

AN INVESTIGATION INTO THE THEORY OF RESONANCE TESTING

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Many vibrating systems are subject to damping forces arising from the structure of the system itself, and in the simpler theories this damping is assumed to be linear. The first part of the paper (§§1 to 5) is devoted to a discussion of systems with n degrees of freedom subject to linear damping. In the remainder of the paper various resonance testing techniques are assessed in the light of this general theory.

1. INTRODUCTION

Generally speaking, a resonance test may be carried out to determine

- (a) the principal modes of a system,
- (b) the associated natural frequencies, and
- (c) the damping.

There are several reasons for which these data may be required.

In aeronautical engineering, in particular, resonance tests provide essential information for flutter calculations. The tests are conducted

- (i) on the ground in still air (with models and aircraft suitably supported), and
- (ii) in flight (or in wind tunnels) with similar systems.

The purpose of tests of type (ii) is mainly to keep track of complex roots as airspeed is increased; i.e. they relate mainly to aspects (b) and (c) mentioned above. This paper is chiefly concerned with tests of type (i).

A common method of identifying modes is to excite the structure concerned and to measure the total forced vibration at enough points so as to be sure that all modes will display their resonant characteristics in at least one of these responses. A 'response curve' of total amplitude plotted against driving frequency is then constructed and the required information is then extracted from this curve. From a purely theoretical point of view, this 'Peak-Amplitude' method is inadequate and may be misleading. This was recognized by Kennedy & Pancu (1947) who proposed an alternative, and theoretically more attractive method.

The proposals made by Kennedy & Pancu are based solely upon theoretical arguments buttressed by numerical examples. Experiments are referred to in their paper but are not reported in detail, although the authors claim that their technique is workable. Since the appearance of their paper a good deal of work has been done on the experimental side—though unfortunately little has been published—and this seems largely to substantiate the claim of feasibility made by Kennedy & Pancu. Broadbent & Hartley (1958), who have examined the possibility of extending the technique to flight flutter tests (i.e. resonance tests of type (ii)) remark that 'this method appears to be the best way of estimating damping in ground resonance tests'. It should be mentioned, however, that the practical side of this, as of every other testing technique, does involve considerable difficulties. The phase differences between forces and responses, which form a vital part of the information required for the application of the technique, change very rapidly as the driving frequency passes through a natural frequency of the system. If they are to be measured then the frequency must be accurately controllable, particularly near resonances. Asher (1958) has shown how important it is to eliminate all harmonic distortion in the forcing when driving a lightly damped system at resonance.

While Kennedy & Pancu seem to have few misgivings about the practical feasibility of their technique, they express some doubts about the validity of their fundamental assumption that principal modes are not coupled by damping forces. Although they promise to present a more thorough theoretical study of the effect of damping, the writers are unable to trace any such publication.

The purpose of this present paper is to provide a theoretical background to the problem of resonance testing—not just to the method of Kennedy & Pancu. The first part of the paper is devoted largely to a discussion of the 'characteristic phase lag theory' of damped motion due to Fraeijs de Veubeke (1948, 1956). In the second part the various testing techniques are discussed in the light of the general theory, and particular care is taken to expose their underlying assumptions. (For notation used in this paper see p. 280).

2. THE DAMPED FORCED VIBRATION OF SYSTEMS HAVING ONE DEGREE OF FREEDOM

A full discussion of the damped forced motion of systems with one degree of freedom subject to viscous or hysteretic (structural) damping will be found in chapters VIII and IX of Bishop & Johnson (1960). It is generally agreed that, of the two models for the damping, the hysteretic one agrees more closely with experiment; it gives an energy dissipation per cycle which is independent of the frequency, whereas the viscous model gives an energy dissipation proportional to the frequency. Therefore it will generally be assumed in this paper that the systems considered are subject to hysteretic damping.

The equation of motion for a hysteretically damped system is

$$m\ddot{z} + (h/\omega)\dot{z} + kz = F \sin \omega t, \tag{2.1}$$

where z is the displacement in the single degree of freedom, and the solution is

$$z = \frac{F \sin(\omega t - \eta)}{\{h^2 + (k - m\omega^2)^2\}^{\frac{1}{2}}}, \tag{2.2}$$

where

$$\tan \eta = h/(k - m\omega^2). \tag{2.3}$$

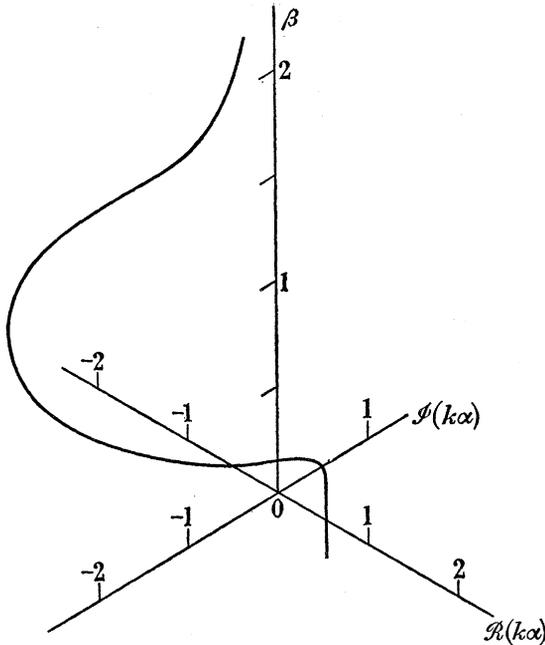


FIGURE 1

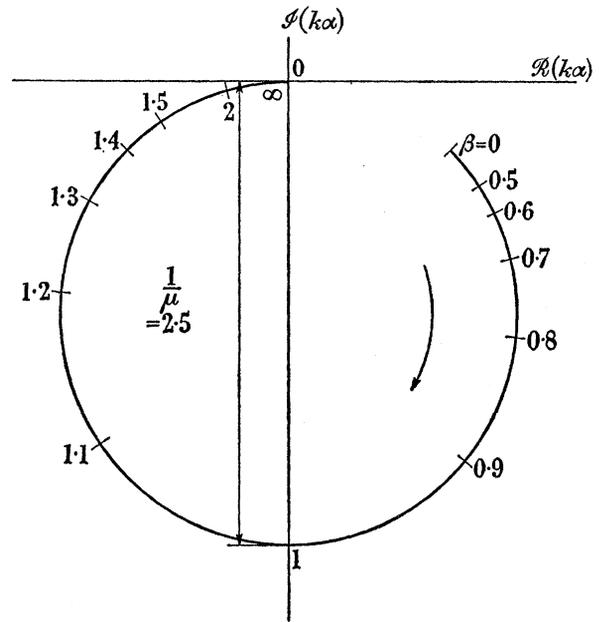


FIGURE 2

If the complex representation

$$z = \mathcal{I}(u) \tag{2.4}$$

is introduced then the equation satisfied by u is

$$m\ddot{u} + (h/\omega)\dot{u} + ku = F e^{i\omega t}. \tag{2.5}$$

It is convenient to introduce dimensionless parameters, β, μ , where

$$\omega_1 = \left(\frac{k}{m}\right)^{\frac{1}{2}}, \quad \beta = \frac{\omega}{\omega_1}, \quad \mu = \frac{h}{k}, \tag{2.6}$$

and in this notation the solution for u is

$$u = \alpha F e^{i\omega t}, \tag{2.7}$$

where the receptance α is given by

$$\alpha = \frac{\sin \eta e^{-i\eta}}{k} = \frac{e^{-i\eta}}{k\{\mu^2 + (1 - \beta^2)^2\}^{\frac{1}{2}}} \tag{2.8}$$

and

$$\tan \eta = \mu/(1 - \beta^2). \tag{2.9}$$

The variation of the complex receptance α with frequency can be represented in a number of ways. Thus figure 1 is a three-dimensional sketch of the dimensionless quantity $k\alpha$ for the system, corresponding to the damping $\mu = 0.4$. It will be noticed that, if the damping

of the system is decreased—and the damping factors encountered in practice are often less than 0.04—the curve becomes greater in diameter and less spread out along the axis. A clear mental picture of this curve will help the reader considerably in subsequent discussions.

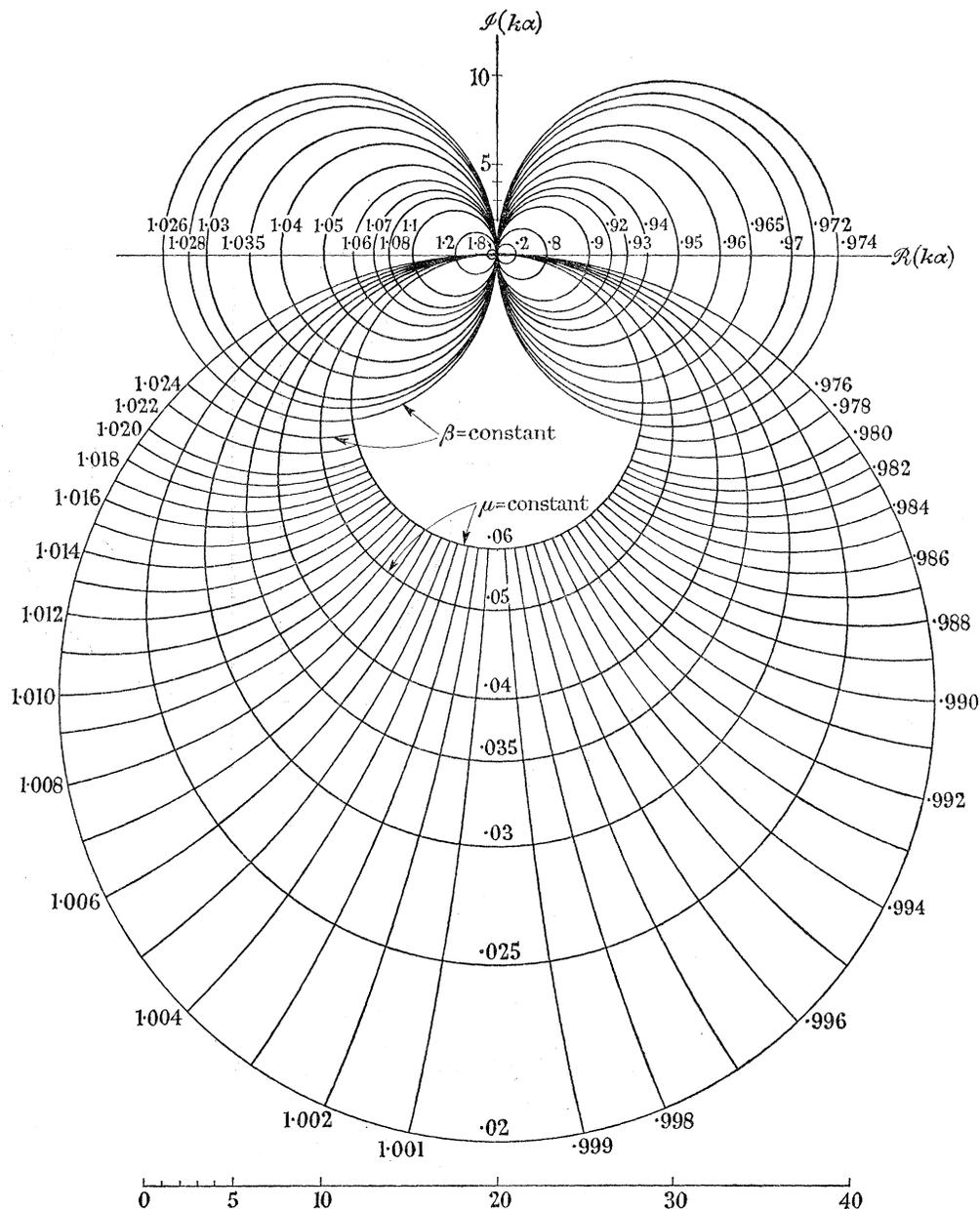


FIGURE 3

The plan of the three-dimensional curve of figure 1 is the locus of figure 2, which has the equation

$$x^2 + \left(y + \frac{1}{2\mu}\right)^2 = \left(\frac{1}{2\mu}\right)^2. \quad (2.10)$$

The marks on the circular locus have been made at equal increments of β . At the two ends of the locus—and it will be noticed that it is not quite a full circle—small distances along the locus represent considerable variations in β . Near resonance, however, the scale opens

out; the spacing is a maximum on the vertical axis where β has the value unity, corresponding to the natural frequency.*

As β (i.e. ω) varies, the locus of the end of the response vector $k\alpha$ is the circle given by equation (2.10). The locus of the same point as μ varies is another circle, namely

$$\left[x - \frac{1}{2(1-\beta^2)} \right]^2 + y^2 = \left[\frac{1}{2(1-\beta^2)} \right]^2. \quad (2.11)$$

These two equations may be used to construct the device shown in figure 3; this shows the sets of circles plotted for ranges of values of β and μ , and may be used to obtain the response to any hysteretic damping.

A third representation of the complex receptance is given in figure 9.2.6. of Bishop & Johnson (1960), in which the amplitude $k|\alpha|$ and phase η of $k\alpha$ are plotted against β separately for various values of μ .

The equation governing the forced vibration of a system with viscous damping is

$$m\ddot{z} + b\dot{z} + kz = F \sin \omega t. \quad (2.12)$$

The discussion of systems with viscous damping is very similar to that already given for hysteretically damped systems. A full discussion will be found in chapter VIII of Bishop & Johnson (1960), but in any case the distinction between the two types of damping is almost negligible for the damping magnitudes usually encountered (see Broadbent & Hartley 1958).

3. THE DAMPED FORCED VIBRATION OF SYSTEMS HAVING n DEGREES OF FREEDOM

The following discussion relates to the forced vibration of a system with hysteretic damping. The equation of motion to be considered is therefore

$$A\ddot{q} + (D/\omega)\dot{q} + Cq = \Phi \sin \omega t, \quad (3.1)$$

in which q is the column matrix of generalized displacements, A and C are the inertia and stability matrices, respectively, and D is the hysteretic damping matrix. Like the matrices A and C , the matrix D is a square, symmetric, real matrix. It is usually positive-definite, but may be positive semi-definite. Equation (3.1) may be compared with the equation for forced motion with viscous damping which is

$$A\ddot{q} + B\dot{q} + Cq = \Phi \sin \omega t, \quad (3.2)$$

where B is a positive-definite or semi-definite matrix. The reasons why the damping will be assumed to be hysteretic have been given in § 2. However, it will be found that whether the damping is viscous, hysteretic or a mixture of both, the analysis is very much the same; comparisons will be made at various points in the analysis.

In order to solve equation (3.1) for a general forcing matrix, a restricted form of the equation will first be considered. It will be enquired whether there are (real) forcing matrices Φ such that the displacements q_1, q_2, \dots, q_n are all in phase, though not necessarily in phase with the force. For such a set of displacements the matrix q will be of the form

$$q = \kappa \sin (\omega t - \theta), \quad (3.3)$$

* To be exact, if s denotes distance along the curve then it is $ds/d\beta^2$, not $ds/d\beta$ which has its maximum at $\beta = 1$. However, the proportional error incurred by locating the natural frequency at the maximum of $ds/d\beta$ is approximately $\mu^2/8$, and therefore negligible.

where κ is an unknown column matrix of amplitudes and θ is an unknown phase lag. Substitution of this trial solution into equation (3.1) yields the equations

$$\cos \theta (C - \omega^2 A) \kappa + \sin \theta D \kappa = \Phi, \quad (3.4)$$

$$-\sin \theta (C - \omega^2 A) \kappa + \cos \theta D \kappa = 0. \quad (3.5)$$

Provided that $\cos \theta \neq 0$, the second of these equations may be written in the form

$$[\tan \theta (C - \omega^2 A) - D] \kappa = 0. \quad (3.6)$$

For this equation to have a solution it is necessary and sufficient that

$$|\tan \theta (C - \omega^2 A) - D| = 0 \quad (3.7)$$

and this is an algebraic equation of order n in $\tan \theta$. For each root $\tan \theta_r$, there is a corresponding modal vector $\kappa^{(r)}$ satisfying the equation

$$[\tan \theta_r (C - \omega^2 A) - D] \kappa^{(r)} = 0. \quad (3.8)$$

This equation will define the 'shape' of the modal vector $\kappa^{(r)}$, but will leave it undefined to the extent of an arbitrary multiplying factor. It may be shown that the roots $\tan \theta_r$, and the corresponding modes $\kappa^{(r)}$ are all real.

These particular values of θ_r and $\kappa^{(r)}$ may now be substituted back into equation (3.4) to give the corresponding forcing vectors Φ , $\Sigma^{(r)}$ say. These will be given by

$$\cos \theta_r (C - \omega^2 A) \kappa^{(r)} + \sin \theta_r D \kappa^{(r)} = \Sigma^{(r)}. \quad (3.9)$$

The vector $\Sigma^{(r)}$ represents the forcing that is required if the system is to vibrate in the r th mode at the frequency ω .

The original trial solution (3.3) has led to the discovery of n modes of distortion corresponding to the given value of the frequency ω . These modes are quite different from the principal modes of the undamped system in that both the phase angle θ_r and the shape of any one of them varies with the frequency ω .

It should be noted that the modes κ depend only on the 'shape' of the damping and not on its intensity. In other words, if the damping is increased everywhere by a factor γ , so that the new damping matrix is γD , the modes will be unaffected. This follows from equation (3.8), which shows that $\tan \theta_r$ also increases by the factor γ so that the modal shape $\kappa^{(r)}$ determined by the equation remains the same.

This argument may be taken a stage further to give the interesting conclusion that the shape of a κ mode does not depend upon whether the damping is viscous or hysteretic. For the equation for viscous damping, corresponding to (3.7), will be

$$|\tan \theta (C - \omega^2 A) - \omega B| = 0 \quad (3.10)$$

so that if B has the same 'shape' as D , that is if

$$B = \gamma D, \quad (3.11)$$

then the roots of equation (3.10) will be

$$\tan \phi_r = \gamma \omega \tan \theta_r. \quad (3.12)$$

This means that the equation

$$[\tan \phi_r (C - \omega^2 A) - \omega B] \kappa^{(r)} = 0 \quad (3.13)$$

is exactly the same as (3.8), so that the modes $\kappa^{(r)}$ are the same as before.

The way in which the κ and the θ are labelled and the scale factor is assigned to $\kappa^{(r)}$ deserves to be considered further. When equation (3.6) was derived from equation (3.5) it was assumed that $\cos \theta \neq 0$. If, however, $\cos \theta = 0$, equation (3.5) becomes

$$(C - \omega^2 A) \kappa = 0 \quad (3.14)$$

and the condition for κ to be non-trivial is that

$$|C - \omega^2 A| = 0. \quad (3.15)$$

This means that ω must be a natural frequency of the (undamped) system. If then $\omega = \omega_s$, the κ mode corresponding to this value of ω and the solution $\cos \theta = 0$ may be identified with the s th principal mode $\Psi^{(s)}$. If the θ solution and the corresponding value of κ are labelled θ_s and $\kappa^{(s)}$, respectively, then one may write

$$\theta_s = \frac{1}{2}\pi, \quad \kappa^{(s)} = \Psi^{(s)}, \quad \text{when } \omega = \omega_s. \quad (3.16)$$

For this value of ω there will also be $n-1$ other $\kappa^{(r)}$ modes corresponding to the remaining $n-1$ roots $\tan \theta_r$ of equation (3.7).

Equation (3.16) may be used to give a consistent way of labelling the θ and the κ for values of ω other than the natural frequencies. Each of the roots θ of equation (3.7) is a continuous function of ω , so that $\theta = \theta(\omega)$. Equation (3.8) shows that

$$\tan \theta_r = \frac{\kappa^{(r)} D \kappa^{(r)}}{\kappa^{(r)} (C - \omega^2 A) \kappa^{(r)}}, \quad (3.17)$$

so that, when $\omega = 0$, θ_r is a small positive angle.

As ω becomes larger and approaches ω_1 , one of the roots $\theta(\omega)$ will approach the value $\frac{1}{2}\pi$; let this root be labelled $\theta_1(\omega)$. For values of ω greater than ω_1 the denominator of the expression for $\tan \theta_1$ will have negative values while the numerator will still be positive; $\theta_1(\omega)$ will thus be greater than $\frac{1}{2}\pi$. As ω is increased indefinitely, $\theta_1(\omega)$ will approach the value π . The remaining $n-1$ roots $\theta_s(\omega)$ may be labelled in a similar way; $\theta_s(\omega)$ is that root which has the value $\frac{1}{2}\pi$ when $\omega = \omega_s$.

The behaviour of the phase angle θ_r may be contrasted with the behaviour of the corresponding angle ϕ_r for a system with viscous damping. Equation (3.13) shows that $\tan \phi_r$ (which, together with $\kappa^{(r)}$, is real) is given by

$$\tan \phi_r = \frac{\omega \kappa^{(r)} B \kappa^{(r)}}{\kappa^{(r)} (C - \omega^2 A) \kappa^{(r)}}. \quad (3.18)$$

This shows that, when $\omega = 0$, $\tan \phi_r$ is not just small, but zero. There is also a contrast between θ_r and ϕ_r when ω is large. For large values of ω , $\tan \phi_r$ decreases with $1/\omega$, whereas $\tan \theta_r$ decreases with $1/\omega^2$. The angle θ_r therefore approaches π faster than ϕ_r .

The difference between heavily damped and lightly damped systems lies mainly in the behaviour of the phase angles θ_r (or, with viscous damping, ϕ_r). In a lightly damped system the angles θ_r are either small or near π at off-resonant frequencies so that, in a sense, the modes cluster together and the response is either in phase or antiphase with the excitation. At a resonant frequency one mode detaches itself, so to speak, and, over a very small range of frequency sweeps from near zero to near π . In a heavily hysteretically damped system the angles $\theta_r(\omega)$ have greater initial values and sweep through the natural frequencies

more gradually. However, even in a heavily damped system the angles θ_r approach π quite soon after passing through the natural frequency.

The way in which the angles θ_r have been labelled may be carried over to the modes—at any frequency ω , the shape of the s th mode is given by the solution $\kappa^{(s)}(\omega)$ of

$$[\tan \theta_s(C - \omega^2 A) - D] \kappa^{(s)} = 0. \quad (3.19)$$

The scale factor for $\kappa^{(s)}(\omega)$ may also be chosen consistently. Instead of choosing one factor for each value of ω separately, one may choose a scale factor for $\Psi^{(s)}$ and then choose that for $\kappa^{(s)}(\omega)$ so that each component of $\kappa^{(s)}(\omega)$ is a continuous function of ω equal to the corresponding component of $\Psi^{(s)}$ when $\omega = \omega_s$. In particular, if there is a component of $\kappa^{(s)}(\omega)$ which is non-zero for all values of ω then the mode may be scaled so that that component is unity for all values of ω . When the modes are scaled in either of these ways the transition to the undamped system ($D = 0$) means

$$\theta_r(\omega) = 0, \quad \kappa^{(r)}(\omega) = \Psi^{(r)}. \quad (3.20)$$

The existence of modes of this general type appears to have been pointed out first by Fraeijs de Veubeke (1948, 1956). The modes $\kappa^{(r)}$ will be referred to as ‘forced modes’ while the sets of amplitudes $\Sigma^{(r)}$ will be termed ‘forced modes of excitation’ in what follows.

It may be shown that the modes satisfy the orthogonality conditions

$$\left. \begin{aligned} \kappa^{(s)}(C - \omega^2 A) \kappa^{(r)} &= 0 \\ \kappa^{(s)} D \kappa^{(r)} &= 0 \end{aligned} \right\} \quad (r \neq s), \quad (3.21)$$

and these conditions imply that

$$\kappa^{(r)} \Sigma^{(s)} = 0 = \kappa^{(s)} \Sigma^{(r)} \quad (r \neq s). \quad (3.22)$$

Since the column vectors $\Sigma^{(1)}, \Sigma^{(2)}, \dots, \Sigma^{(n)}$ are linearly independent (see Fraeijs de Veubeke 1948) any set of forces $\Phi \sin \omega t$ —that is, any set of forces which are all in phase—may be expressed in the form

$$\Phi \sin \omega t = \{\lambda_1 \Sigma^{(1)} + \lambda_2 \Sigma^{(2)} + \dots + \lambda_n \Sigma^{(n)}\} \sin \omega t. \quad (3.23)$$

The constants $\lambda_1, \lambda_2, \dots, \lambda_n$ in this equation may be found by premultiplying the equation by $\kappa^{(r)}$ and using the result (3.22). By this means it is found that

$$\lambda_r = \frac{\kappa^{(r)} \Phi}{\kappa^{(r)} \Sigma^{(r)}}. \quad (3.24)$$

The excitation $\Sigma^{(r)} \sin \omega t$ gives the displacement $\kappa^{(r)} \sin(\omega t - \theta_r)$, and therefore the excitation $\Phi \sin \omega t$ gives the displacement

$$q = \sum_{r=1}^n \left\{ \frac{\kappa^{(r)} \kappa^{(r)}}{\kappa^{(r)} \Sigma^{(r)}} \sin(\omega t - \theta_r) \right\} \Phi. \quad (3.25)$$

This may be expressed in the alternative form

$$q = \sum_{r=1}^n \left\{ \frac{\kappa^{(r)} \kappa^{(r)} \sin \theta_r}{\kappa^{(r)} D \kappa^{(r)}} \sin(\omega t - \theta_r) \right\} \Phi. \quad (3.26)$$

Algebraically, the right-hand side of this equation represents a square symmetric matrix (each of whose elements is represented by n terms added together) post-multiplied by a column matrix Φ . It is of interest to extract the general term; suppose that it is wished to

measure the generalized displacement q_i which is caused by a generalized forced $\Phi_j \sin \omega t$. This is given by

$$q_i = \sum_{r=1}^n \left\{ \frac{\kappa_i^{(r)} \kappa_j^{(r)} \sin \theta_r}{\kappa^{(r)} D \kappa^{(r)}} \sin (\omega t - \theta_r) \right\} \Phi_j. \tag{3.27}$$

The response to a set of harmonic forces which are not all in phase may be obtained by using the complex exponential. If the vector of generalized forces is

$$\begin{bmatrix} \Phi_1 \sin (\omega t + \epsilon_1) \\ \Phi_2 \sin (\omega t + \epsilon_2) \\ \dots\dots\dots \\ \Phi_n \sin (\omega t + \epsilon_n) \end{bmatrix},$$

then the response vector q is given by

$$q = \mathcal{J}(u), \tag{3.28}$$

where u is the solution of the equation

$$A\ddot{u} + (D/\omega) \dot{u} + Cu = \Phi_0 e^{i\omega t} \tag{3.29}$$

and Φ_0 is the complex force vector

$$\Phi_0 = \begin{bmatrix} \Phi_1 e^{i\epsilon_1} \\ \Phi_2 e^{i\epsilon_2} \\ \dots\dots\dots \\ \Phi_n e^{i\epsilon_n} \end{bmatrix}. \tag{3.30}$$

The trial solution

$$u = \kappa e^{i(\omega t - \theta)} \tag{3.31}$$

leads to the general result

$$u = \sum_{r=1}^n \left\{ \frac{\kappa^{(r)} \kappa_j^{(r)} \sin \theta_r e^{-i\theta_r}}{\kappa^{(r)} D \kappa^{(r)}} \right\} \Phi_0 e^{i\omega t}. \tag{3.32}$$

The only major difference between the present case and that dealt with above is that the numerical multipliers λ may now have to be taken as complex. The cross-receptance α_{ij} which gives the displacement q_i due to the excitation $\Phi_j e^{i\omega t}$ is now

$$\alpha_{ij} = \sum_{r=1}^n \frac{\kappa_i^{(r)} \kappa_j^{(r)} \sin \theta_r e^{-i\theta_r}}{\kappa^{(r)} D \kappa^{(r)}}. \tag{3.33}$$

This is one element of the square symmetric receptance matrix α . The expression for α_{ij} corresponding to equation (2.8) is

$$\alpha_{ij} = \sum_{r=1}^n \frac{\kappa_i^{(r)} \kappa_j^{(r)} e^{-i\theta_r}}{\{[\kappa^{(r)} D \kappa^{(r)}]^2 + [\kappa^{(r)} (C - \omega^2 A) \kappa^{(r)}]^2\}^{\frac{1}{2}}}. \tag{3.34}$$

4. DAMPED PRINCIPAL CO-ORDINATES

The principal co-ordinates of a system are the natural co-ordinates to use when the system is undamped.* However, for a damped system they do not necessarily introduce any simplification; but it is possible to introduce other co-ordinates, which we shall call

* A discussion of the undamped vibration of systems with n degrees of freedom will be found in many standard works, for example, Kármán & Biot (1940) and Bishop & Johnson (1960).

'damped principal co-ordinates'. At any moment the motion of the system may be split up into its component motions in each of the forced modes; thus the vector q may be written

$$q = \nu_1 \kappa^{(1)} + \nu_2 \kappa^{(2)} + \dots + \nu_n \kappa^{(n)}. \quad (4.1)$$

The multipliers $\nu_1, \nu_2, \dots, \nu_n$ are the damped principal co-ordinates. These co-ordinates need to be normalized in some way since the vectors $\kappa^{(r)}$ are undefined to the extent of an arbitrary multiplying factor.

The damped principal co-ordinates are fundamentally different from the ordinary principal co-ordinates. If an undamped system is made to vibrate in its r th principal mode, then the displacement vector q is given by

$$q = p_r \Psi^{(r)} = \Pi_r \Psi^{(r)} \sin \omega t. \quad (4.2)$$

If now the frequency is allowed to vary, then (provided that Π_r is constant) q is constant in amplitude—that is, each component of q is constant in amplitude. But now suppose that in a damped system q is given by

$$q = \nu_r \kappa^{(r)} = \tau_r \kappa^{(r)} \sin \omega t, \quad (4.3)$$

then, even if τ_r is kept constant, the amplitude of q will vary as ω varies.

If a square matrix K is introduced which has the κ as columns, then equation (4.1) may be written

$$q = Kv. \quad (4.4)$$

The inertia, stability and damping matrices for the damped principal co-ordinates will be

$$\bar{A} = K'AK, \quad \bar{C} = K'CK, \quad \bar{D} = K'DK \quad (4.5)$$

and the orthogonality relations (3.21) show that $\bar{C} - \omega^2 \bar{A}$ and \bar{D} are both diagonal matrices. It must be remembered, however, that neither \bar{C} nor \bar{A} is in general diagonal.

The equation for forced motion is

$$A\ddot{q} + (D/\omega) \dot{q} + Cq = \Phi \sin \omega t, \quad (4.6)$$

and the corresponding complex equation is

$$A\ddot{u} + (D/\omega) \dot{u} + Cu = \Phi e^{i\omega t}. \quad (4.7)$$

If

$$v = \mathcal{F}(w) \quad (4.8)$$

so that

$$u = Kw, \quad (4.9)$$

then the equation for w may be obtained by substituting Kw for u in equation (4.7) and pre-multiplying by K' . This gives

$$\bar{A}\ddot{w} + (\bar{D}/\omega) \dot{w} + \bar{C}w = K'\Phi e^{i\omega t} = We^{i\omega t}, \quad (4.10)$$

for which the solution is

$$w_r = \frac{W_r e^{i\omega t}}{\bar{c}_r - \omega^2 \bar{a}_r + i\bar{d}_r}, \quad (4.11)$$

where the notation used is

$$\bar{C} - \omega^2 \bar{A} = \text{diag}(\bar{c}_r - \omega^2 \bar{a}_r), \quad (4.12)$$

$$\bar{D} = \text{diag} \bar{d}_r. \quad (4.13)$$

Equation (4.11) may be written

$$w_r = \alpha_r W_r e^{i\omega t}. \quad (4.14)$$

Equation (3.17) gives

$$\tan \theta_r = \frac{\bar{d}_r}{\bar{c}_r - \omega^2 \bar{a}_r}, \tag{4.15}$$

so that

$$\alpha_r = \frac{\sin \theta_r e^{-i\theta_r}}{\bar{d}_r} \tag{4.16}$$

Equations (4.15) and (4.16) are of exactly the same form as equations (2.9) and (2.8).

There is thus a complete mathematical analogy between these results and those found in § 2. The behaviour of the system is built up from its behaviour in each of its damped principal modes, that in each mode being analogous to the response of a single-degree-of-freedom system.

From the point of view of theoretical analysis, damped principal co-ordinates appear to have much to commend them. The question of how useful they might be in flutter theory appears to be worth some investigation.

5. NO COUPLING BETWEEN PRINCIPAL CO-ORDINATES

It is shown in standard treatises, for example, Rayleigh (1894) and Bishop & Johnson (1960), how a set of generalized co-ordinates q is related to a set of principal co-ordinates p . If X is the square matrix having the principal modes $\Psi^{(r)}$ as columns then the relation between the co-ordinates q_1, q_2, \dots, q_n and the principal co-ordinates p_1, p_2, \dots, p_n may be expressed by the matrix equation

$$q = Xp. \tag{5.1}$$

The equation governing the principal co-ordinates in damped vibration may be found by pre-multiplying equation (3.1) by X' and using equation (5.1). It is

$$L\ddot{p} + (H/\omega)\dot{p} + Np = \Xi \sin \omega t, \tag{5.2}$$

where

$$L = X'AX, \quad N = X'CX \tag{5.3}$$

are the diagonal matrices

$$\left. \begin{aligned} L &= \text{diag}(a_1, a_2, \dots, a_n), \\ N &= \text{diag}(c_1, c_2, \dots, c_n), \end{aligned} \right\} \tag{5.4}$$

and

$$H = X'DX, \quad \Xi = X'\Phi. \tag{5.5}$$

The theory of § 3 is still valid provided that L, H and N are used instead of A, D and C respectively (and, of course, p and Ξ instead of q and Φ).

The determinantal equation giving the quantities $\tan \theta_r$ is now

$$|\tan \theta(N - \omega^2 L) - H| = 0, \tag{5.6}$$

and, since L and N are diagonal, this is

$$\left| \begin{array}{cccc} \tan \theta(c_1 - \omega^2 a_1) - h_{11} & -h_{12} & \dots & -h_{1n} \\ -h_{21} & \tan \theta(c_2 - \omega^2 a_2) - h_{22} & \dots & -h_{2n} \\ \dots & \dots & \dots & \dots \\ -h_{n1} & -h_{n2} & \dots & \tan \theta(c_n - \omega^2 a_n) - h_{nn} \end{array} \right| = 0. \tag{5.7}$$

If H is not diagonal then the roots of this equation are not easily found. But if H is diagonal and

$$H = \text{diag}(d_1, d_2, \dots, d_n), \tag{5.8}$$

then

$$\tan \theta_r = d_r / (c_r - \omega^2 a_r). \tag{5.9}$$

The forced modes κ , which are now the solutions of

$$[\tan \theta_r(N - \omega^2 L) - H] \kappa^{(r)} = 0, \quad (5.10)$$

are the same as the principal modes because $\kappa_r^{(r)}$ —the r th element in $\kappa^{(r)}$ —is the only non-zero element in $\kappa^{(r)}$. Moreover, the modes are no longer dependent on driving frequency.

In this case the damped principal co-ordinates are plainly identical with the ordinary principal co-ordinates. The matrices \bar{A} , \bar{C} , \bar{D} are the same as L , N and H while X and K are also the same.

When H is diagonal* it may be shown that the receptance α_{ij} giving the (complex) response at q_i due to a harmonic force at q_j is given by the comparatively simple expression

$$\alpha_{ij} = \sum_{r=1}^n \frac{(\partial q_i / \partial p_r) (\partial q_j / \partial p_r) \sin \theta_r e^{-i\theta_r}}{d_r}. \quad (5.11)$$

If the notation

$$\mu_r = \frac{d_r}{c_r}, \quad \beta_r = \frac{\omega}{\omega_r} \quad (5.12)$$

is introduced, α_{ij} may be written in the alternative form

$$\alpha_{ij} = \sum_{r=1}^n \frac{(\partial q_i / \partial p_r) (\partial q_j / \partial p_r) e^{-i\theta_r}}{c_r \{\mu_r^2 + (1 - \beta_r^2)^2\}^{\frac{1}{2}}}, \quad (5.13)$$

which is exactly analogous to equation (2.8).

6. A STATEMENT OF THE PROBLEM OF RESONANCE TESTING

The first part of this paper has been concerned with the nature of the forced damped vibration of a system with n degrees of freedom. For it is the authors' thesis that the various techniques used for resonance testing can be evaluated only if the motion of the system under test is properly understood.

The problem facing the tester is as follows. He has before him a continuous body such as an aeroplane and he wishes to discover its dynamical properties when it is vibrating in still air. The first step is to treat the body as one possessing finite freedom rather than as a continuous body. The properties of a continuous body can be simulated to any desired accuracy by a system possessing a finite number, n , of degrees of freedom, provided only that n is large enough. Once the tester has replaced the body by a finite-freedom system the data he requires are the inertia, stability and damping matrices of the system; these, or some other quantities giving equivalent information, are the quantities he will need in any subsequent flutter analysis.

In the first part of this paper the vibration of a system has been deduced from its matrices A , C and D ; the problem in resonance testing is the inverse of this, to deduce the matrices from the vibration. The matrices A , C and D cannot be measured directly—one might say that they are not 'observable'. They have to be deduced from observable and measurable quantities, i.e. from the natural frequencies, modal shapes, and dampings of the system. Even these are extremely difficult to identify because, as has been shown in §§ 3 and 4, the

* Caughey (1960) has given very general conditions under which A , C , D may be reduced to diagonal form simultaneously; the conditions are shown to be sufficient, but not necessary.

vibration of a damped system has not the clear-cut nature which undamped vibration has. The result is that some of the methods used are rather indirect, and depend on comparatively obscure properties of a vibrating system.

The various techniques which will now be described may be differentiated according to their approaches to three main problems. The problems are:

(1) The manner in which the body is made to vibrate and the means which are used to achieve it.

(2) The physical quantities which are measured, and the ways in which the experimental data are displayed, i.e. the graphs which are plotted.

(3) The ways in which the graphs are analyzed.

Corresponding to these three aspects of the problem, there will be three kinds of errors. Errors may arise because the exciter is faulty; the most common faults are concerned with frequency control and distortion. Then there may be errors arising from the measuring equipment. Finally, there may be errors in the analysis.

It is the writers' belief that there is a marked need for critical investigation of several aspects of resonance testing. This seems particularly apparent on the side of experimentation. A number of problems become apparent during the discussion which follows.

7. THE PEAK-AMPLITUDE METHOD

In this method of resonance testing, the structure is excited harmonically and the amplitudes at various points are measured. Total amplitude is then plotted against driving frequency for various points. The graphs may resemble that shown in figure 4. Difficulties arising from the possibility of placing a vibrator or a pick-up at a node may be overcome by using several pick-ups and by changing the position of the exciter. This is the most widely used method. From a theoretical standpoint it is rather crude since it is difficult to analyze the results exactly. Briefly, the drawbacks of the method are that too little is measured, and what is measured is displayed badly.

To evaluate the method it is necessary to refer back to the theory given in §§ 3 and 5. Suppose that the excitation is applied at a generalized co-ordinate q_j and that the amplitude is measured at the generalized co-ordinate q_i . The quantity which is being measured is the modulus of the complex receptance α_{ij} given in equation (3.33), that is

$$|\alpha_{ij}| = \left| \sum_{r=1}^n \frac{\kappa_i^{(r)} \kappa_j^{(r)} \sin \theta_r e^{-i\theta_r}}{\kappa^{(r)} D \kappa^{(r)}} \right|. \quad (7.1)$$

This equation shows that the receptance α_{ij} is made up from contributions from each of the forced modes, and that each of these contributions has its own phase lag behind the excitation. This method, by measuring $|\alpha_{ij}|$ (a real quantity obtained by lumping together n complex quantities) takes little or no account of the various phase lags. This is a serious drawback because the change in the phase lag can be most informative, especially near the natural frequencies. It will be remembered that θ_r changes rapidly from a small angle to almost π in a relatively small interval about the frequency ω_r . The failure to register these changes of phase lag is particularly unfortunate as it is just in the neighbourhood of the natural frequencies that the amplitude curve is likely to be inaccurate because of experimental factors.

The expression in equation (7.1) is that for the amplitude if no assumptions are made about the damping except that it is hysteretic. The problem facing the analyst is to extract from the various amplitude-frequency curves the values of the natural frequencies, the mode shapes and the damping matrix D .

(a) *The natural frequencies*

The natural frequencies are usually identified as the values of ω at which the peaks are attained. In addition to the considerable difficulties, both experimental and otherwise, involved in locating the peaks of the curve, there is also a minor analytical difficulty. This

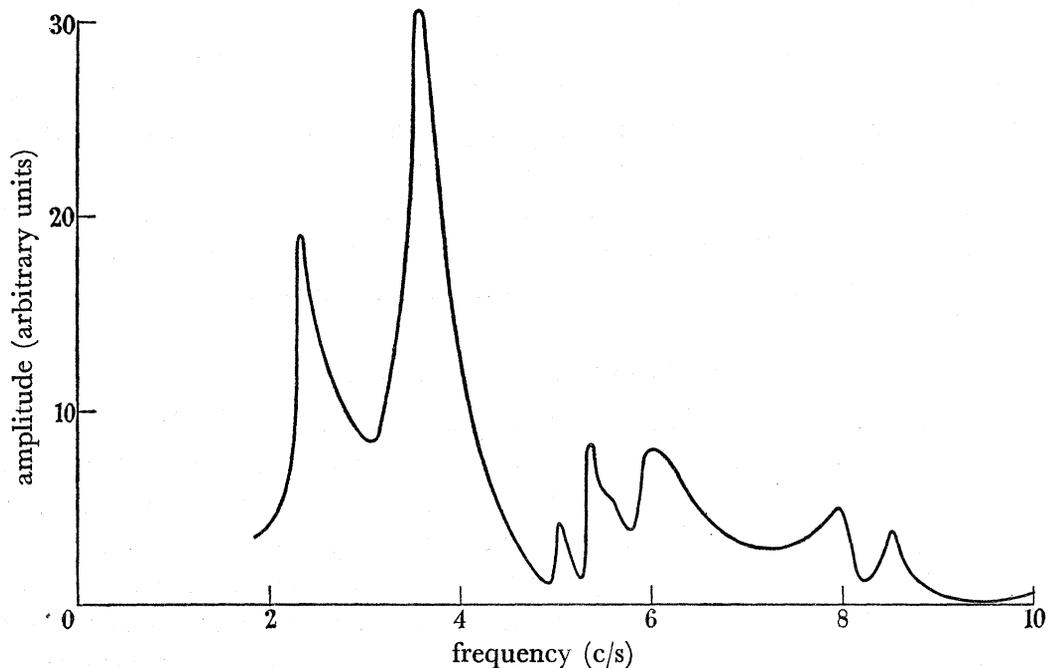


FIGURE 4

is simply that the peaks do not occur exactly at the natural frequencies, but at frequencies displaced one way or another from them. Thus even the r th term in the receptance, which may be written as

$$\frac{\kappa_i^{(r)} \kappa_j^{(r)} e^{-i\theta_r}}{\{[\kappa^{(r)} D \kappa^{(r)}]^2 + [\kappa^{(r)} (C - \omega^2 A) \kappa^{(r)}]^2\}^{\frac{1}{2}}},$$

does not attain its maximum amplitude at ω_r , and in addition there is the effect of the much smaller contribution from the other terms in the expression. It seems to be difficult to determine, even for a simple system with two degrees of freedom, which way the r th peak frequency will be displaced from ω_r , either owing to the deviation of $\kappa^{(r)}$ from $\Psi^{(r)}$, or owing to the variation in the contributions from other modes.

If the damping matrix is assumed to be diagonal when referred to the principal coordinates, the r th term in the receptance will be

$$\frac{(\partial q_i / \partial p_r) (\partial q_j / \partial p_r) e^{-i\theta_r}}{c_r \{\mu_r^2 + [1 - (\omega / \omega_r)^2]^2\}^{\frac{1}{2}}}.$$

The peak due to this term will be at ω_r , but there will still be the effect of the other terms in the receptance and again it seems difficult to determine which way they will displace

the peak. It should be emphasized that whether the damping matrix is assumed to be diagonal or not the analytical errors will usually be considerably smaller than the experimental errors involved in locating the peaks.

(b) *The damping coefficients*

We shall find that it is convenient to discuss how the damping coefficients are found before describing how the shapes of the principal modes are determined.

It is virtually impossible to obtain any reliable knowledge about the damping in the system if the general expression (3.34) for the receptance α_{ij} is used. It was shown in § 5 that the difference between (3.34) and the simpler expression

$$\alpha_{ij} = \sum_{r=1}^n \frac{(\partial q_i / \partial p_r) (\partial q_j / \partial p_r) \sin \theta_r e^{-i\theta_r}}{d_r} \quad (7.2)$$

is caused solely by the presence of off-diagonal terms in the damping matrix related to the principal co-ordinates.

Since neither the diagonal nor the off-diagonal terms are known at the start it is reasonable to assume that the damping is diagonal; otherwise it is impossible to make any headway at all. It has not been sufficiently emphasized in the past that when this assumption is made it is solely for convenience in analyzing the results, and is not based on physical arguments.

When the damping matrix is diagonal there is a single damping factor, d_r , corresponding to each mode. Even so, it is difficult to estimate these damping factors without making further assumptions. The simplest method is to assume that the peak around ω_r arises solely from the r th term in the receptance α_{ij} . If this is assumed then the damping may be calculated from the sharpness of the peak. This assumption is equivalent to the assumption that the three-dimensional receptance curve is of the form sketched in figure 5 and consists of a number of loops attached to the ω axis. This diagram may be contrasted with figure 6 which gives a more realistic representation of a three-dimensional receptance curve. It may be shown that if this assumption is made, however, and the r th peak is as shown in figure 7, then the dimensionless measure μ_r of the damping is given by

$$\mu_r = \frac{d_r}{c_r} = \frac{\Delta}{\omega_r}, \quad (7.3)$$

where the measurement of Δ is made as shown.

It is not easy to apply the assumption made by Kennedy & Pancu (1947) in their method (see § 9) to the present technique. They assume—quite reasonably—that the off-resonant vibration is constant (in amplitude and phase) as the system passes through a resonance. But this does not mean that the contribution to the *amplitude* from the off-resonant vibration is constant. In fact, because of the change in the phase of the resonant vibration, the effect of the off-resonant vibration on the amplitude above the resonant frequency will differ from the effect below, and the resultant curve will not be symmetrical. In figure 8 is shown the effect of adding off-resonant vibration represented by OA to vibration in the resonant mode above, OB , and below, OC , the resonant frequency. The resultant amplitudes are OD and OE .

If, in spite of this, it is assumed that the effect of the off-resonant vibration on the amplitude is constant (and how true this is in any example will be shown by how symmetrical the peak is) then the contribution to the peak from the resonant mode may be found by the method described in Gladwell (1962).

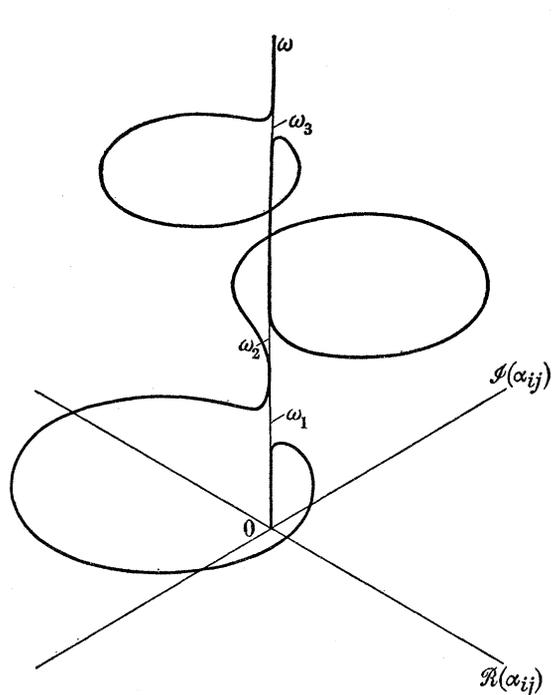


FIGURE 5

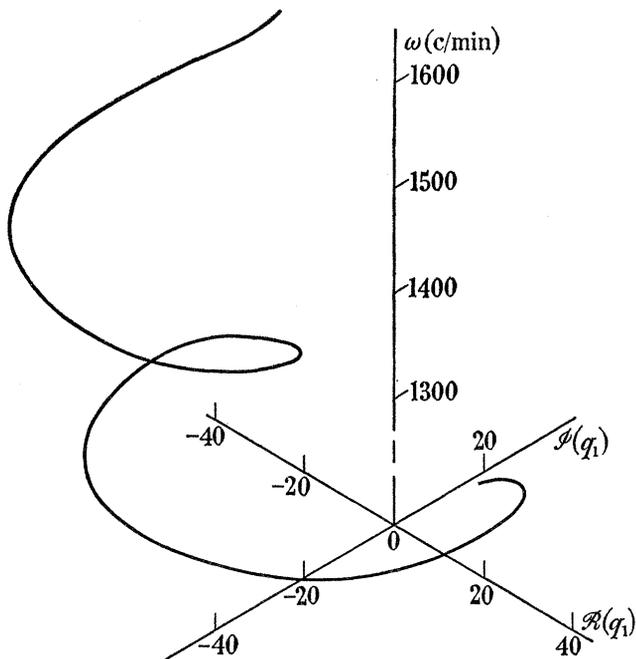


FIGURE 6

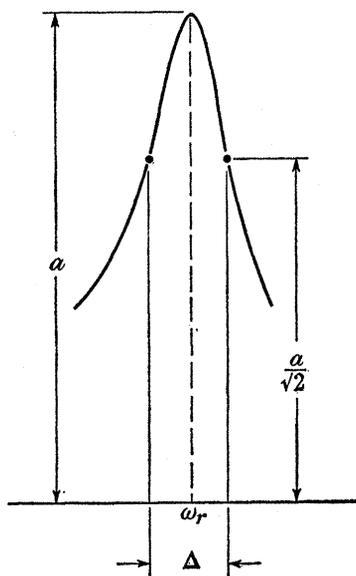


FIGURE 7

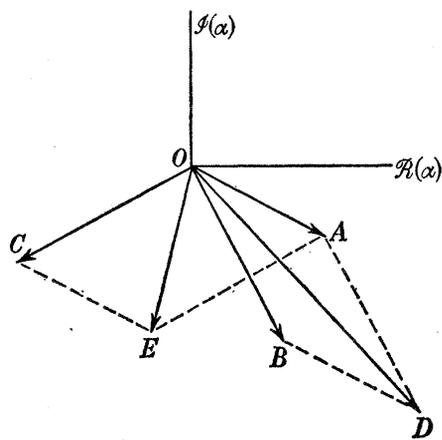


FIGURE 8

(c) *The principal modes*

The shapes of the principal modes are calculated from the ratios of the amplitudes at various points when the structure is being driven at a natural frequency. Mode shapes are notoriously difficult to obtain accurately and there are several possible reasons for this. First, there is the considerable experimental difficulty involved in keeping the excitation

steady while measuring the peak amplitude; but there is also an analytical difficulty of comparable magnitude.

When the system is driven at its r th natural frequency, the shape of the r th forced mode, $\kappa^{(r)}$, is certainly the same as the r th principal mode $\Psi^{(r)}$ (see equation (3.16)), and this is true whether the damping couples the modes or not. The contribution to α_{ij} from the r th mode will therefore be

$$\frac{\Psi_i^{(r)}\Psi_j^{(r)}e^{-\frac{1}{2}i\pi}}{\Psi^{(r)}D\Psi^{(r)}},$$

which represents a vibration in quadrature with the excitation. If the contributions from the r th terms only are considered then the ratios of the vibration amplitudes at various co-ordinates q_i due to excitation at a given co-ordinate q_j will give the shape of the r th mode. However, unless other modes are eliminated there will be other terms of unknown magnitudes in the receptance amplitudes and these may give rise to considerable inaccuracy in the mode shape. Whereas the damping coefficient μ_r has to be calculated from one peak about which there is some uncertainty, the mode shape $\Psi^{(r)}$ has to be calculated from the ratios of a number of peaks about each of which there is uncertainty. This means that if an error is made in estimating the contribution to any peak from the resonant mode, the error in the principal mode is likely to be many times greater than that in the damping coefficient.

If one wishes to avoid having to make a complicated analysis of the amplitude curves, to determine, for example, which part of each peak is due to the resonant mode and which to the others, then one must find some means of exciting a structure in a pure principal mode. Some methods for achieving this will be discussed later, but it can be stated without any hesitation that vibration in modes other than the resonant mode can be eliminated only by using some systematic procedure and exercising considerable care.

The drawbacks which have already been mentioned may be aggravated by a number of related factors, which will now be considered.

(i) *Closeness of natural frequencies*

This is undoubtedly the most important factor influencing the distortion of the amplitude-frequency curves. In structures like airframes—as opposed to flexing single-span beams for instance—natural frequencies are irregularly spaced and may be close together. The kind of effect produced by close natural frequencies may be strikingly illustrated by comparing figures 9 and 10. These curves are adapted from curves given by Kennedy & Pancu (1947). They were obtained by calculating the vibration amplitudes for systems with known dynamical properties and are not plots of experimental data. The damping has been assumed to be diagonal so that the amplitudes have been calculated by substituting particular numerical values into the equation

$$|\alpha_{ij}| = \left| \sum_{r=1}^n \frac{(\partial q_i / \partial p_r)(\partial q_j / \partial p_r) e^{-i\theta_r}}{c_r \{ \mu_r^2 + [1 - (\omega / \omega_r)^2]^2 \}^{\frac{1}{2}}} \right|. \tag{7.4}$$

The curve in figure 9 corresponds to a system whose natural frequencies and damping coefficients are

$$\left. \begin{aligned} \omega_1 &= 1400 \text{ c/min,} & \omega_2 &= 1500 \text{ c/min,} \\ \mu_1 &= 0.03, & \mu_2 &= 0.05. \end{aligned} \right\} \tag{7.5}$$

The curve is the plot of $|q_1|$, where

$$q_1 = \frac{2e^{-i\theta_1}}{\{\mu_1^2 + [1 - (\omega/\omega_1)^2]^{2\frac{1}{2}}\}^{\frac{1}{2}}} + \frac{3e^{-i\theta_2}}{\{\mu_2^2 + [1 - (\omega/\omega_2)^2]^{2\frac{1}{2}}\}^{\frac{1}{2}}}, \quad (7.6)^*$$

and it has the shape expected for a system with two degrees of freedom. The second curve also appears to be a curve corresponding to a system with two degrees of freedom. In fact it is a curve for a system with three degrees of freedom and the relevant data are

$$\left. \begin{array}{l} \omega_1 = 1400 \text{ c/min, } \omega_2 = 1450 \text{ c/min, } \omega_3 = 1500 \text{ c/min,} \\ \mu_1 = 0.03, \quad \mu_2 = 0.03, \quad \mu_3 = 0.05. \end{array} \right\} \quad (7.7)$$

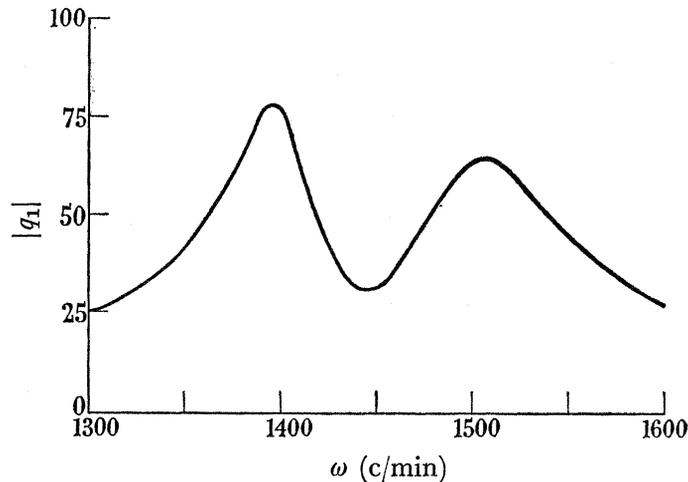


FIGURE 9

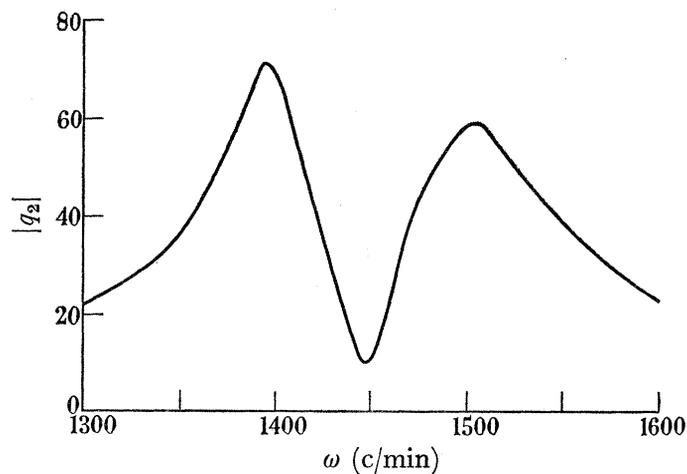


FIGURE 10

In calculating the second curve neither the exciter nor the pick-up was placed near a node of the second mode, and in fact the curve is the plot of $|q_2|$, where

$$q_2 = \frac{2e^{-i\theta_1}}{\{\mu_1^2 + [1 - (\omega/\omega_1)^2]^{2\frac{1}{2}}\}^{\frac{1}{2}}} - \frac{0.75e^{-i\theta_2}}{\{\mu_2^2 + [1 - (\omega/\omega_2)^2]^{2\frac{1}{2}}\}^{\frac{1}{2}}} + \frac{3e^{-i\theta_3}}{\{\mu_3^2 + [1 - (\omega/\omega_3)^2]^{2\frac{1}{2}}\}^{\frac{1}{2}}}. \quad (7.8)^*$$

The first effect of close natural frequencies is therefore that a mode may be missed altogether. This difficulty may be overcome in practice by using a number of exciters and

* The curve was plotted with the aid of the calculating device shown in figure 3.

pick-ups. If they are distributed over the structure it is likely that some of them will be at nodes of overshadowing modes (like the first and third in the system of figure 10) so that obscured modes will be brought to light. Often a mode can be eliminated by making use of any symmetry or anti-symmetry properties it possesses. This matter is considered in detail in Kennedy & Pancu (1947). Further examples of amplitude-frequency curves will be found in chapter XIII of Bisplinghoff, Ashley & Halfman (1955).

The second effect of close natural frequencies may be to produce greater discrepancies between the natural frequencies and the frequencies at which the peak amplitudes occur. This effect is illustrated in both figures 9 and 10; the original two frequencies in the curve in figure 9 are in fact quite close.

The third effect of close natural frequencies is the effect on the calculation of mode shapes and damping coefficients. It is essential that one mode of a close pair be eliminated when calculating the mode shape and damping coefficient of the other. Unless this is done the assumption made earlier, namely, that the effect of other modes on the amplitude of a resonant mode is negligible or constant, become unrealistic, and a more sophisticated analysis becomes necessary.

(ii) *Heavy damping*

Heavy damping will have two main effects on the ease and accuracy with which amplitude-frequency curves can be analyzed. First of all, a heavily damped mode may be obscured altogether from some or all of the curves. Unfortunately, heavily damped modes cannot be disregarded in resonance testing of aircraft, for modes which are heavily damped in still-air vibration may in fact become critical modes for flutter analysis. This means that it is all the more essential to have some systematic means by which unwanted modes can be eliminated.

The second effect of heavy damping is that at the resonant frequency of a heavily damped mode, extraneous vibration from other modes may be comparable in magnitude to the vibration in the resonant mode. This means that unless one can analyze the peak amplitudes accurately into their resonant and off-resonant parts there will be large errors in the relevant damping coefficient and principal mode.

Finally, all these difficulties will be accentuated when the effects of heavy damping and close natural frequencies are combined.

8. THE METHOD OF KENNEDY & PANGU

The method of Kennedy & Pancu (1947) differs from the peak-amplitude method in its approach to the second and third of the problems mentioned in § 6. Instead of measuring just the amplitude of the vibration, the amplitude and phase are measured. Then, instead of plotting two graphs—one for amplitude and one for phase—the complex receptance α_{ij} is plotted on an Argand plane. The resulting curve is found to be relatively easy to analyze.

Kennedy & Pancu make the assumption that, when referred to the principal co-ordinates, the damping matrix is diagonal. Actually, to say that they make this assumption is perhaps misleading. They do not deal with matrices at all. They just consider the undamped motion of a system, introduce principal co-ordinates, and then assume that any effects of damping

can be accounted for by introducing a single damping factor for each mode. They give no indication that it could be otherwise. The theory they are using is really that given in § 5, although the end-product of that theory—the expression (5.13) for α_{ij} —does not appear explicitly in the paper.

It is convenient to start from the expression for the receptance α_{ij} which is given by the theory of § 5, namely

$$\alpha_{ij} = \sum_{r=1}^n \frac{(\partial q_i / \partial p_r) (\partial q_j / \partial p_r) \sin \theta_r e^{-i\theta_r}}{d_r} \quad (8.1)$$

This equation gives the receptance as the sum of n terms. If the r th term is plotted against ω in the way that was used in § 2 the curve will be exactly similar to that shown in figure 1. The curve will have a single loop which will occur around the natural frequency ω_r . The plan of this curve will be exactly similar to that shown in figure 2. The centre of the circle will be at the point

$$\left(0, \frac{-1}{2d_r} \frac{\partial q_i}{\partial p_r} \frac{\partial q_j}{\partial p_r}\right)$$

and the circle will pass through the origin.

If the whole receptance (8.1) is plotted against ω , the resulting curve will resemble that shown in figure 11, which is a receptance curve for a hypothetical system with three degrees of freedom and with natural frequencies 1, 1.1 and 1.21. This curve is actually that of the expression for the direct receptance of the system used for illustrative purposes in § 12. There will be a number of loops of varying diameters and thicknesses. If the whole receptance is plotted on the Argand plane and the values of ω are marked on the curve, the curve will resemble that shown in figure 12, which is the plan of figure 11. If the general level of damping is low and the natural frequencies are well spaced each loop will be very nearly circular, though of course none of the loops will be a full circle. If the damping is heavy, and particularly if there are close natural frequencies, the loops will be more or less distorted.

It is emphasized that these comments refer to the actual curves calculated from specific values of the various constants a_r , c_r , d_r ($r = 1, 2, \dots, n$); they do not refer to the curves obtained from experimental data which will be considered later. There is an important distinction to be noticed. The curves of the type sketched in figure 12 may be thought of as being made up by superimposing n circular loops; but the actual loops which appear in the final curve are not themselves circular.

Having described the form of the response curves given by the theory we must now consider the inverse problem—the analysis of curves obtained from experiment. Kennedy & Pancu do not present any experimental results although they discuss various experiments which they carried out. They illustrate the methods they put forward for the analysis of receptance curves by analyzing curves calculated from specific numerical values of the various parameters a_r , c_r , d_r . They show that, as far as the analysis of such curves is concerned, their method is superior to the peak-amplitude technique, and they give estimates of the errors resulting from using the methods in one or two examples. Broadbent & Hartley (1958) state that ‘practical experience seems to show that this type of plot (a vector response curve) considerably reduces the likelihood of missing a resonance and also improves the accuracy of determining the resonant frequency. This in itself leads to modes being measured which are a better approximation to the true normal modes than is usually

possible from amplitude plots alone. In addition the structural damping can be estimated directly for each resonance'.

It is now necessary to explain the method of extracting the required data from suitable curves of α_{ij} plotted, as in figure 12, in the Argand plane.

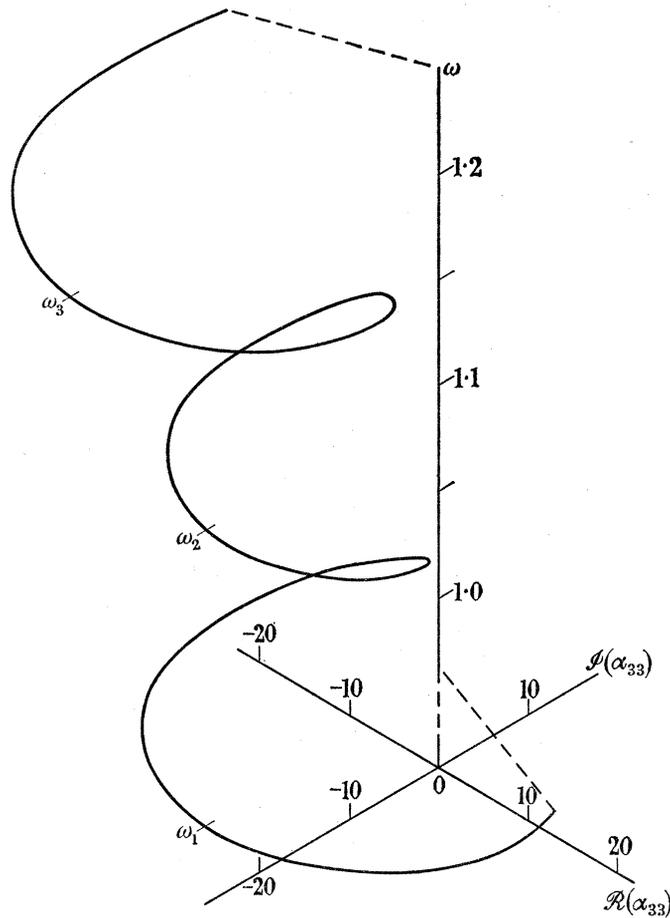


FIGURE 11

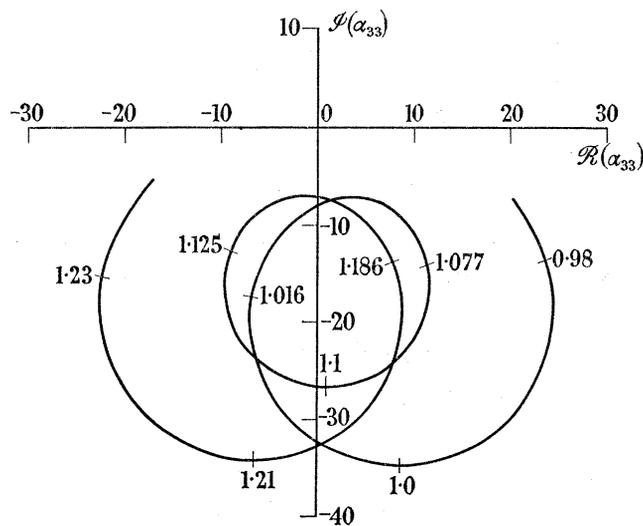


FIGURE 12

(a) The natural frequencies

The natural frequencies of the system are located by finding the points having minimum frequency gradient on the curve. More exactly, they are located by finding the values of ω for which $d\omega^2/ds$ has local minima (or $ds/d\omega^2$ has local maxima). Kennedy & Pancu show that this method is reliable in showing up the existence of modes, and that the accuracy with which the natural frequencies can be obtained seems to be less affected by the presence of other modes than in the peak-amplitude method. They show that in certain cases in which the peak-amplitude method gives almost exact results, the error resulting from their's would in fact be greater. But in other cases in which the peak-amplitude method would give no result at all—simply because there was no peak—their method would still give a reasonably accurate estimate of the natural frequency involved.

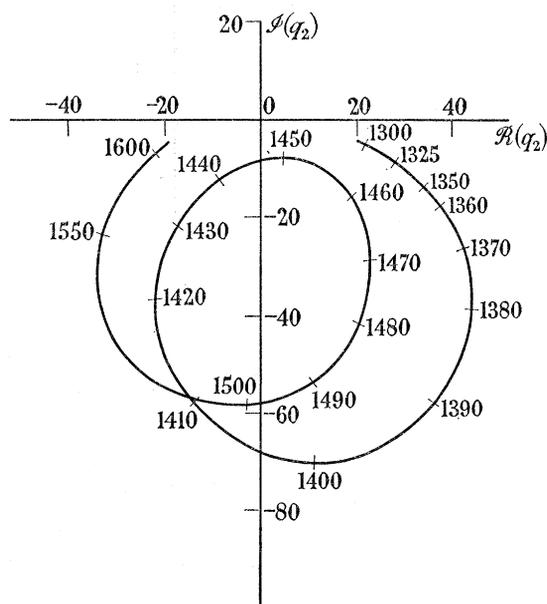


FIGURE 13

One of the examples that Kennedy & Pancu give to illustrate this is the system of figure 10. They show that if the vector response curve is drawn the three minima of the frequency gradient are clearly distinguishable and can be located quite accurately. The curve is shown in figure 13.

The first advantage of the method is therefore its ability to show up the existence of modes. This will be particularly important when, for some reason or other, it is not possible to employ multi-point excitation, and information has to be obtained by means of a simple drive system. Liner (1960) states:

‘This has often been a real problem: simple drive systems combined with amplitude response plots have on many occasions failed to show up the existence of an important mode.

Experience with a dynamically complicated system has shown that, provided some care is used in selecting the drive point, it is virtually impossible to miss a mode’ [by using the Kennedy & Pancu technique].

The theoretical basis for this method of locating the natural frequencies is as follows. For a system with a single degree of freedom the minimum of $d\omega^2/ds$ is attained at ω_1 . This was mentioned in § 2. Therefore for the r th circular loop from which the vector locus is built up the minimum of $d\omega^2/ds$ will occur at ω_r . Therefore for the whole locus the minima of $d\omega^2/ds$ will occur near the natural frequencies, but may be displaced therefrom due to the presence of close natural frequencies or heavy damping.

(b) *The principal modes*

It will be remembered that one of the major causes of inaccuracy in the estimation of the shapes of the principal modes by means of the peak-amplitude technique, was the presence near the resonant frequencies of unknown amounts of vibration in the off-resonant modes.

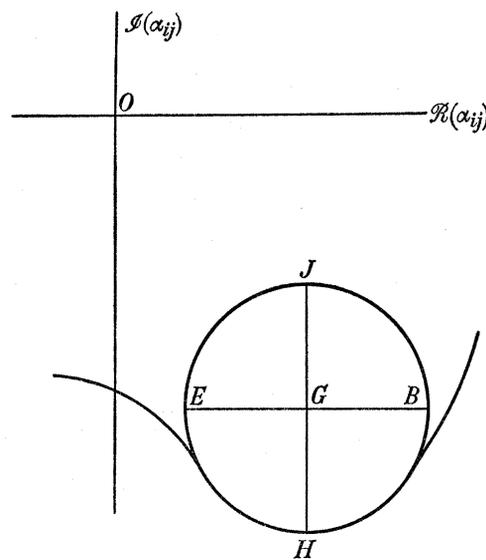


FIGURE 14

This other vibration gave a false base to the peak amplitude. The second advantage of the method of Kennedy & Pancu is that it provides a simple way of determining with fairly good accuracy the true peak amplitude in the resonant mode at a resonant frequency.

The procedure for finding the true peak amplitude in a given mode is as follows. The relevant natural frequency is found by the method outlined above. Then the best circle is fitted to the arc, placing particular emphasis on the part of it in the immediate vicinity of the natural frequency (figure 14). The peak amplitude will then be given by the diameter HJ of the circle.* The point J is called the 'displaced origin' for the mode. The vector OJ represents the extraneous vibration at the resonant frequency.

The theoretical basis for the method is as follows. It is assumed that in some range of frequency around the natural frequency the contribution from the off-resonant modes is effectively constant. How far this assumption departs from the truth in any instance will be shown by how far the loop for the particular mode departs from a circle. Now clearly the smaller the range of frequency considered the more nearly will the contribution of the

* In practice, the curves which will be drawn will be those of response, rather than of receptance. For a constant amplitude of excitation, however, the difference is merely one of scale, and receptance curves will be considered here.

off-resonant modes be constant over the range. Therefore the curve will be most nearly circular in the immediate vicinity of the resonant frequency, where the curve is swept out most rapidly with respect to variation in ω .

The circles which are obtained in this way from the curves for any receptances α_{ij} are the approximations to the circles given by

$$\frac{\partial q_i}{\partial p_r} \frac{\partial q_j}{\partial p_r} \frac{\sin \theta_r e^{-i\theta_r}}{d_r}.$$

The diameter of the circle will be the approximation to

$$\left| \frac{\partial q_i}{\partial p_r} \frac{\partial q_j}{\partial p_r} / d_r \right|$$

and the sign of the numerator may be found from the way in which the circle is attached to the curve. The sign will be positive if the circle is attached to the locus at the point on itself which has the least value of $\mathcal{I}(\alpha_{ij})$, and negative if it is attached at the point having the greatest value of $\mathcal{I}(\alpha_{ij})$. Thus the circle in figure 14 corresponds to a positive numerator. Once the true peak amplitudes are obtained, the shapes of the principal modes may be found by finding the ratios of the peak amplitudes at various co-ordinates q_i due to a given excitation at a fixed station q_j ; for these ratios are determined by the $\partial q_i / \partial p_r$ ($i = 1, 2, \dots, n$)—that is by the diameters of the appropriate circles.

(c) *The damping coefficients*

It will be remembered that the inaccuracy in the damping coefficients d_r found from the peak-amplitude technique resulted from the same cause as the inaccuracy in the mode shapes. They had to be determined by assuming that the peak amplitude was due solely to the vibration in the resonant mode. The third advantage of the method of Kennedy & Pancu is that, by giving a means of extracting the vibration in the resonant mode from the total vibration, it allows a better estimate to be made of the damping coefficients.

The damping coefficient is found from the scale of ω which is required on the response locus. Let BE (figure 14) be the diameter of the r th circle which is parallel to the real axis and let ω_B, ω_E be the values of the driving frequency corresponding to B and E . It may be shown that the dimensionless measure μ_r of the damping in this mode is given by

$$\mu_r = \frac{d_r}{c_r} = \frac{\omega_E - \omega_B}{\omega_r}, \quad (8.2)$$

where ω_r is the natural frequency, corresponding to the point H in the figure. This expression may be shown to be equivalent to that whose use was illustrated in figure 7; but now the peak of that figure is, in effect, given a fresh base.

In practice the value of μ will usually be less than 0.04. Equation (8.2) shows that for $\mu = 0.04$ half the circle is traversed in the range

$$0.98 \leq \omega/\omega_r \leq 1.02,$$

while if $\mu = 0.01$, half is traversed in the range

$$0.995 \leq \omega/\omega_r \leq 1.005.$$

9. THE THEORY OF PURE MODE EXCITATION

It will have been noticed that almost all the difficulties which arise in connexion with the analysis of response curves are due to the fact that vibration is taking place in several modes simultaneously. There are a number of devices for eliminating unwanted modes. Such modes may be eliminated by placing pick-ups or shakers at nodes, or if they are symmetrical or anti-symmetrical they may be eliminated by using anti-symmetrical or symmetrical forcing distributions. Accounts of the technical details of such devices will be found in Kennedy & Pancu (1947) and Bisplinghoff *et al.* (1955). Unfortunately, the application of such devices is seldom sufficient for the excitation of a pure mode in a complex structure, and for this reason other more systematic methods have been proposed. In this section we shall give an account of the theoretical problems involved, in preparation for the discussion of the techniques proposed by Lewis & Wrisley (1950) which will be described in the next section.

The equation governing the motion of a hysteretically damped system is

$$A\ddot{q} + (D/\omega)\dot{q} + Cq = Q, \quad (9.1)$$

where it is assumed that Q represents a harmonic excitation of frequency ω . From a theoretical point of view the problem involved in exciting a system in one of its principal modes is the problem of finding the correct force distribution Q . If the system is to vibrate in its r th principal mode $\Psi^{(r)}$ at some frequency ω , not necessarily the natural frequency ω_r , then the displacement matrix q will be

$$q = \Pi_r \Psi^{(r)} \sin(\omega t - \theta), \quad (9.2)$$

where Π_r is some amplitude factor. The correct forcing distribution is therefore

$$Q = \Pi_r \{(C - \omega^2 A) \Psi^{(r)} \sin(\omega t - \theta) + D \Psi^{(r)} \cos(\omega t - \theta)\} \quad (9.3)$$

and it is made up of two parts. The first is the force necessary to cancel the elastic and inertia forces, and is in phase with the displacement. The second is the force needed to counteract the damping and is $\frac{1}{2}\pi$ ahead of the displacements.

Equation (9.3) shows that, in general, the forces to be applied at the various parts of the structure will not be in phase. From a practical point of view it will be almost impossible to excite a pure mode if not only the amplitudes of the forces to be applied at the various points of the structure have to be found, but also their phases. It therefore becomes essential to enquire whether it is ever possible to excite a pure mode by using forces which are all in phase. There are actually three occasions on which this is possible.

Suppose, then, that the forcing is

$$Q = \Phi \sin \omega t. \quad (9.4)$$

(a) If there is no damping then any principal mode can theoretically be excited at any frequency by a force distribution in phase with it and given by

$$\Phi = \Pi_r (C - \omega^2 A) \Psi^{(r)} = \Pi_r (\omega_r^2 - \omega^2) A \Psi^{(r)}. \quad (9.5)$$

(b) If there is damping present and the principal mode is excited at its resonant frequency, then

$$(C - \omega^2 A) \Psi^{(r)} = 0. \quad (9.6)$$

The phase lag θ in equation (9.2) is then $\frac{1}{2}\pi$ and

$$\Phi = \Pi_r D \Psi^{(r)}. \quad (9.7)$$

This corresponds to the fact (see equation (3.16)) that at the r th natural frequency the forced mode $\kappa^{(r)}$ is the same as the principal mode $\Psi^{(r)}$, and the phase lag θ_r is $\frac{1}{2}\pi$.

(c) If the damping matrix is diagonal when referred to the principal co-ordinates of the system, the theory of § 5 applies and equation (9.1) can be replaced by

$$L\ddot{p} + (H/\omega)\dot{p} + Np = \Xi \sin \omega t, \quad (9.8)$$

in which L , H and N are diagonal matrices whose r th terms are a_r , d_r and c_r , respectively. If the system is to vibrate in its r th principal mode then it is necessary that

$$p_r = \Pi_r \sin(\omega t - \theta), \quad p_s = 0 \quad (r \neq s). \quad (9.9)$$

Substituting these values into equation (9.8) and using the notation (5.12), we obtain

$$\Xi_r = c_r [\mu_r^2 + (1 - \beta_r^2)^2]^{\frac{1}{2}} \Pi_r, \quad \Xi_s = 0 \quad (r \neq s) \quad (9.10)$$

and
$$\theta = \theta_r = \tan^{-1} \left(\frac{\mu_r}{1 - \beta_r^2} \right), \quad (9.11)$$

where θ_r denotes the characteristic phase lag encountered in § 5.

Equations (9.9) and (9.10) show that when the damping matrix is diagonal it is possible to excite a principal mode at any frequency by a distribution of forces which are all in phase.

Equation (9.10) also shows that the excitation should be applied 'at the principal co-ordinate', but this statement is in rather technical language; to translate it into ordinary language we proceed as follows. The column matrix Φ , which gives the generalized forces to be applied at the co-ordinates q , is related to Ξ by equation (5.5), namely,

$$\Xi = X' \Phi. \quad (9.12)$$

To obtain Φ we must pre-multiply this equation by the inverse of X' , which the first of equations (5.3) shows is

$$(X')^{-1} = AXL^{-1}. \quad (9.13)$$

Using this inverse we find that
$$\Phi = AXL^{-1}\Xi, \quad (9.14)$$

and in the particular case when only Ξ_r is non-zero this equation yields

$$\Phi = a_r^{-1} \Xi_r A \Psi^{(r)}, \quad (9.15)$$

where we have used the fact that the columns of the matrix X are the modal vectors $\Psi^{(r)}$. If this equation is combined with equation (9.10) the result is

$$\Phi = \Pi_r [\mu_r^2 + (1 - \beta_r^2)^2]^{\frac{1}{2}} \omega_r^2 A \Psi^{(r)}. \quad (9.16)$$

We notice, then, that when the damping matrix is diagonal, the forcing required to excite the vibration

$$q = \Pi_r \Psi^{(r)} \sin(\omega t - \theta_r) \quad (9.17)$$

is given by
$$Q = \Phi \sin \omega t = \Pi_r [\mu_r^2 + (1 - \beta_r^2)^2]^{\frac{1}{2}} \omega_r^2 A \Psi^{(r)} \sin \omega t, \quad (9.18)$$

where θ_r has the value given in equation (9.11). In particular the forcing required to excite the resonant vibration

$$q = -\Pi_r \Psi^{(r)} \cos \omega_r t \quad (9.19)$$

is
$$Q = \Pi_r \mu_r \omega_r^2 A \Psi^{(r)} \sin \omega_r t. \quad (9.20)$$

Equation (9·18) shows that the forces must be proportional to the inertia forces corresponding to the generalized displacements.

The results which have just been proved provide a means of determining whether or not the damping in a system couples the modes. If the damping matrix is diagonal, that is if there is just one damping coefficient connected with each mode, then the system may be made to vibrate in a principal mode at any frequency by a set of forces all in phase. This is the meaning of equations (9·17) and (9·18). These equations show that the ratios of the various forces to be applied at the various points of the structure do not depend on the frequency, so that they are given, in particular, by the ratios of the forces required to excite the mode at its resonant frequency.

On the other hand, if the damping is not diagonal, the situation is quite different. It is then possible to find a set of forces all in phase which will excite the r th principal mode only when the system is driven at the r th natural frequency ω_r . The forcing will then be in quadrature with the vibration and it will be given by equation (9·7). It does not seem to be fully recognized that, even when damping of the most general type is present, the system *can* vibrate in a principal mode at a natural frequency without there being any phase differences between the displacements at various points.* Nor is it realized that this result is an exact one, and not an approximate one depending on the smallness of the damping.

At frequencies other than the natural frequencies the only in-phase vibrations which may be excited by in-phase force distributions are the forced modes $\kappa^{(r)}$ discussed in §3. It will be remembered that there are n such modes for each frequency, they lag behind the excitation by angles θ_r , and they are related to the necessary excitation by equation (3·9).

10. THE METHOD OF LEWIS & WRISLEY

Lewis & Wisley (1950) have developed a systematic procedure for exciting pure natural modes and have devised equipment for its implementation. At the beginning of their paper they present the assumptions upon which they worked. We shall quote from their own words. 'As a reasonable but simple assumption, we can take the concept of structural damping literally and assume that each elastic shear and moment in the structure has associated with it a proportional damping shear and moment in quadrature phase relationship....' This means that 'the oscillatory forces acting at a natural frequency must be adjusted in magnitude to be proportional to the product of the mass of the structure on which it acts and the amplitude of that mass in the mode being excited'. When translated into the language of the present paper the fundamental assumption seems to be that

$$D = \gamma C, \quad (10\cdot1)$$

where γ is some factor of proportionality.

If, with this assumption, the motion is resonant, such that

$$q = -\Pi_r \Psi^{(r)} \cos \omega_r t, \quad (10\cdot2)$$

* Cf. Plantin (1952) 'structural damping does cause phase displacements throughout the structure, and in future it may be necessary to take account of the actual phase angles in order to obtain a true picture of the mode'. This seems to suggest that when the system is vibrating in a mode the displacements of the various parts of the system are not all in phase; however, this deduction may be a misjudgement on the part of the present writers.

then the excitation required will be

$$\begin{aligned} Q &= A\dot{q} + (D/\omega) \dot{q} + Cq \\ &= -\Pi_r(C - \omega_r^2 A) \Psi^{(r)} \cos \omega_r t + \Pi_r D \Psi^{(r)} \sin \omega_r t \\ &= \gamma \Pi_r C \Psi^{(r)} \sin \omega_r t, \end{aligned} \quad (10.3)$$

or
$$Q = \gamma \Pi_r \omega_r^2 A \Psi^{(r)} \sin \omega_r t, \quad (10.4)$$

so that the matrix of generalized forces must be proportional to $A\Psi^{(r)}$; and this, it will be noted, is a statement in our language of the way in which Lewis & Wrisley say the forcing must be applied.

The assumption in equation (10.1) is of course a particular case of the assumption we made in the last section, namely, that the damping matrix is diagonal when referred to the principal co-ordinates. Clearly if D is given by equation (10.1) then the damping matrix for the principal co-ordinates will be

$$H = X'DX = \gamma X'CX = \gamma N, \quad (10.5)$$

so that
$$d_r = \gamma c_r \quad \text{and} \quad \mu_r = \gamma \quad (r = 1, 2, \dots, n). \quad (10.6)$$

The assumption (10.1) is therefore equivalent to the assumption that all the dimensionless damping coefficients μ_r are equal.

The result proved in the last section is that, to excite the r th principal mode alone, the excitation Φ must be proportional to $A\Psi^{(r)}$ whatever the damping is, so long as it does not couple the modes. Equation (9.18) shows, moreover, that this holds whatever the frequency at which the mode is excited. This will mean, as we shall see, that the method which Lewis & Wrisley propose should be valid for any damping, provided it does not couple the modes, and not just for damping of the type given in equation (10.6).

We are not concerned with the technical details of the apparatus used by Lewis & Wrisley. It is fully described in the original paper and is also discussed at length in chapter XIII of Bisplinghoff *et al.* (1955). We shall content ourselves with giving a bare outline of the technique.

Briefly the method consists of an iterative procedure for obtaining the force distribution

$$\Phi \sim A\Psi^{(r)}$$

in phase with velocity. The equipment consists of twenty-four independently controllable electromagnetic shakers, each with an attached accelerometer. All the shakers are driven in phase or exactly out of phase with each other by a single variable-frequency source. The mass of the structure (giving the matrix A) is considered to be divided among the shakers according to the amount of structure which can reasonably be assigned to each shaker; clearly it may be difficult accurately to divide up the structure in this way. A means is provided for indicating the modified shaking force, that is the ratio of force to mass, for each shaker. Response velocities at the various shakers are derived by integrating the outputs of the accelerometers. The modified shaking force and velocity for any point are displayed on the vertical and horizontal axes of a cathode ray oscilloscope, which will indicate the ratio of the signals and their phase difference. The mode is excited by bringing the structure to the appropriate resonant frequency, bringing the shakers into play one after another and making the ratio of modified shaking force to velocity the same for each point.

Lewis & Wrisley report that 'although phase shifts between the responses at various stations may be large under the excitation of the first applied force even when adjusted to a resonant frequency, as more and more forces are applied and adjusted the more uniform in phase are the various points of the structure'. It is also found that as the adjustments proceed the velocities become more and more in phase with the exciting forces. These two trends, that the displacements are becoming all in phase with each other, and that the forces are becoming in phase with the velocities, are of course definite indications that a principal mode is being approached. Finally, when the pure mode is attained the operator records the responses of all the pick-ups simultaneously on the accompanying recording oscillograph.

One other matter which they report is of particular interest to us; it is 'that with the shakers so adjusted as to phase and force amplitudes, the structural response will be in the one mode regardless of the frequency of excitation'. This is of interest because it agrees with equation (9.18).

Lewis & Wrisley achieved success by using a method which, theoretically, is valid only when the damping does not couple the modes. Their success may therefore be considered as indirect evidence for the damping being of this type—at least in their experimental model. It may, however, be unwise to assert that their success proves the matter. Damping is in its very nature an uncertain and vague phenomenon and such a concept as a damping matrix should not be taken too seriously. Indeed the fact that they were able to achieve success so easily by using a forcing distribution that could have been only approximately proportional to $A\Psi^{(r)}$ shows that the actual vibration of a structure is not nearly so clear-cut and precise as the theory we have been using would suggest. In this connexion it should be noted that they did not use any elaborate procedure to determine the masses which should be associated with each shaker.

Two lines of further investigation are immediately suggested. First, it should be ascertained how nearly a pure mode can be excited at a natural frequency by using Lewis & Wrisley's criterion, $\Phi \sim A\Psi^{(r)}$, even in practical cases where it is known that the damping *does* couple the modes. Secondly, further investigation could be carried out along the following lines. Excite a certain mode by the Lewis & Wrisley method. Then, keeping the ratios of the forces applied at the various points constant, change the forcing frequency. If the damping truly does not couple the modes, and if the forcing distribution is correct, then the vibration should continue to be in the one mode. An acid test for the forcing distribution will be, of course, to adjust the frequency to another *resonant* frequency, for this will show up impurities in the forcing, or at least those impurities which tend to excite that particular mode.

11. THE METHODS OF TRAILL-NASH AND ASHER

Traill-Nash (1958), apparently dissatisfied with the method of Lewis & Wrisley (1950) discussed in the last section, has proposed an alternative method for exciting pure natural modes. Unlike the method of Lewis & Wrisley, his technique should theoretically be applicable whether the damping couples the modes or not. Indeed the damping need not even be hysteretic, and can in fact be some unknown combination of hysteretic and viscous damping. Theoretically, all the equipment needed to implement his technique is a set of (electromagnetic) shakers which can be run either in phase or exact antiphase with each

other and whose frequency can be varied; and a set of pick-ups suitable for measuring the components of the responses in phase with the applied forces.

The procedure is as follows. The natural frequency ω_r of the mode under investigation is first found, using the method of Kennedy & Pancu, for example. Then, if n is the number of degrees of freedom of the system, n linearly independent force distributions

$$f^{(1)} \sin \omega_r t, \quad f^{(2)} \sin \omega_r t, \quad \dots, \quad f^{(n)} \sin \omega_r t$$

are applied to the system in turn, and the *in-phase* components

$$y^{(1)} \sin \omega_r t, \quad y^{(2)} \sin \omega_r t, \quad \dots, \quad y^{(n)} \sin \omega_r t$$

of the corresponding responses are measured.

The symbols $f^{(1)}, f^{(2)}, \dots, f^{(n)}$ therefore denote the column matrices of order n whose elements are the amplitudes of the generalized forces (i.e. forces and couples) applied at the generalized displacements (i.e. displacements and rotations) in each distribution. The n distributions are said to be linearly independent if they cannot be combined to give a null force-distribution—that is, if the equation

$$x_1 f^{(1)} + x_2 f^{(2)} + \dots + x_n f^{(n)} = 0 \quad (11.1)$$

implies that

$$x_1 = 0 = x_2 = \dots = x_n. \quad (11.2)$$

Similarly the symbols $y^{(1)}, y^{(2)}, \dots, y^{(n)}$ denote the column matrices whose elements are the in-phase components of the generalized displacements corresponding to each force distribution.

Theoretically this information is sufficient to enable the analyst to determine what combination of the n forcing distributions is needed for the excitation of the principal mode. The required combination is found by using the following property of damped vibration. ‘A necessary and sufficient condition for the response to be in an undamped natural mode is that this response be in quadrature with the exciting forces.’ This means that

$$\Phi = x_1 f^{(1)} + x_2 f^{(2)} + \dots + x_n f^{(n)} \quad (11.3)$$

is the required combination if and only if the corresponding in-phase response is null, that is, if

$$x_1 y^{(1)} + x_2 y^{(2)} + \dots + x_n y^{(n)} = 0. \quad (11.4)$$

In other words, the modes can be found by finding the solutions of equation (11.4). This equation may be written in the alternative form

$$Yx = 0, \quad (11.5)$$

where $x = \{x_1, x_2, \dots, x_n\}$ and Y is the square matrix whose columns are $y^{(1)}, y^{(2)}, \dots, y^{(n)}$. When a solution x is found—and it will be shown below that there is only one solution of the equation—then the forcing distribution Φ given by equation (11.3) for these values of x_1, x_2, \dots, x_n is applied to the system. The response corresponding to this forcing will be in quadrature with it and will therefore be in the mode required.

The property mentioned above may be proved on the basis of the theory of § 3. Equation (3.23) states that any in-phase force distribution $\Phi \sin \omega t$ may be expressed in the form

$$\Phi \sin \omega t = \{\lambda_1 \Sigma^{(1)} + \lambda_2 \Sigma^{(2)} + \dots + \lambda_n \Sigma^{(n)}\} \sin \omega t, \quad (11.6)$$

where $\Sigma^{(1)}, \Sigma^{(2)}, \dots, \Sigma^{(n)}$ denote the force-distributions required to excite the various forced modes $\kappa^{(1)}, \kappa^{(2)}, \dots, \kappa^{(n)}$, and $\lambda_1, \lambda_2, \dots, \lambda_n$ are real constants. This equation, and in fact the whole of § 3 was based on the assumption that the damping was hysteretic. However, it was pointed out (see the remarks after equation (3·2)) that the analysis of a damped system is much the same whether the damping is hysteretic, viscous or a mixture of both. Whatever the damping, therefore, there will be an equation of the type (11·6) and the response due to the forcing $\Phi \sin \omega t$ will be

$$q = \lambda_1 \kappa^{(1)} \sin(\omega t - \theta_1) + \lambda_2 \kappa^{(2)} \sin(\omega t - \theta_2) + \dots + \lambda_n \kappa^{(n)} \sin(\omega t - \theta_n). \quad (11\cdot7)$$

This equation is really the same as equation (3·25), except that in that equation the explicit values of the λ_r calculated on the basis of hysteretic damping have been used.

The in-phase component of the response given by equation (11·7) is

$$y \sin \omega t = \{\lambda_1 \cos \theta_1 \kappa^{(1)} + \lambda_2 \cos \theta_2 \kappa^{(2)} + \dots + \lambda_n \cos \theta_n \kappa^{(n)}\} \sin \omega t \quad (11\cdot8)$$

and it may easily be proved that if y is zero, then, for some value of r

$$\lambda_1 = 0 = \lambda_2 = \dots = \lambda_{r-1} = \lambda_{r+1} = \dots = \lambda_n, \quad \omega = \omega_r. \quad (11\cdot9)$$

But equation (3·16) states that

$$\kappa^{(r)} = \Psi^{(r)}, \quad \theta_r = \frac{1}{2}\pi \quad \text{when} \quad \omega = \omega_r \quad (11\cdot10)$$

(and again this is true whatever the damping) and substituting these results into equation (11·7) we obtain

$$q = -\lambda_r \Psi^{(r)} \cos \omega_r t, \quad (11\cdot11)$$

which is the required result. The converse property is trivial, for if ω is a natural frequency ω_r and q is wholly in the resonant mode then clearly y will be zero.

The problem of pure mode excitation may be looked at in another way. The problem is to find a combination of the forced distributions $f^{(1)}, f^{(2)}, \dots, f^{(n)}$ which is proportional to the force $\Sigma^{(r)}$. For at the natural frequency ω_r this force will excite only the r th principal mode $\Psi^{(r)}$ in quadrature with the excitation.

There are two points which should be noted. First, the requirement that the response be in quadrature with the excitation not only gives the mode shape, but also determines the natural frequencies. For if ω is not a natural frequency then y in equation (11·8) can be zero only if all the λ 's are zero, in which case there is neither excitation nor response. This means that if ω is not a natural frequency then equation (11·5) will have no solution other than the trivial one $x = 0$. In other words, the natural frequencies are characterized by being the only frequencies for which it is possible to find an in-phase force distribution which will give a response in quadrature. The second point to note is that when $\omega = \omega_r$ there is just one force distribution given by equation (11·6) which will give a response in quadrature; this is the force distribution proportional to $\Sigma^{(r)}$. In other words, when ω is a natural frequency equation (11·5) has just one solution.

The foregoing analysis shows that in essence the method proposed by Traill-Nash is perfectly straightforward and general, and relies on a fundamental property of damped forced motion. Unfortunately, difficulties appear to be likely in any attempt to put the method into operation. These difficulties would arise because, before the modes can be found, equation (11·5) has to be solved. Traill-Nash, realizing the undesirability of having to solve the equations numerically, suggests that a relaxation technique might be

appropriate, and that the relaxation might be applied on the experimental set-up itself. Attractive as this method sounds it might be subject to considerable difficulty arising from the fact that, unless the forcing frequency is *exactly* a natural frequency, the equations which are to be solved, namely (11.5), have no solution. It would therefore be necessary to find the natural frequency very accurately before attempting to start the relaxation technique, unless some method could be devised by which the frequency could be adjusted at the same time as the correct force distribution was found. The primary need is therefore for some equipment to be devised which could effect this.

Up to this point we have considered the technique solely in connexion with a vibrating system having a definite number of degrees of freedom. Before the method can be used to determine the frequencies and modes of a complex structure it is necessary to discuss how it must be adapted to be applicable to a continuous structure. Difficulties arise because for a continuous structure whose vibration characteristics are unknown it is not known beforehand how many significant modes are present in the system. In other words it is not known how many degrees of freedom must be considered if the motion of the system is to be adequately simulated by the motion of a system with a finite number of degrees of freedom. But in the method as we have described it, it is necessary to know this number and to use just so many shakers and pick-ups. This difficulty does not arise in the other methods we have considered, e.g. in those of Kennedy & Pancu, and Lewis & Wrisley.

Traill-Nash does not seem to feel the full force of this difficulty, but he does make a contribution to the solution of the problem in the form of a discussion of the exact number of shakers and pick-ups which are necessary to find a given mode; this discussion actually takes up the major part of his paper. He introduces the idea of the 'effective number of degrees of freedom', n , relevant to a given natural frequency ω_r . The definition which he gives for this term is rather obscure. However, he introduces the term to make precise the idea that at any frequency, and in particular at any natural frequency of a system (whether discrete or continuous), there are a certain number of dominant modes; these are the only modes of the system which can be excited to any appreciable extent at that frequency. The precise idea is that at any natural frequency ω_r the motion of the system is (to all intents and purposes) the same as that of a system with a certain number, n , degrees of freedom. The effective number of degrees of freedom will, of course, vary from frequency to frequency, and will usually be unknown beforehand. He proves the following results, in which n denotes the effective number of degrees of freedom of the mode under investigation:

(a) The method requires at least $(n-1)$ measuring stations, but the use of a greater number introduces no difficulty.

(b) Ideally, there should be n excitation stations. It is not possible to excite the modes with fewer, and the use of a greater number may lead to inefficient use of the forces applied (they may be linearly dependent).

He also gives an account of the modifications which must be made to the analysis when various possible numbers of shakers and pick-ups are used. Traill-Nash concludes by suggesting that, in view of these results, it is best to make an initial underestimate of the effective number of degrees of freedom of a given natural frequency, and then increase the estimate until a solution is obtained. It remains to be seen how this could be applied in practice. Again, the need of experimental evidence is clear.

It is at this point that the technique proposed by Asher (1958) is relevant. Asher uses essentially the same analysis as Traill-Nash and considers an equation equivalent to equation (11.5). But instead of attempting to solve it so that the correct forcing distribution can be found he first uses it to find the natural frequencies of the system; these will be given by the roots of the determinantal equation

$$|Y| = 0. \tag{11.12}$$

This equation is equivalent to the statement that the determinant of the real part of the receptance matrix is zero. For if F denotes the square matrix whose columns are the force distributions $f^{(1)}, f^{(2)}, \dots, f^{(n)}$ then

$$Y = \mathcal{R}(\alpha) F \tag{11.13}$$

and therefore

$$|Y| = |\mathcal{R}(\alpha)| |F|. \tag{11.14}$$

But the force distributions are linearly independent so that the determinant $|F|$, having its columns linearly independent, is non-zero. Therefore equation (11.12) holds if and only if

$$|\mathcal{R}(\alpha)| = 0. \tag{11.15}$$

This is the equation which Asher uses.

He suggests also that the difficulty about not knowing the effective number of degrees of freedom of a given mode might be overcome as follows. The test would start at a low frequency with a single exciter and a single displacement measurement. The frequency at which the real part of the single receptance changed sign would give the approximate position of the first natural frequency. A second shaker and pick-up would then be added so that four receptances can be measured, and the test repeated over the range in which the first natural frequency was observed. The first zero of the (2×2) determinant of the real part of the receptance matrix should give a better approximation to the first natural frequency, and the second zero should give an indication of the second natural frequency. A third exciter and a third pick-up would now be added and the receptance matrix near the frequencies of the first and second modes remeasured to redefine the natural frequencies of these modes. The process would continue in this way throughout the frequency range of interest, and in this way the correct number of exciters could be determined for a system whose vibration characteristics were unknown before the test. Asher suggests that the method might be particularly useful for separating two modes closely spaced in frequency.

12. FURTHER INVESTIGATION OF ASHER'S METHOD

In order to discuss the theoretical basis of the method proposed by Asher it will be necessary to refer back to the theory in § 3. Equation (3.34) shows that when the damping is hysteretic the receptance α_{ij} of a system with n degrees of freedom is

$$\alpha_{ij} = \sum_{s=1}^n \frac{\kappa_i^{(s)} \kappa_j^{(s)} e^{-i\theta_s}}{\{[\kappa^{(s)} D \kappa^{(s)}]^2 + [\kappa^{(s)} (C - \omega^2 A) \kappa^{(s)}]^2\}^{\frac{1}{2}}}. \tag{12.1}$$

This will therefore be the form which the receptance will take when n is the effective number of degrees of freedom near any given natural frequency. If the damping is viscous or a mixture of hysteretic and viscous the expression for the receptance will have to be modified. However, in every case we shall have

$$\mathcal{R}(\alpha_{ij}) = \sum_{s=1}^n v_s \kappa_i^{(s)} \kappa_j^{(s)} \cos \theta_s, \tag{12.2}$$

where the coefficients v_s are *positive* and depend on the elements of A , B , C , D and the frequency. When the damping is hysteretic the angle θ_r will be that defined by equation (3·17), when the damping is viscous the angle will be ϕ_r defined by (3·18), while when the damping is a mixture θ_r will be an angle with similar properties derived from a joint equation involving B and D . For our purposes it is sufficient that θ_r be some phase angle satisfying equation (12·4), below.

Suppose that for simplicity the first displacement measurement and the first shaker are both located at q_1 . Then the quantity that is plotted first is

$$A_1 = \mathcal{R}(\alpha_{11}) = \sum_{s=1}^n v_s [\kappa_1^{(s)}]^2 \cos \theta_s = \sum_{s=1}^n \beta_s \cos \theta_s, \quad (12\cdot3)$$

where the β are positive constants.

Some idea of the way in which the zeros of A_1 are distributed may be found by using the fact that $\cos \theta_m$ changes from positive to negative when ω passes through ω_m , i.e.

$$\cos \theta_m \cong 0 \quad \text{according as} \quad \omega \cong \omega_m. \quad (12\cdot4)$$

This is true whatever the damping. From this we may deduce that

- (i) $A_1 > 0$ for $\omega < \omega_1$;
- (ii) when $\omega = \omega_1$ the first term in A_1 is zero while all the rest are positive (or zero), so that $A_1 \geq 0$;
- (iii) when $\omega = \omega_n$ the last term in A_1 is zero while all the rest are negative (or zero), so that $A_1 \leq 0$;
- (iv) $A_1 < 0$ for $\omega > \omega_n$.

This means

(a) the first zero of A_1 gives an overestimate for ω_1 and the last gives an underestimate for ω_n ;

(b) A_1 has an odd number of zeros and they all lie between ω_1 and ω_n inclusive.

In addition, it may be shown that (at least for any damping which does not couple the modes)

(c) A_1 has at most $2n-1$ zeros.

These results may be illustrated by considering a simple system with three degrees of freedom and which is subject to hysteretic damping which does not couple the modes. The receptance α_{ij} of such a system will be given by

$$\alpha_{ij} = \sum_{s=1}^3 \frac{(\partial q_i / \partial p_s) (\partial q_j / \partial p_s) e^{-i\theta_s}}{c_s [\mu_s^2 + (1 - \beta_s^2)^2]^{\frac{1}{2}}}, \quad (12\cdot5)$$

and this gives

$$\mathcal{R}(\alpha_{ij}) = \sum_{s=1}^3 \frac{\partial q_i}{\partial p_s} \frac{\partial q_j}{\partial p_s} \frac{\gamma_s}{c_s}, \quad (12\cdot6)$$

where

$$\gamma_s = \frac{\cos \theta_s}{[\mu_s^2 + (1 - \beta_s^2)^2]^{\frac{1}{2}}} = \frac{1 - \beta_s^2}{\mu_s^2 + (1 - \beta_s^2)^2}. \quad (12\cdot7)$$

The quantity γ_r is dimensionless and is identical in form with the real part of the receptance of a system with a single degree of freedom.

For simplicity consider the system with the parameters

$$\left. \begin{aligned} \omega_1 = 1\cdot0, \quad \mu_1 = 0\cdot03, \quad c_1 = 1; \\ \omega_2 = 1\cdot1, \quad \mu_2 = 0\cdot04, \quad c_2 = 1; \\ \omega_3 = 1\cdot21, \quad \mu_3 = 0\cdot03, \quad c_3 = 1. \end{aligned} \right\} \quad (12\cdot8)$$

Suppose also that the exciters and pick-ups are placed at the points q_1, q_2 and q_3 which, when referred to the principal co-ordinates of the system are

$$q_1 = (1, 0, 2), \quad q_2 = (0, 1, \sqrt{2}), \quad q_3 = (1, 1, 1), \quad (12\cdot9)$$

so that the matrix X of equation (5.1) is

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & \sqrt{2} \\ 1 & 1 & 1 \end{bmatrix}. \quad (12\cdot10)$$

For these values $A_1 = \mathcal{R}(\alpha_{11}) = \gamma_1 + 4\gamma_3,$ (12.11)

and it may be shown that A_1 has three zeros. This shows that the system has *at least two* degrees of freedom, for if it had only one it could have at most one zero by result (c) above.

The zeros are at $\omega = 1\cdot007, \quad 1\cdot027, \quad 1\cdot20971,$ (12.12)

which agrees with result (a). This is the result that would be found using just one shaker, applied at q_1 .

A different result would be found by using one shaker only, placed at q_3 (say), for which

$$\mathcal{R}(\alpha_{33}) = \gamma_1 + \gamma_2 + \gamma_3. \quad (12\cdot13)$$

This has five zeros, showing that the system has at least three degrees of freedom, and they are at

$$\omega = 1\cdot004, \quad 1\cdot036, \quad 1\cdot101, \quad 1\cdot160, \quad 1\cdot206. \quad (12\cdot14)$$

So far we have considered only A_1 , and we have shown that, from the positions of the first and last zeros, we can obtain estimates of the lowest and greatest natural frequencies, respectively. In addition, from the number of zeros, we can find an underestimate of the number of degrees of freedom; the rule is that if a certain A_1 has $2m - 1$ zeros then the system has at least m degrees of freedom. It will now be shown, however, that unless one knows the value of n , the effective number of degrees of freedom, and plots the determinant of the real part of a receptance matrix of order n , one can obtain very little certain information about the values of the intermediate natural frequencies.

Asher suggests that after plotting A_1 one should plot the determinant of the real part of a 2×2 receptance matrix, A_2 say, of the system, and he states that the first zero of this should give a better approximation to ω_1 , and the second should give an estimate for ω_2 . Suppose that the second shaker and pick-up are placed at q_2 , the first being kept at q_1 ; then

$$A_2 = \left| \mathcal{R} \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \right| = \begin{vmatrix} \sum_{s=1}^n \nu_s \kappa_1^{(s)} \kappa_1^{(s)} \cos \theta_s & \sum_{s=1}^n \nu_s \kappa_1^{(s)} \kappa_2^{(s)} \cos \theta_s \\ \sum_{s=1}^n \nu_s \kappa_2^{(s)} \kappa_1^{(s)} \cos \theta_s & \sum_{s=1}^n \nu_s \kappa_2^{(s)} \kappa_2^{(s)} \cos \theta_s \end{vmatrix}, \quad (12\cdot15)$$

and it may be shown that this is

$$A_2 = \sum_{\substack{s, t=1 \\ s < t}}^n \beta_{st} \cos \theta_s \cos \theta_t, \quad (12\cdot16)$$

where the β_{st} are positive quantities given by

$$\beta_{st} = \nu_s \nu_t \left| \begin{matrix} \kappa_1^{(s)} & \kappa_1^{(t)} \\ \kappa_2^{(s)} & \kappa_2^{(t)} \end{matrix} \right|^2. \quad (12\cdot17)$$

We may now prove Asher's first conjecture. Equation (12·16) shows that, when $\omega \leq \omega_1$, A_2 is positive or zero. On the other hand, A_2 can be written

$$A_2 = \mathcal{R}(\alpha_{11})\mathcal{R}(\alpha_{22}) - [\mathcal{R}(\alpha_{12})]^2. \quad (12\cdot18)$$

This shows that A_2 is negative (or zero) whenever

$$\mathcal{R}(\alpha_{11}) = 0 \quad \text{or} \quad \mathcal{R}(\alpha_{22}) = 0. \quad (12\cdot19)$$

Therefore A_2 is negative (or zero) when ω has the value given by the lesser of the first zero of $\mathcal{R}(\alpha_{11})$ and the first zero of $\mathcal{R}(\alpha_{22})$. Therefore the first zero of A_2 must lie between ω_1 and the lesser of the first zeros of $\mathcal{R}(\alpha_{11})$ and $\mathcal{R}(\alpha_{22})$, and therefore must give a better approximation to ω_1 than either $\mathcal{R}(\alpha_{11})$ or $\mathcal{R}(\alpha_{22})$. It may be proved in a similar way that the last zero of A_2 will give a better approximation to ω_n than either $\mathcal{R}(\alpha_{11})$ or $\mathcal{R}(\alpha_{22})$.

Unfortunately Asher's second suggestion—that concerning ω_2 —does not appear to be sound. For there can in fact be three zeros of A_2 all less than ω_2 . To show this we return to our numerical example, for which

$$A_2 = \gamma_1\gamma_2 + 2\gamma_1\gamma_3 + 4\gamma_2\gamma_3. \quad (12\cdot20)$$

This has four zeros and they are at

$$\omega = 1\cdot003, \quad 1\cdot052, \quad 1\cdot097, \quad 1\cdot20976, \quad (12\cdot21)$$

and the first three of these are all less than $\omega_2 = 1\cdot1$. We note however, that the first and last zeros do give better approximations to ω_1 and ω_3 than were obtained from A_1 .

Let A_r denote the determinant of the real part of a receptance matrix of order r . The following results may be proved:

(i) A_r has an even or odd number of zeros according to whether r is even or odd, and all the zeros lie between ω_1 and ω_n inclusive;

(ii) as r increases, the first and last zeros of A_r become progressively closer approximations to the first and last natural frequencies;

(iii) when $r = n - 1$ there will be an odd number of zeros of A_r between any pair of consecutive natural frequencies; in addition (at least when the damping does not couple the modes) there will be just one zero of A_{n-1} between any pair of consecutive natural frequencies, with the possible exception of one, between which there may be three;

(iv) when $r = n$ there are just n zeros and they are at the natural frequencies of the system, whatever n points are chosen; this is the only case in which the zeros of A_r do not depend on the points chosen.

The last result may be illustrated by our numerical example, for which

$$A_3 = \left| \mathcal{R} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \right| = (3 + 2\sqrt{2})\gamma_1\gamma_2\gamma_3; \quad (12\cdot22)$$

the zeros of this expression are at the natural frequencies.

In addition it appears that, except for the case of $r = n$, the intermediate zeros of A_r do not give any reliable information concerning the values of the intermediate natural frequencies.

We conclude by suggesting an alternative procedure for finding the natural frequencies using receptance matrices. First, plot $\mathcal{R}(\alpha_{ii})$ for various points q_i . Each plot will have an odd number of zeros. Suppose $2r - 1$ is the greatest number of zeros occurring in any plot; then r will be the best underestimate for the number of degrees of freedom of the system. Now choose r points and plot

$$A_r = \mathcal{R} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1r} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2r} \\ \dots & \dots & \dots & \dots \\ \alpha_{r1} & \alpha_{r2} & \dots & \alpha_{rr} \end{bmatrix}. \tag{12.23}$$

If this has r zeros it is possible that $r = n$, which would mean that the zeros were the natural frequencies. But A_r can have r zeros even when $r < n$, and even when $r < n - 1$. However, if A_r has the same r zeros whatever r points are chosen then this means that r is equal to n and the zeros are the natural frequencies. If A_r does not have r zeros, but has more or less than this means that $r < n$ and one must choose $r + 1$ or more points and plot the curves for them, until one finds as many zeros as exciters.

Finally, once the number of degrees of freedom and the values of the natural frequencies are known the method of Traill-Nash might be applied as indicated earlier in this section. It remains to be seen how practicable the method is.

13. CONCLUSIONS

The fundamental conclusion of this report is that the whole problem of resonance testing revolves around the nature of the damping of the system under test.

It appears that little experimental work has been carried out with the aim of discovering the mechanism and effects of damping. Because of this lack of experimental evidence there is insufficient information on which to decide which of the many possible mathematical devices that may be used in the equations of motion most nearly represents its action. This means that the choice of mathematical representation is governed very largely by the simplicity of the resulting equations and by little else.

There is another complication. Damping is an inherently imprecise factor and therefore no simple mathematical device can be expected to represent it properly, for any such representation will be precise. Bearing these remarks in mind we consider the various possible ways of representing the damping.

For systems with a single degree of freedom there are two simple representations in common use; it may be 'viscous' or 'hysteretic', and so far as structural vibration is concerned, what little experimental evidence there is comes down in favour of the latter. Of course, the damping may be more nearly represented by non-linear terms, but in this report this possibility has not been taken into account—solely because this representation would not lead to simple equations.

The generalization of viscous and hysteretic damping from one to n degrees of freedom may be made in two ways, one of which is more general than the other. The more general representation of the damping has the effect of coupling the principal modes of the system, so that a force which would have excited just one principal mode in the undamped system

now excites other modes as well. The less general kind does not couple the modes; with this damping the modes are independent and in each mode the system behaves as a system executing damped vibration, as if with a single degree of freedom. As far as the writers are aware there is no conclusive experimental evidence (so far as any particular type of structure is concerned) to show whether the damping does in fact couple the modes or not. Until such evidence is produced it is reasonable to use the simpler theory and assume that it does not, as most of those working in the subject have done.

After these remarks on damping we turn to the problem of resonance testing. The three main aspects of resonance testing are:

- (1) The manner in which the body is made to vibrate.
- (2) The physical quantities that are measured and the graphs that are plotted.
- (3) The ways in which the graphs are analyzed.

The various techniques which are described in the paper are concerned with one or more of these aspects. Thus Lewis & Wrisley (1950) are concerned mostly with the first, while Kennedy & Pancu (1947) are concerned largely with the other two.

The first technique which is described is the Peak Amplitude method. This is essentially a 'common-sense' method in which the distinction between the two kinds of damping is not made. When it is being used one naturally assumes that the damping is of the simpler type, which does not couple the modes. On this assumption the curves of deflexion amplitude may be analyzed to give reliable results provided that the natural frequencies are not close or the damping heavy—these are the two conditions which have to be satisfied for the peaks to be clearly defined.

Kennedy & Pancu (1947) require more information for their method—they need both the amplitude and the phase of the deflexion. But, assuming that the damping does not couple the modes, one may analyze the vector loci they suggest to give reliable results even when the frequencies are close and the damping is heavy. This seems to be the best technique available at present.

Lewis & Wrisley (1950) are concerned more with a method of excitation than with analysis. They suggest a multi-point excitation technique for exciting a principal mode of the system, and it is shown in the report that theoretically the loading they suggest will excite a principal mode provided, again, that the modes are not coupled. They themselves thought that the damping would have to be of a more restricted type for their method to work. Were it not for the complicated instrumentation required for the excitation of the system their method would be ideal. The success of their method provides evidence, though meagre, that the damping does not couple the modes, or at least does not couple them to an appreciable extent.

Only in the last two methods, those proposed by Traill-Nash (1958) and Asher (1958), is the restriction on the damping lifted. But, unfortunately, in both these methods, there seem to be considerable difficulties, both theoretical and practical. The main theoretical difficulty is the same in both cases. The two methods are based on multi-point excitation but, unlike the method of Lewis & Wrisley, they require a definite number of exciters. For a system with n degrees of freedom the Traill-Nash method would require n exciters, and for such a system the method should theoretically give reliable results. In trying to extend the method to continuous systems, which will be the ones encountered in practice, Traill-Nash

uses the fact that in any limited frequency range there are only a few modes of any continuous system which need to be taken into account. But unfortunately this 'effective number of degrees of freedom' is seldom known beforehand, and this is where the method breaks down.

In a sense Traill-Nash goes to the opposite extreme to Lewis & Wrisley. They use a complicated excitation system and isolate a pure mode, which means that they have to do very little calculation. Traill-Nash just measures the in-phase deflexion at a certain number of points when the system is being run at a natural frequency. Thereafter, the computer takes over; and it is clear that, even if the difficulty of the 'effective number of degrees of freedom' is overcome, the great amount of computation that must inevitably be done will be a grave disadvantage in the method.

Asher's method may be regarded as being essentially an attempt to overcome Traill-Nash's difficulty. The method requires the computation of the zeros of the determinants of certain response matrices, the matrices gradually increasing in size. This is really a process of successive approximation and the general idea behind it appears to be attractive. However, in attempting to analyze the curves representing the response determinants one runs into considerable difficulty, and although a method has been suggested here by which the difficulty may be overcome, the modified method is no longer a straightforward process of successive approximation. The method might be workable, but that can be decided only by experiment. It would involve a considerable amount of computation, and would have to be developed further if it is to provide the principal modes and damping factors, as at present it is a method for finding just the natural frequencies.

We conclude by stressing the need for more experimental work. Among particular matters which need to be investigated experimentally we mention the following: the nature of damping; the existence and properties of damped principal modes; the method of Lewis & Wrisley; the idea of the 'effective number of degrees of freedom'; and Asher's method.

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NOTATION

The following are the principal symbols used.

A	inertia matrix
A_r	number defined in equation (12·23)
a_r	r th element in leading diagonal of L
B	viscous damping matrix
C	stability (stiffness) matrix
c_r	r th element in leading diagonal of N
D	hysteretic damping matrix
d_r	r th element in leading diagonal of H if H is a diagonal matrix
H	hysteretic damping matrix referred to principal co-ordinates
L	diagonal inertia matrix
N	diagonal stability (stiffness) matrix
n	number of degrees of freedom
p	column vector of principal co-ordinates $\{p_1, p_2, \dots, p_n\}$
Q	column vector of generalized forces
q	column vector of generalized co-ordinates
s	distance measured along response locus
T	kinetic energy
u	complex column vector of which q is the imaginary part
V	potential energy
X	square matrix whose columns are $\Psi^{(1)}, \Psi^{(2)}, \dots, \Psi^{(n)}$
α_{ij}	cross-receptance between q_i and q_j
$\beta_r = \omega/\omega_r$	
θ_r	r th 'characteristic phase lag'
$\kappa^{(r)}$	amplitude vector of r th 'forced mode' (see equation (3·8))
λ_r	r th multiplier in equation (3·23)
$\mu_r = d_r/c_r$	
Ξ	amplitude vector of generalized forces corresponding to p
Π_r	amplitude of r th element of p
$\Sigma^{(r)}$	amplitude vector of r th 'forced mode of excitation'
Φ	amplitude vector of matrix Q
$\Psi^{(r)}$	r th principal mode
ω_r	r th natural frequency