CE 573 – Structural Dynamics

## Solution procedure for step-by-step numerical integration based on the average-acceleration method

Given: Equation of motion: $m \cdot \ddot{x}(t) + c \cdot \dot{x}(t) + k \cdot x(t) = F(t)$
m, c, k, F(t), and initial conditions x(0) and $\dot{x}(0)$

If the problem was cast in a continuous-time form, convert it into a discrete-time form by choosing appropriate time-steps $\Delta t_i$ (i=1, …, N). Often times a constant time-step, i.e. $\Delta t_i = \Delta t$, is chosen. A rule-of-thumb for $\Delta t_i$ is that one should have about 10 points measured or computed during each "cycle" in the excitation or the response. In other words, $\Delta t_i \le \frac{T}{10}$ where T is the smaller of the natural period of the structure or the smallest period in the excitation.

The problem in discrete-time form: $m \cdot \ddot{x}_i + c \cdot \dot{x}_i + k \cdot x_i = F_i$ ; i=1, …, N (Eq. 1)

1) Solve for $\ddot{x}(0)$, i.e. $\ddot{x}_1$, from Eq.1: $\ddot{x}_1 = \frac{1}{m}(F_1 - c \cdot \dot{x}_1 - k \cdot x_1)$

2) Compute $k_i^* = k + (\frac{2 \cdot c}{\Delta t_i}) + (\frac{4 \cdot m}{\Delta t_i^2})$

3) Compute $\Delta F_i^* = \Delta F_i + (\frac{4 \cdot m}{\Delta t_i} + 2 \cdot c)\,\dot{x}_i + 2 \cdot m \cdot \ddot{x}_i$

4) Solve for $\Delta x_i$ from $k_i^* \cdot \Delta x_i = \Delta F_i^*$

5) Solve for $\Delta \dot{x}_i$ from $\Delta \dot{x}_i = (\frac{2}{\Delta t_i})\,\Delta x_i - 2 \cdot \dot{x}_i$

and for $\Delta \ddot{x}_i$ from $\Delta \ddot{x}_i = (\frac{4}{\Delta t_i^2}) \cdot (\Delta x_i - \dot{x}_i \cdot \Delta t_i) - 2 \cdot \ddot{x}_i$

6) Compute x, $\dot{x}$, $\ddot{x}$ for the next step from

$$x_{i+1} = x_i + \Delta x_i$$
$$\dot{x}_{i+1} = \dot{x}_i + \Delta \dot{x}_i$$
$$\ddot{x}_{i+1} = \ddot{x}_i + \Delta \ddot{x}_i$$

7) If you have constant $\Delta t$, go to step 3); otherwise, go to step 2. Repeat until i=N.

from Chopra, Dynamics of Structures, 1995.



**TABLE 5.4.2 NEWMARK'S METHOD: LINEAR SYSTEMS**

Special cases

(1) Average acceleration method ($\gamma = \frac{1}{2}$, $\beta = \frac{1}{4}$)

(2) Linear acceleration method ($\gamma = \frac{1}{2}$, $\beta = \frac{1}{6}$)

1.0 *Initial calculations*

1.1 $\ddot{u}_0 = \dfrac{p_0 - c\dot{u}_0 - ku_0}{m}.$

1.2 Select $\Delta t$.

1.3 $\hat{k} = k + \dfrac{\gamma}{\beta\,\Delta t}c + \dfrac{1}{\beta(\Delta t)^2}m.$

1.4 $a = \dfrac{1}{\beta\,\Delta t}m + \dfrac{\gamma}{\beta}c$; and $b = \dfrac{1}{2\beta}m + \Delta t\left(\dfrac{\gamma}{2\beta} - 1\right)c.$

2.0 *Calculations for each time step, i*

2.1 $\Delta\hat{p}_i = \Delta p_i + a\dot{u}_i + b\ddot{u}_i.$

2.2 $\Delta u_i = \dfrac{\Delta\hat{p}_i}{\hat{k}}.$

2.3 $\Delta\dot{u}_i = \dfrac{\gamma}{\beta\,\Delta t}\Delta u_i - \dfrac{\gamma}{\beta}\dot{u}_i + \Delta t\left(1 - \dfrac{\gamma}{2\beta}\right)\ddot{u}_i.$

2.4 $\Delta\ddot{u}_i = \dfrac{1}{\beta(\Delta t)^2}\Delta u_i - \dfrac{1}{\beta\Delta t}\dot{u}_i - \dfrac{1}{2\beta}\ddot{u}_i.$

2.5 $u_{i+1} = u_i + \Delta u_i$, $\dot{u}_{i+1} = \dot{u}_i + \Delta\dot{u}_i$, $\ddot{u}_{i+1} = \ddot{u}_i + \Delta\ddot{u}_i$.

3.0 *Repetition for the next time step*. Replace $i$ by $i + 1$ and implement steps 2.1 to 2.5 for the next time step.

Non-iterative/Explicit formulations.