CHAPTER III

GEOMETRY OF DEFORMATIONS - VIRTUAL WORK

General Deduction - Displacement. The fundamental relations which determine in structures the distortions produced by internal strains may be stated in the mathematical language of the calculus and of geometry and in the engineering terminology of moment and shear, product of inertia, and moment of inertia. These theorems are, however, correlated by use of the principle of virtual work. While essentially a principle of geometry, the principle of virtual work has proved the most powerful tool ever applied to the analysis of statically indeterminate structures. Assume any rigid body acted on by a force and reactions and let it be required to determine the deflection Δ in the direction ab of any point a. Consider first the stretch--call it δ -- of any differential fibre. Now if we consider a unit hypothetical resistance to motion at a to produce a stress u in the fibre, then the external work will be $1 \times \Delta$ and the internal work will be $u \times \delta$, then,

$$\Delta = u\delta$$

Each fibre produces its independent deflection and hence,

$$\text{Total } \Delta = \Sigma u\delta$$

If the internal hypothetical stress resists the distortion, the internal work will be positive and the external movement opposite in direction to the hypothetical load. The same result follows from the more convenient rule which treats lengthening and tension as positive, and indicates a positive displacement in the direction of the hypothetical force when the algebraic summation, $\Sigma u\delta$ is positive.

Reference has been made to the stretch of a differential fibre. We might, without affecting the argument, have referred equally well to the shearing distortion of a differential cube, to the rotational distortion of a differential cylinder, to internal angular displacement, or to any internal distortion.

Also, it is possible to deal with external rotation at a instead of translation. The hypothetical unit force is then a unit moment. Then

$$\text{Rotation at } \underline{a} = \Sigma u\delta$$

in which u is the stress in each small particle due to a unit moment at a and is the distortion actually existing in the particle.

Since the equation of virtual work as thus stated involves no work done by the reactions from the hypothetical unit force, it must have been assumed that they do no work, and hence that they do not move. Or, it may be stated that these reactions do not have any motion which appears as a part of the described displacement, and hence the reactions to the hypothetical unit force exist at those points which fix the line with reference to which the desired displacement is measured.

Commonly this line is the line joining the supports of the structure, but it might equally well be the line joining any two points on the deflected structure or it might be any tangent to the deflected structure. In the first case the unit load or moment acts on a structure simply supported at the two given points; in the latter case, the structure is fixed, or cantilevered, at the point of tangency of the fixed tangent.

The General Principle of Virtual Work. This principle may be summarized as follows: Displacement, linear or angular, at any point in a structure is equal to the sum of the product of the internal distortions by imaginary internal resistances to such distortion produced by a unit hypothetical force of displacement, - i.e., a unit force at the point and in the direction of the displacement. The reactions to this hypothetical force of displacement will fix the reference by which the displacement may be measured.

The internal distortions may be due to any condition of stress and they may be either elastic or plastic, or they may be due to temperature or inaccurate workmanship. The external movement may be either of rotation or of translation. Rotation may be measured with reference to any line in the deflected structure. Translation may be measured in any direction with reference to any given line in the deflected structure and any given point on that line.

Virtual Work Applied to Trusses. The broad statement of the relations of external movements to internal distortions thus presented leads directly to the usual theorems dealing with slopes and deflections. In trusses it takes the form,

$$\Delta = \Sigma \frac{SuL}{AE}$$

except that u needs to be defined with reference to the fixed or reaction points. Thus, it is possible to find the deflection of any panel point A with reference to any other two panel points B and C by considering the hypothetical unit load applied at A, the truss being supported at B and C. Either B or C may be taken as fixed, the other point being for hypothetical loads, on rollers. Similarly, the relative deflection of opposite corners of any quadrilateral of a truss may be found by taking $\Sigma \delta u$ where u is the stress in any bar due to a unit load at one corner in the direction of the other when the truss is supported at the other corner. This is a familiar problem in internal indetermination. The trusses in Fig. 42 illustrate in a simple way some of the applications of this theorem.

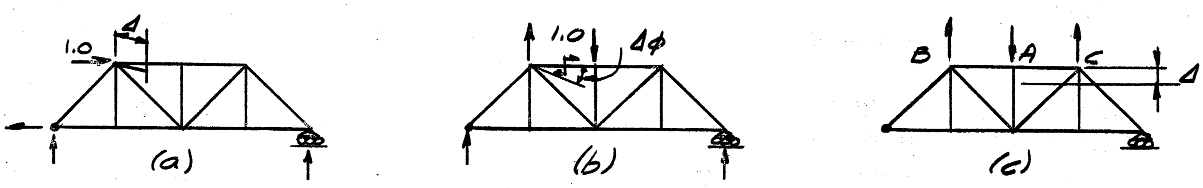

FIG. 42

Virtual Work Applied to Beams. In the case of beams the displacements due to moment only are:

$$\Delta = \int M' d\phi$$

in which, Δ is the external relative movement of any point, $d\phi$ is the rotation (produced by the bending moments) in any differential length, and M' is the imaginary moment over this differential length produced by a unit resistance to the external movement. See Fig. 43.

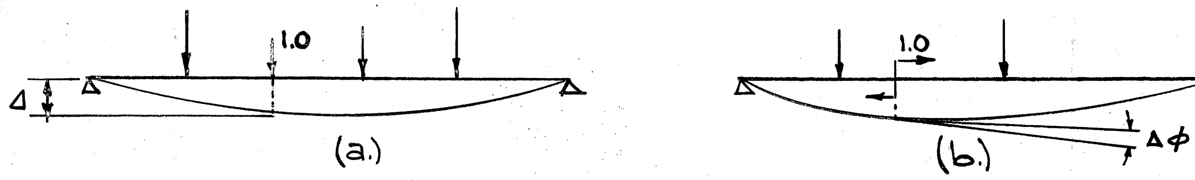

FIG. 43

If the deflection at any point from a chord of a structure, due to moments only, is to be found, the values, M', will be identical with the ordinates to an influence line for bending moment at this point on the chord. From this it follows that the deflections away from its original position produced by slightly curving a line will be the bending moments on the chord, treated as a beam, simply supported at its ends, due to the angle changes as loads. This is applicable in any case to a beam straight or curved, a floor line, or a truss chord, if the angle changes can be computed. From it the moment-area theorems of Mohr and Greene may be shown to follow directly, or they may be taken as direct corollaries of the principle of virtual work, as will be shown.

Moreover, the curve of displacements due to transverse distortions (shear slip, for example), may be computed as a moment diagram due to the displacement considered as a moment load on the beam. If Δ is the transverse distortion at A, Fig. 44, u is the shear at A due to a unit load at the point where the deflection is desired, B. But the shear at A due to a load at B is the same as the shear at B due to a load at A. The deflection diagram then is the curve of shears due to the displacement as a load at A, which is the same as the curve of moments due to the displacement as a moment load at A.

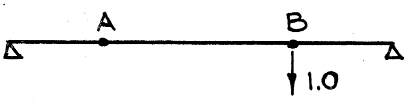

FIG. 44

This theorem would give a convenient method of including shearing distortions in deflection computation, if it were worth doing.

The angle change in a differential length of the axis of a beam, if plane sections remain plane, is the strain of an outer fibre, divided by the distance of this fibre from the neutral axis. The strain of the outer fibre is $\frac{f}{E}\,dl$, in which f is the intensity of stress in the outer fibre. Then

$$\Delta\phi = \frac{f}{Ey}\,dl$$

in which y is the distance from outer fibre to neutral axis.

In all the theorems dealing with moment-areas, the term, f/y, may be substituted for M/I. Sometimes this is convenient. It also makes possible a clearer understanding of the deflection of reinforced concrete beams and, in general, gives a clearer view of the assumptions that are involved in the analysis of indeterminate structures.

If the beam formula applies, then,

$$f = \frac{My}{I}\,;\quad d\phi = \frac{f}{Ey}\,dl = \frac{M}{EI}\,dl$$

Hence,

$$\Delta = \int M'M\,\frac{dl}{EI}$$

If applied to a part of a beam containing a frictionless hinge, this expression is evidently indeterminate, since at the hinge both M and I are zero. Moment-area theorems are, therefore, indeterminate for those parts of a beam in which a hinge occurs unless the change of angle at the hinge is computed independently.

The Reciprocal Theorem. If the expressions, $\sum \frac{SuL}{AE}$ for trusses and, $\int M'M \frac{dl}{EI}$ for beams, be used to find absolute displacements due to loads on the structure, the interchangeability of the terms, S and M, which are due to the loads, with u and M', respectively, which are due to the hypothetical external unit resistances to displacement, indicates at once the general theorem of reciprocal displacements. If "displacement" is interpreted in a general sense as either linear or angular, and "load" in a general sense as either force or moment, then in any structure the displacement at A due to a load at B has the same value as the displacement at B due to the same load at A, provided that both at A and B the force and the displacement are of the same nature, linear displacement corresponding to force along its line of action, and rotation corresponding to moment. This is illustrated in Fig. 44a.

See Meissner & Zwaluk

$\delta_{AB} = \delta_{BA}$

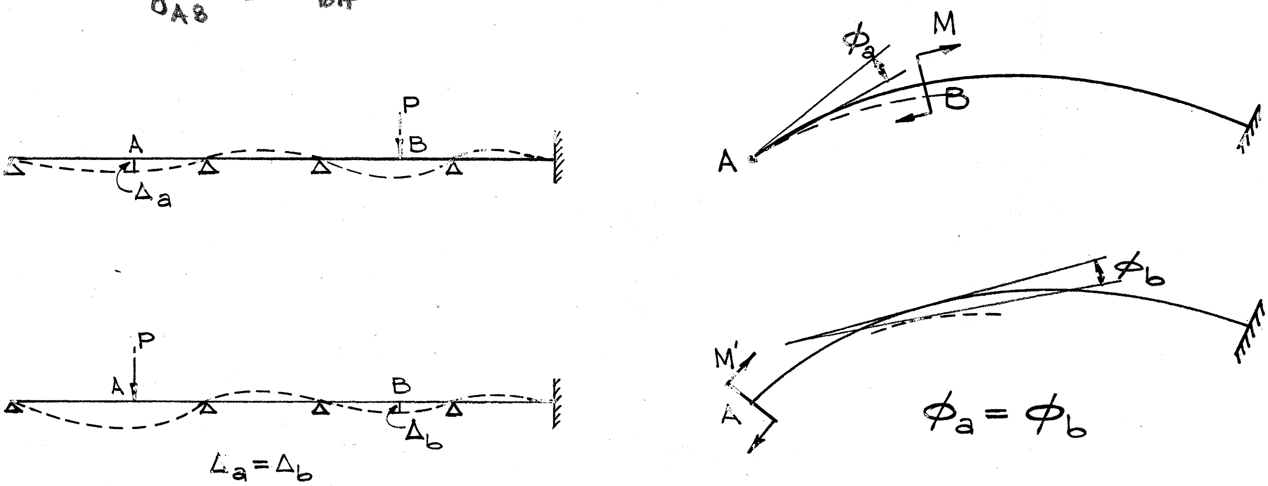

FIG. 44a.

This theorem is useful principally in interpreting as influence ordinates those displacements of the load line of a structure which would be produced by an imaginary unit internal distortion corresponding to the function for which the influence line is desired.

Area Moments. Greene's Theorems. If the relative rotation at one point on a beam referred to the tangent at another point is desired, consider the beam cantilevered from one point and loaded with a unit moment at the other point. Then, M' = constant = 1, and

$$\frac{MM'}{EI}\, dl = \frac{1}{EI} \times \text{(area under moment curve between the points of reference).}$$

If, however, the deflection of the second point with reference to a tangent from the first is wanted, consider the beam to be loaded and cantilevered from the first point, with a unit load at the second. Then M' equals the distance of each section from the second point and $(\frac{M}{EI}\, dl)\, M'$ is evidently $\frac{1}{EI}$ times the statical moment of the area of the moment curve about the second point.

Mohr's Theorems. In the case of a beam on fixed supports the change of slope at any point, A, relative to the line joining these supports may be found by applying a unit moment at the point, the beam being simply supported at the fixed points. The moment curve for M' will be found to be identical with an influence line for shear at A. Hence,

$$\frac{MM'}{EI}\, dl = \int \frac{M\, dl}{EI} \times \text{(shear at A due to a unit load at each section)}$$

$$= \int \text{(shears at A due to each } \frac{M\, dl}{EI}\text{)}$$

$$= \text{shear at A on a simple beam due to the } \frac{M}{EI} \text{ curve considered as a load.}$$

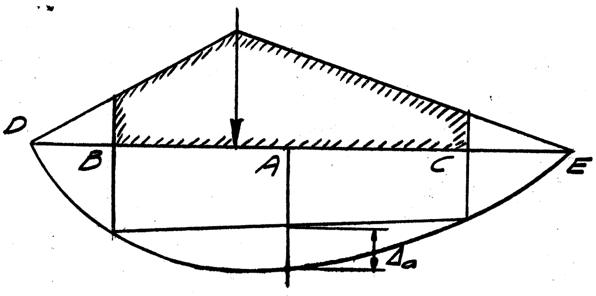

Fig. 45

For the deflection at A, on a simply supported beam, DE, apply a unit load at A. The curve for M' is then identical with an influence line for bending moment at A due to the M/EI curve considered as a load.

Evidently, any line on the beam other than DE may be considered as fixed. Thus, consider a beam bent as shown in Fig. 45. The deflection of point A with reference to the line BC - Δ_a in the diagram - is the bending moment at A produced by the M/EI curve between B and C on a simply supported beam, BC. This is sometimes a convenient theorem.

Also, the slope of the tangent at A with reference to the line, BC, is the shear at A on the simply supported beam, BC due to the M/EI - curve between B and C acting as a load.

Angle Changes in Trusses. Another interesting application of the principle of virtual work is in finding the change, $\Delta\alpha$ of any angle, α of a triangle (Fig. 46) due to the stresses in the three sides.

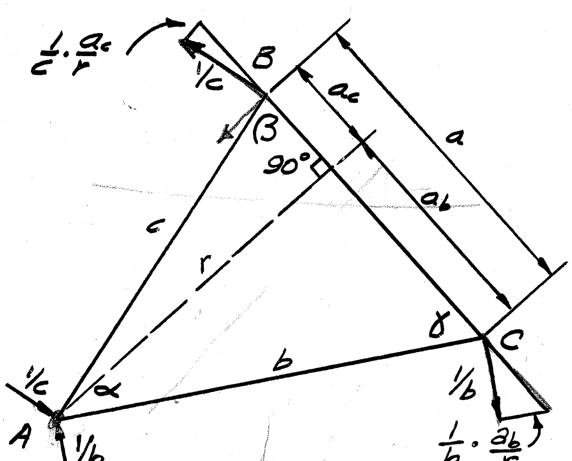

Fig. 46

Apply at A and C the elements of a unit couple, causing reactions such as to hold AB fixed in direction. Tension and increase of angle are taken as positive. Then,

$$\Delta\alpha = \Sigma\,\delta u = \Sigma\,\frac{fL}{E}\cdot u$$

$$E\,\Delta\alpha = \Sigma\, f L u$$

Resolving the forces at joints B and C,

$$u_a = \frac{1}{r}\,;\quad u_b = -\frac{1}{b}\cdot\frac{a_b}{r}\,;\quad u_c = \frac{1}{c}\cdot\frac{a_c}{r}$$

$$E\,\Delta\alpha = f_a\cdot a\cdot\frac{1}{r} - f_b\cdot b\cdot\frac{1}{b}\cdot\frac{a_b}{r} - f_c\cdot c\cdot\frac{1}{c}\cdot\frac{a_c}{r}$$

$$= f_a\cdot\frac{a}{r} - f_b\cdot\frac{a_b}{r} - f_c\cdot\frac{a_c}{r}$$

$$= \left[f_a - \left(f_b\frac{a_b}{a} + f_c\frac{a_c}{a}\right)\right]\frac{a}{r}$$

This is probably the most convenient formula for the angle changes. It may be modified to:

$$E\Delta\alpha = f_a \frac{a_b}{r} + f_a \frac{a_c}{r} - f_b \frac{a_b}{r} - f_c \frac{a_c}{r}$$

$$= (f_a - f_b)\frac{a_b}{r} + (f_a - f_c)\frac{a_c}{r}$$

$$= (f_a - f_b)\cot\gamma + (f_a - f_c)\cot\beta$$

This is the familiar formula used in secondary stress computations. Evidently, the method may be readily extended to the quadrilaterals which occur in sub-divided trusses and K-trusses. (See Chapter IX)

Slope-Deflection. The application of this method to the special case of a beam or part of a beam loaded only with moments at its two ends gives directly either the end slopes in terms of the end moments or the latter in terms of the former. In the first case given the loads on the beam, the reactions (end shears) are desired; in the second, the magnitude of the loads is to be determined for given reactions. By taking moments about B (Fig. 47), there follows:

$$\theta_a = \frac{2}{3} M_a \frac{L}{EI} - \frac{1}{3} M_b \frac{L}{2EI} = \frac{L}{6EI}(2M_a - M_b)$$

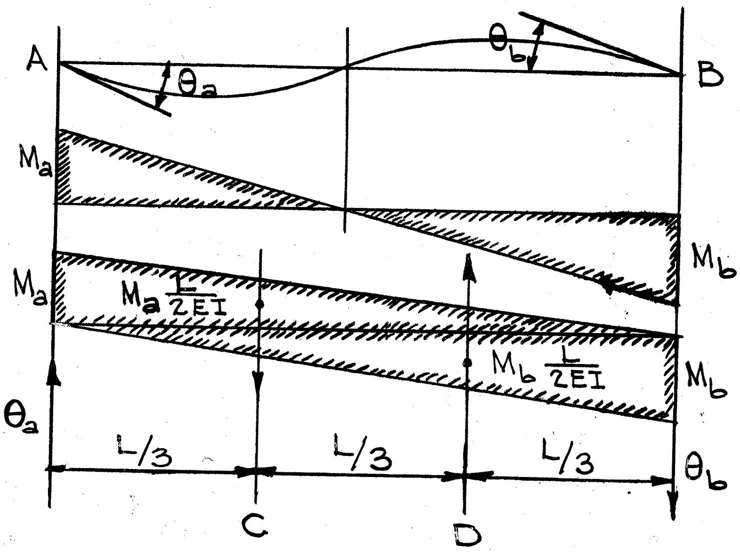

FIG. 47

By taking moments about D,

$$\frac{M_a \cdot L}{2EI} = 2\theta_a + \theta_b$$

or,

$$M_a = \frac{2EI}{L}(2\theta_a + \theta_b)$$

Both forms of the equation have been found convenient in evaluating secondary stresses. The moments due to the displacements of the joints may be found by the column analogy in which the displacement ψL, Fig. 48 is equivalent to a

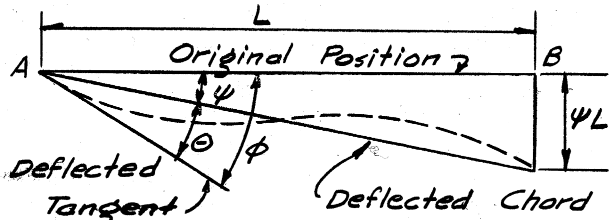

FIG. 48

bending moment about the parallel axis or, -

$$M = f = \frac{6M}{Ad} = \frac{6\psi L}{L/_{EI} \cdot L} = 6E\psi K$$

Combining this with the moments due to the rotations of the joints and noting that $\theta = \phi - \psi$ we have, -

$$M_a = 2EK(2\phi_a + \phi_b - 3\psi)$$

In these equations

- θ is the deflection of the tangent from the chord in the deflected structure - the primary angle in the analysis of continuous structures.
- ϕ is an angle of reference - the deflection of the tangent from the original position, or any assumed line of reference.
- ψ is likewise an angle of reference - the deflection of the chord from an assumed original position.

Angle Weights. The principle of virtual work furnishes directly a method of computing angle weights for determining the deflected load line for trusses. Assume that it is desired to draw the deflected load line for a unit load as shown in Fig. 49(a), that is, an influence line for horizontal reaction. The angle change at a may be computed by applying a unit moment resisting this angle change. This is effected by applying loads as shown in Fig. 49(b), acting at the points on the load line or floor as indicated by circles, and then computing $\sum \delta u$, in which, δ is the change in length of any bar due to the horizontal reaction, and u is the stress for the loading shown. These angle changes may then be treated as loads at the panel points and, when corrected for the deflection of the ends of the load line, the moment curve thus produced will have the shape of the influence line.

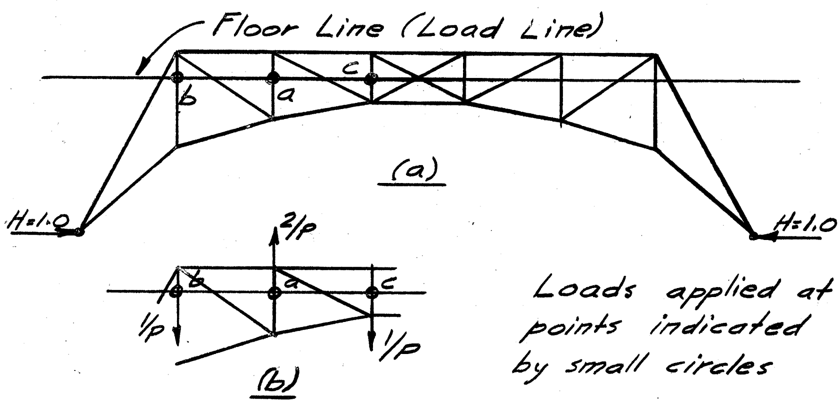

FIG. 49

This method presents advantages in directness in some cases as where the floor-beams frame into the verticals between upper and lower panel points.

Practical Considerations.

These illustrations indicate the broad usefulness of the general principle of virtual work. More definite applications of the principle will be made as various problems are discussed in succeeding chapters. The purely geometrical nature of all the correlated theorems needs emphasis. None of these theorems bears on the accuracy of the physical assumptions regarding the action of the structures.

To apply them to engineering structures, it is simply necessary either to show that for given values of the moments, shears, and thrusts, the strains can be predicted, or, if exact prediction is not possible, to determine by direct computation what error results from the inexactness. In analyzing indeterminate structures it is also to be noted that usually the relative and not the absolute values of these strains are in question.

Because of the definiteness of the moments, shears, and thrusts, strain measurements on statically determinate structures seem for this purpose more valuable and dependable than similar data derived from measurements on indeterminate frames. When it is established that, for any type of construction, these strains - or their relative values - can be definitely predicted, the whole theory of indeterminate stress analysis follows from the relations of the distortions as a matter of geometry. In order to be of real value in designing indeterminate structures, however, the load-strain relations must be those for conditions approaching failure and not merely those that exist at working stresses. An understanding of these facts will make clearer the limitations of the theory of elasticity as applied in much of the literature dealing with indeterminate structures of reinforced concrete, will make possible the application of more correct theory to such structures, and will give greater confidence in the results obtained by its use.

Internal Work - Least Work. It may be well here to distinguish virtual work, internal work and least work. The internal work done in a structure is evidently the continued product of the internal forces by the internal distortions. In the case of a beam the work of the moments is the sum of the products of the differential rotations by the bending moments. Each section of length ds has a relative rotation of its two ends $\frac{mds}{EI}$ and the work done in it is $\frac{m}{2}\frac{m}{EI}$ ds, if the load, and hence the moment, be gradually applied. The total internal work then, is $\frac{m}{2}\frac{m}{EI}$ ds, which is also, in the case of a prismatical beam, the statical moment about the base of the moment curve times 1/EI, provided we consider all statical moments as positive.

Now suppose that any structure has acting on it forces and reactions which satisfy the laws of statics. As a consequence of the law of conservation of energy the redundant reactions will so adjust themselves that the internal work stored in the structure must be a minimum. This is the principle of least work. Quantitatively this means that the first derivative of the internal work with reference to any internal stress or external force considered as redundant is zero.

The equations derived by least work are identical with those given by a direct consideration of the distortions, but, except in the case of trusses, they are likely to be unwieldy. The theorem has, however, great analytical value in some cases. One illustration of its use is the analysis of arches as given by Professor Spofford. Perhaps the best illustration is the proof of the so-called line of pressure method of arch analysis.

As distinguished from the internal work done in a structure, which we rarely have any occasion to compute, and from the principle of least work, which cannot be used to compute the internal work, though it deals with it, the principle of virtual work has nothing to do with the true internal work, but is simply a mental device for deducing certain purely geometrical relations. The internal distortions may be either elastic or plastic, and may or may not be accompanied by internal work.

compare with Salig & Smith

Summary of Principles.

An effort has been made to make this chapter brief. The important points are

First - Virtual work is a convenient tool in developing the theory of displacements.

Second - In its direct form, as used chiefly in trusses, we apply a unit dummy load corresponding to the desired displacement and then find the sum of the products of internal distortions times the stresses produced by the dummy load, $\Delta = \Sigma u\delta$

Third - In beams the equation $\Delta = \Sigma u\delta$ takes the form $\Delta = \Sigma m\phi$

Fourth - A corrolary of this is that the deflection from the original position due to slightly bending any line can be found as the bending moment on the chord as a simple beam due to the angle changes in the arc considered as loads, provided the ends do not move.

Fifth - In structural engineering these angle changes may be $m\frac{ds}{EI}$, $f\frac{ds}{Ey}$ or may be the angle changes occurring between adjacent chord members in a truss.

Sixth - From this follows very simply that

(a) The total change in slope along a beam equals

$$\int d\phi = \int mw = \int \frac{mds}{EI} = \int \frac{fds}{Ey}$$

(b) The deflection of a point on a beam away from a tangent to the beam is the statical moment about the point of the area under the curve of $d\phi$ or of mw or of m/EI or of f/EY

(c) The slope at any point of a flexed beam with reference to its line of supports is the shear at that point due to the m/EI curve (or its equivalent) as a load figured on a simple beam.

(d) The deflection at any point of a flexed beam with reference to its line of supports is the bending moment at that point due to the curve of angle changes (or its equivalent) as a load - which, of course, is the original theorem.

Seventh - Virtual work has nothing to do with the true internal work of the structure.

Eighth - The principle of least work - that the redundants so adjust themselves that the total internal work is a minimum consistent with statics - leads to exactly the same results as virtual work, but the two should not be confused.

Ninth - The fundamental relations here stated are entirely geometrical. Differences in stating them result from a search for convenience of application. The relations of external displacements to internal distortions are, however, not subject to dispute; what these internal distortions are is subject to dispute; what the significance of these distortions may be is even more subject to dispute.

The relations here presented could be elaborated historically, philosophically and mathematically. The history would be interesting and extensive and would include many famous names, Claperon, Menabrae, Castigliano, Clerk Maxwell, Mueller-Breslau, Fraenkel, Otto Mohr and a long list of others; certain philosophical aspects are evident; the mathematical elaboration may be - has been - very extensive. The important fundamentals are clear, the elaboration found in the literature results from efforts to restate these fundamentals in such a way as to reduce somewhat the tediousness of the computations involved. The value to the engineer of such modification is to be judged almost entirely on this basis.