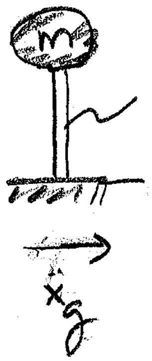
Fundamentals of Vibration (Structural Dynamics)

Linear Elastic SDOF

$\mapsto x$ (relative to ground)



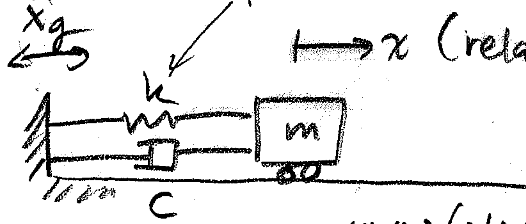
k (stiffness) ← elastic forces are proportional to displacement (relative distortion)

c (damping - viscous damping) ← viscous forces are proportional to velocity

Mech Eng.

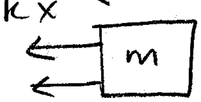
initially unstretched (when $x=0$)

$\mapsto x$ (relative to ground)



$m(\ddot{x}+\ddot{x}_g)$ (D'Alembert force) ⟶ accelerating towards $+x$

FBD



Equil. Eqn ⇒ Dynamic: $-kx - c\dot{x} - (m\ddot{x}+\ddot{x}_g) = 0$

$$\underbrace{+kx(t)}_{\text{elastic restoring forces (column/walls)}} + \underbrace{c\dot{x}(t)}_{\text{energy dissipating elements}} + \underbrace{m\ddot{x}(t)}_{\text{inertia force}} = \overbrace{-m\ddot{x}_g(t)}^{\text{ground acceleration}}$$

inertia force (exists due to dynamic/time-variant loading)

elastic restoring forces (column/walls)

energy dissipating elements (modeled as viscous damping viscoelasticity)

in-element

leak to surroundings? (soil-structure interaction?)

Look @ the homogeneous part

Soln consists of $x_{homog.}$ & $x_{particular}$

initial conditions

steady-state (transient)

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad (*)$$

Homogeneous 2nd order Ordinary Diff. Eqn.

Try $x(t) = A e^{\lambda t}$ as a soln

$$\Rightarrow (m\lambda^2 + c\lambda + k)\underbrace{A e^{\lambda t}}_{\neq 0 \text{ (otherwise trivial soln)}} = 0$$

$$\Downarrow$$

$$= 0$$

$$m\lambda^2 + c\lambda + k = 0 \qquad \text{characteristic eqn.}$$

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$\boxed{x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}}$$ General soln.

A_1, A_2 to be found from initial conditions (disp & vel.)

The term in $\lambda_{1,2} \Rightarrow \sqrt{c^2 - 4mk}$

if $c^2 - 4mk < 0$ $\quad \lambda_{1,2} \Rightarrow$ complex conjugates

$= 0$ $\quad \lambda_{1,2} \Rightarrow$ repeated roots

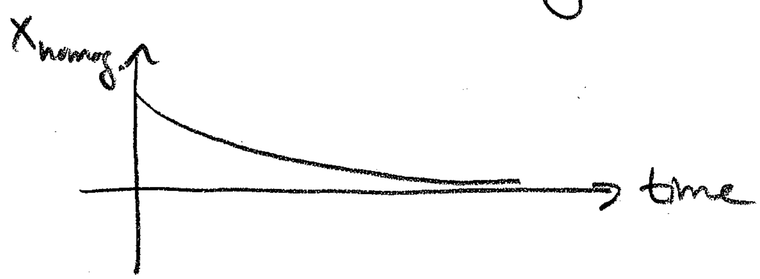
> 0 $\quad \lambda_{1,2} \Rightarrow$ real & distinct (unequal) roots

negative

(critically damped system)

(*) $c^2 - 4mk = 0 \Rightarrow c_{cr} = 2\sqrt{km}$ critical damping ratio.

c_{CR} is the smallest linear viscous damping ratio that prohibits "oscillatory" response in the SDOF system $\left(x_{homog}(t) = A_1 e^{-\frac{c}{2m}t} + A_2 t e^{-\frac{c}{2m}t} \right)$



if $c > c_{CR} \rightarrow$ super-critically damped system

no oscillation

$x_h(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$

if $c < c_{CR} \rightarrow$ sub-critically damped

most man-made structures are sub-critically damped

(actually 2-10% of critical damping)

$x_h(t) = e^{-\frac{c}{2m}t} \left(B_1 \cos \frac{q c_{cr}}{2m} t + B_2 \sin \frac{q c_{cr}}{2m} t \right)$

($x_{complementary}$)

(*) Now, expressing damping coeff. as $\underline{c = \xi\, c_{cr}}$

$$c = \xi \cdot 2\sqrt{km}$$

and calling $\underline{\sqrt{\dfrac{k}{m}} = \omega_n}$ (undamped circular natural frequency)

$$m\ddot{x} + c\dot{x} + kx = -m\ddot{x}_g$$

$1/m$ becomes

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = -\ddot{x}_g(t)$$

$\Rightarrow$ characteristic eqn becomes $\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 = 0$

and for underdamped systems $\boxed{\lambda_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{1-\xi^2}}$

$$x_h(t) = e^{-\xi\omega_n t}\underbrace{\left(B_1\cos\omega_d t + B_2\sin\omega_d t\right)}_{C\,\sin(\omega_d t + \varphi)}$$

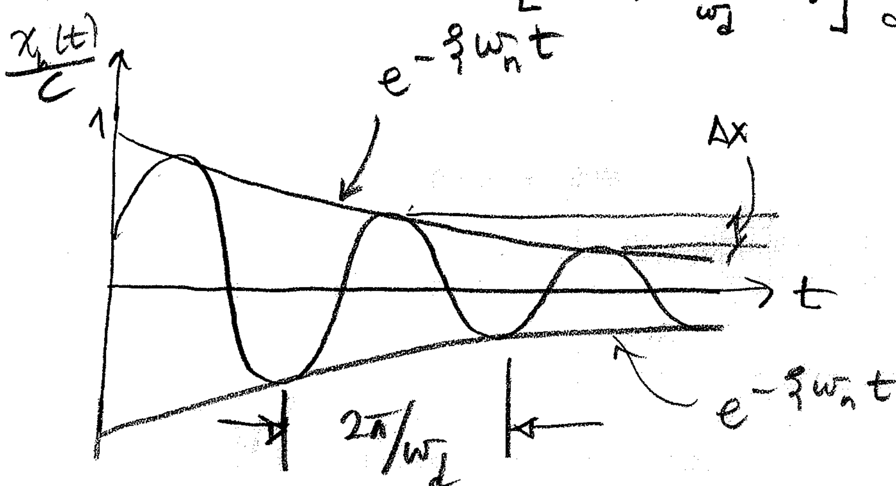
$x_{complementary}$

$$C = \left[x_0^2 + \left(\frac{\dot{x}_0 + \xi\omega_n x_0}{\omega_d}\right)^2\right]^{1/2}$$

$\boxed{\varphi = \tan^{-1}\dfrac{B_1}{B_2}}$

damped natural freq

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$



ASIDE: Damping in bldgs (free vibration)

Consider the relationship btw successive "peaks" of the response.

Define these peaks as the points $\sin(\omega_d t + \phi) = 1$

$$x_n = C\, e^{-\xi \omega_n t_n}$$

$$x_{n+1} = C\, e^{-\xi \omega_n \left(t_n + \frac{2\pi}{\omega_d}\right)}$$

$$\frac{x_n}{x_{n+1}} = e^{+2\pi \xi \frac{\omega_n}{\omega_d}} = e^{+2\pi \frac{\xi}{\sqrt{1-\xi^2}}}$$

take log. $\Rightarrow$ logarithmic decrement, $\delta \equiv \ln\left(\frac{x_n}{x_{n+1}}\right)$

$$\delta = \frac{2\pi \xi}{\sqrt{1-\xi^2}}$$

for $\xi \ll 1$ $\boxed{\delta \cong 2\pi \xi}$

furthermore, $\delta = \ln\left(\frac{x_{n+1} + \Delta x}{x_{n+1}}\right) = \ln\left(1 + \frac{\Delta x}{x_{n+1}}\right) = \frac{\Delta x}{x_{n+1}} + \mathcal{O}\left(\frac{\Delta x}{x_{n+1}}\right)^2$

$$\delta \cong \frac{\Delta x}{x_{n+1}} \quad \text{for } \xi \ll 1.$$

$$\Rightarrow 2\pi \xi = \frac{\Delta x}{x_{n+1}} \Rightarrow \boxed{\xi \cong \frac{1}{2\pi} \frac{\Delta x}{x_{n+1}}}$$

Aside:

Estimation of free vibration:

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{kg}{mg}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{g}{W/k}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{g}{\Delta_{static}}}$$

$$f_n\,(Hz) = \frac{3.13}{\sqrt{\Delta_{static}\,(in)}}$$