual external work done by it going through θ_A rotation in the real system. However, $\theta_A = 0$ in the real system and therefore there is no contribution to the virtual energy (internal virtual strain energy or external virtual work) by the virtual moment *A*. That's why we can ignore the moment at *A* and work with the released/reduced system as done above.

c) Virtual system is a cantilever beam with a unit vertical load applied at B.



$$W_v = U_v$$

$$1 \times \Delta_B = \frac{1}{6}(3)(2 \times 16.875 - 14.0625)\frac{1}{EI} \cdot 3 = \frac{29.53125}{EI}$$

$$\Rightarrow \boxed{\Delta_B = \frac{29.53125}{EI} \downarrow}$$

Same result as before. The released redundant force (reaction at *C*) does not do any external virtual work or cause any internal virtual strain energy because the vertical displacement at *C* in the real system is zero.

In summary, all virtual systems we have considered gave the same result.

Note that in finding displacements/rotations using virtual force approach, if one considers axial and shear deformations besides curvature, i.e., axial and shear behavior as well as flexural behavior, same results will be found using stable simplified virtual systems as those found using original real system structure as the virtual system structure, which may be statically indeterminate and, hence, not easy to analyze to find the internal virtual forces/moments.