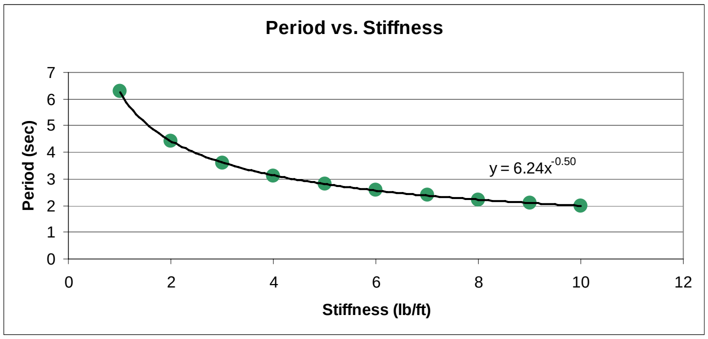
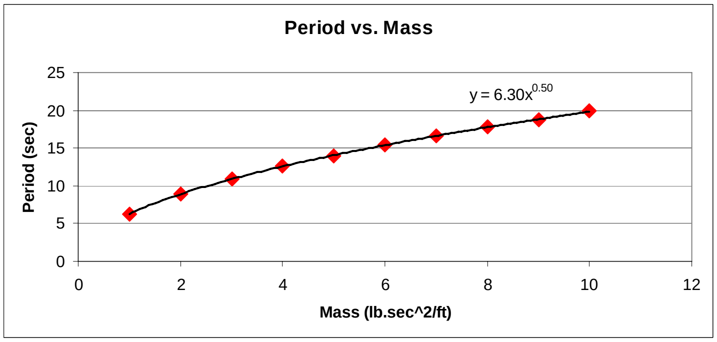
CE 573 – Structural Dynamics
Homework #2

1-a) In each case, the natural period of oscillations is estimated from the numerical simulations of freely vibrating undamped SDOF system.

First, let's keep the mass constant (say, m = 1 lb sec sec / ft) and vary the stiffness over a range of values.
The "data points" and a best-fit to the period vs. stiffness using a power relationship are shown in the next figure.

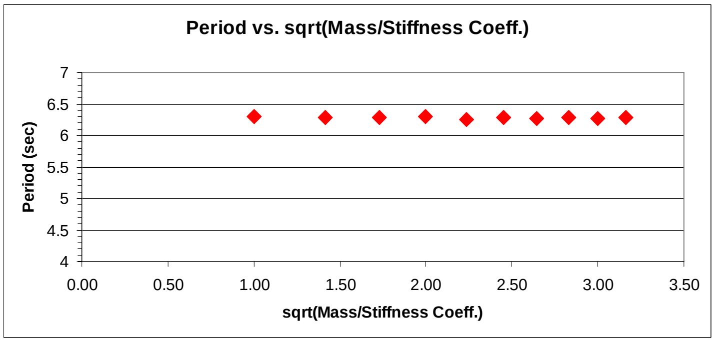


Then, let's keep the stiffness constant (say, k = 1 lb / ft) and vary the mass over a range of values. The "data points" and a best-fit to the period vs. mass using a power relationship are shown in the next figure.



It appears that the period is proportional to $m^{0.5}$ and $\frac{1}{k^{0.5}}$. That is, period is proportional to $\sqrt{\frac{m}{k}}$.

Now, let's find the constant coefficient. Using several data points from our numerical simulations, we can estimate the constant coefficient/multiplier in the relationship.

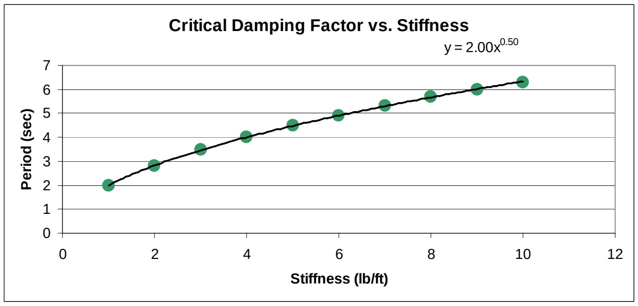


We see that the multiplier is around 6.3 (=2π).

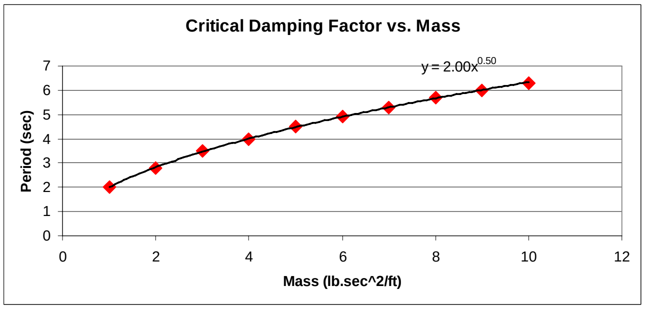
So, we can estimate the relationship between period, mass, and stiffness as $T = 6.3\sqrt{\frac{m}{k}}$.

1-b) In each case, the critical damping for the given system is estimated from numerical simulations of response of damped SDOF system to initial displacement. For a system with given *m* and *k*, the lowest damping coefficient that makes the system go to zero/equilibrium state without any oscillation is found by trial and error.

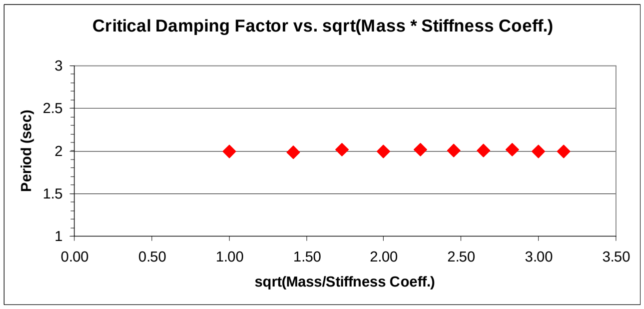
First, let's keep the mass constant (say, m = 1 lb sec sec / ft) and vary the stiffness over a range of values.
The "data points" and a best-fit to the critical damping factor vs. stiffness using a power relationship are shown below.



Then, keep the stiffness constant (say, k = 1 lb / ft) and vary the mass over a range of values. A graph of the "data points" and a best-fit to the critical damping factor vs. mass using a power relationship are shown below.



It appears that the period is proportional to $m^{0.5}$ and $k^{0.5}$. That is, period is proportional to $\sqrt{m \cdot k}$.
Using data points from our numerical simulations, the constant coefficient/multiplier can be found as follows.



The multiplier is around 2. Putting all together, we estimate the relationship between critical damping factor, mass, and stiffness to be $c_{critical} = 2\sqrt{m \cdot k}$.

2) Let's solve for the maximum displacement time the harder way:

$m = 20$ kg

$k = 64{,}000$ N/m

$c = 400$ N·sec/m

$x(0) = 0$

$\dot{x}(0) = 100$ m/sec

$$m\ddot{x} + c\dot{x} + kx = 0$$

normalizing w.r.t. mass, $\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0 \quad (*)$

$$\omega_n = \sqrt{\frac{k}{m}} = 56.56 \text{ rad/sec}$$

$$\xi = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}} = \frac{400}{2\sqrt{64{,}000 \times 20}} = 0.177 < 1$$

the system is subcritically damped

Sol^n to $(*)$ is

$$x(t) = \frac{\dot{x}(0)}{\omega_d} e^{-\xi\omega_n t} \sin\omega_d t \qquad (**)$$

let t_1 be the time required for the board to reach the maximum displacement at which instant the velocity is zero. That is,

$$\dot{x}(t_1) = \underbrace{\frac{\dot{x}(0)}{\omega_d} e^{-\xi\omega_n t}}_{\neq 0} \underbrace{\left[-\xi\omega_n \sin\omega_d t_1 + \omega_d \cos\omega_d t_1\right]}_{\text{has to be } = 0} = 0$$

$$\Rightarrow -\xi\omega_n \sin\omega_d t + \omega_d \cos\omega_d t_1 = 0$$

$$\Rightarrow \tan\omega_d t_1 = \frac{\omega_d}{\xi\omega_n} = \frac{\sqrt{1-\xi^2}}{\xi} = 5.57 \quad \Rightarrow \omega_d t_1 = 1.393 \text{ rad}$$

$$\omega_d = \omega_n \sqrt{1-\xi^2} = 56.56 \times \sqrt{1-0.177^2} = 55.68 \text{ rad/sec}$$

$$\Rightarrow t_1 = \frac{1.393}{55.68} = 0.025 \text{ sec} \;//$$

How about the easier way? Well… From our numerical simulations we know that an initially at rest damped SDOF system acquires its maximum displacement at a quarter of damped period time when hit impulsively (i.e. acquire initial velocity very rapidly) and let respond freely. $T_d = \frac{2\pi}{w_d} = 0.11\text{sec}$ so $t_1 = \frac{T_d}{4} \approx 0.025$ sec.

from $(**)$, $x_{max} = x(0.025) = 1.376$ m //

Forces acting on mass @ $t = t_1$

$$f_{spring} = k * x(t_1) \cong 88{,}093 \text{ N} \leftarrow$$

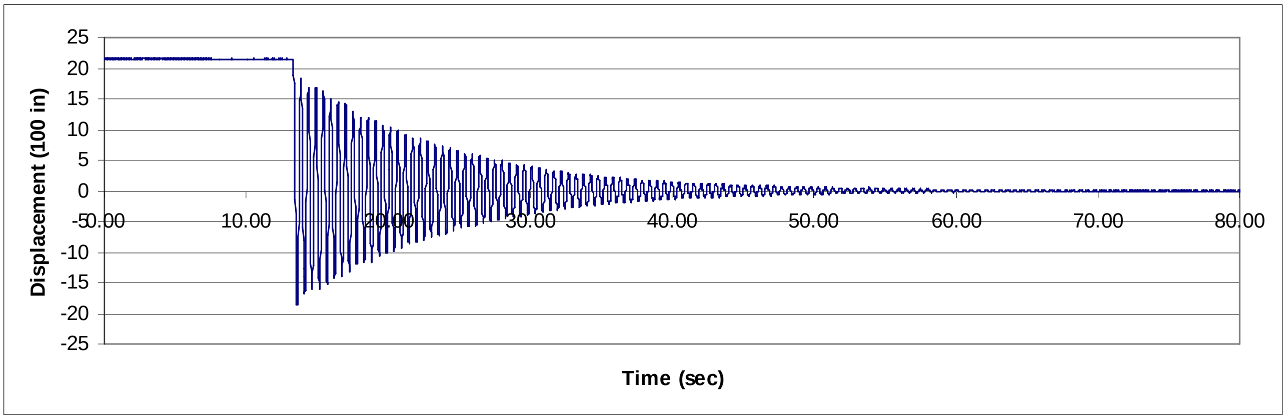
$$f_{dashpot} = c * \dot{x}(t_1) = 0$$

$$f_{inertia} = m * \ddot{x}(t_1) = 88{,}093 \text{ N} \rightarrow \;//$$

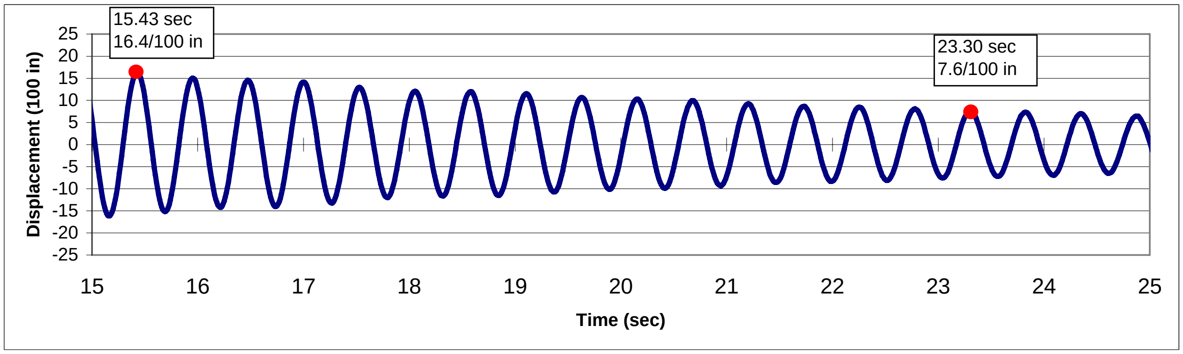
$$\ddot{x}(t_1) = \frac{\dot{x}(0)}{\omega_d} e^{-\xi\omega_n t_1}\Big[-\xi\omega_n\left(-\xi\omega_n \sin\omega_d t_1 + \omega_d \cos\omega_d t_1\right) - \xi\omega_n\omega_d \cos\omega_d t_1 - \omega_d^2 \sin\omega_d t_1\Big]$$

$$\ddot{x}(0.025) = 4{,}404.7 \text{ m/sec/sec} \cong 44\, g \;!$$

3) Graphing the displacement response, we see that the structure was initially pulled to about 0.20 inches and released to vibrate freely. Around 30-35 seconds into the free vibration (about 50 seconds into the record), the displacements fall below 1/1000 inches, the accuracy of the displacement-meter (LVDT – linear variable displacement transducer).



Let’s look at the time window 15 sec to 25 sec of the record.



We see that between 15.43 sec and 23.30 sec, the building makes 15 cycles of oscillations and the peak response drops from 16.4/100 inches to 7.6/100 inches.

Calculating the period: $T = \dfrac{(23.30\,\text{sec} - 15.43\,\text{sec})}{15} \cong 0.5\,\text{sec}$

Assuming that the damping is small, i.e. the building is very lightly damped, we can calculate the damping ratio as $\xi \cong \dfrac{1}{15}\dfrac{\ln\left(\dfrac{16.4}{7.6}\right)}{2\pi} \cong 0.8\%$. 0.8% is small damping, indeed; so our assumption was correct.

4)

Free vibration. Response drops from ~8 in to ~4 in in 9 cycles.
assume small damping ($\xi \ll 1$) : $\omega_d = \sqrt{1-\xi^2}\,\omega_n \approx \omega_n$

logarithmic decrement, $\delta = \frac{1}{n}\ln\left(\frac{x_n}{x_{n+9}}\right) \cong 2\pi\xi$

9 cycles   damping ratio

Draft for Public Comment   Moire Analysis of WTC 2

$\Rightarrow T = \frac{129-27}{9} = \frac{102\text{ sec}}{9\text{ cycles}} \cong 11.3\text{ sec}$ // ← fundamental period

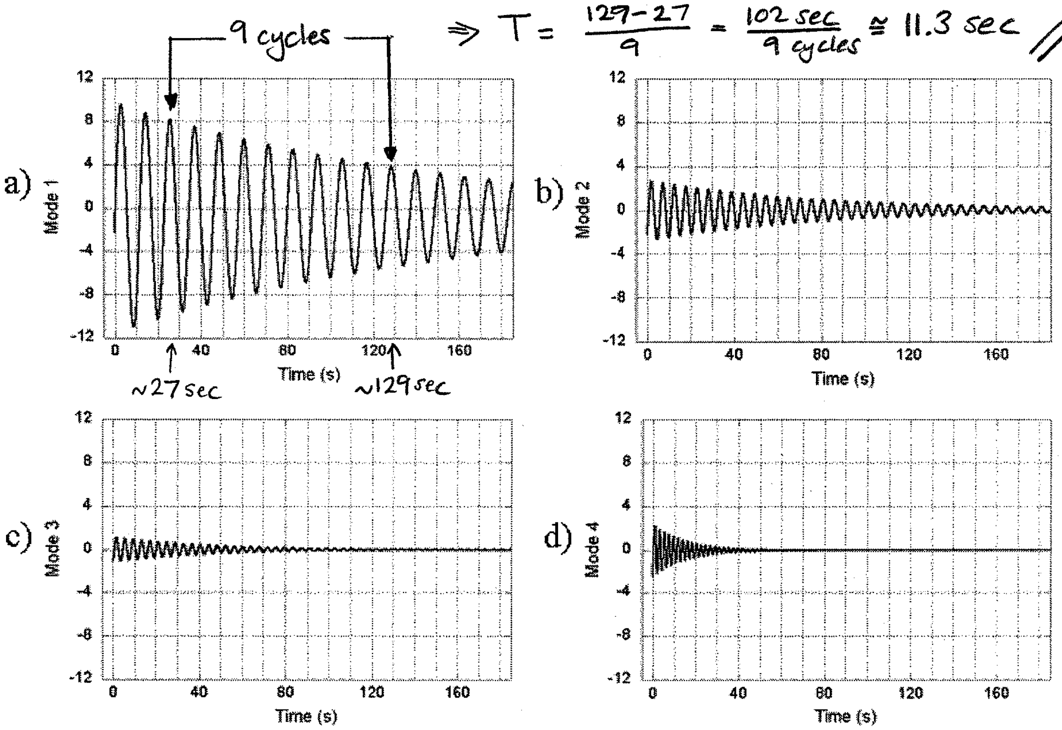


Figure K–14. Modes from analysis of displacement data, with periods of a) 11.39 s, b) 5.26 s, c) 3.90 s, and d) 2.15 s.

$x_n = 8$ in , $x_{n+9} = 4$ in $\Rightarrow \frac{1}{9}\ln\left(\frac{8}{4}\right) = 7.7\times10^{-2} \cong 2\pi\xi$

$\Rightarrow \xi \cong 1.2\times10^{-2} = 1.2\% \ll 1$ //

assumption valid.

The "decay rate" in NIST report, $D = \xi\,\omega_n$

$T_n \approx 11.4\text{ s} \Rightarrow \omega_n = 2\pi/T_n = 0.551$ rad/sec

$\Rightarrow \xi_{NIST} = \frac{D}{\omega_n} = \frac{0.0065}{0.551} \cong 1.2\times10^{-2} = 1.2\%$

They got it right!