

# Exact and Numerically Stable Closed-Form Expressions for Potential Coefficients of Rectangular Conductors

Jitesh Jain, Cheng-Kok Koh, and Venkataramanan Balakrishnan

**Abstract**—Existing exact closed-form expressions for the scalar mutual and self-potential coefficients for rectangular conductors may be ill-conditioned for certain geometries. We propose new, exact, closed-form expressions for potential coefficients that are much better-conditioned. The basic idea is to express all potential coefficients as weighted sums of mutual and self-potential coefficients of suitably defined virtual plates. Experimental results are presented to demonstrate the improved numerical stability of the new formulas.

**Index Terms**—Capacitance extraction, closed-form expressions, interconnect, potential coefficient.

## I. INTRODUCTION

CAPACITANCE extraction has been a fundamental problem in the modeling and analysis of VLSI interconnects. A number of approaches are available for the extraction of capacitance for a general structure of  $n$  conductors [1], [2], [5], [7]. To calculate capacitance, Laplace equations have to be solved numerically over a charge free region, with the conductors providing the boundary conditions. Though there are a number of numerical methods that could be employed for the solution of Laplace equations involved, the usual approach is to use a boundary element technique for solving the integral form of Laplace equations [2], [5]. Here, the conductor surfaces are divided into rectangular panels, and their surface charge density  $\sigma$  is computed by solving the equation

$$\varphi(x) = \int_{\text{surfaces}} G(x, x') \sigma(x') da' \quad (1)$$

where  $x, x' \in R^3$ ,  $\varphi$  is the known surface potential, and  $G(x, x')$  is Green's function. Given the surface charge density,  $\sigma$ , the total charge on the conductor can be calculated by summing it over the panels that comprise the entire surface. If conductor  $j$  is raised to a unit potential, the partial capacitance between the  $i$ th and  $j$ th conductors  $C_{ij}$  is simply equal to the charge on conductor  $i$ .

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The approach presented in [5] is to divide the surface into a large number of panels  $A_i$  so that the charge on each panel can be assumed to be constant. The charge voltage relationship for the panels can then be written as matrix equation of the form  $Pq = v$ , where  $v$  is the vector of panel potentials,  $q$  is the vector of panel charges, and  $P$  is the potential coefficient matrix of the system. The capacitance matrix is now obtained by summing the panel charges. The potential coefficient between panels  $\mathcal{P}_i$  and  $\mathcal{P}_j$  is given by

$$P_{ij} = \frac{1}{A_i A_j} \int_{x_i \in \mathcal{P}_i} \int_{x_j \in \mathcal{P}_j} \frac{1}{4\pi\epsilon_0 \|x_i - x_j\|} da_j da_i \quad (2)$$

where  $A_i$  and  $A_j$  denote the areas of panels  $\mathcal{P}_i$  and  $\mathcal{P}_j$ , respectively.

In [2], the potential coefficients are approximated by

$$P_{ij} = \frac{1}{\text{area}(\mathcal{P}_j)} \int_{x_i \in \mathcal{P}_i} \frac{1}{4\pi\epsilon_0 \|x' - x_i\|} da' \quad (3)$$

where  $x'$  is the center of panel  $j$ .

For rectangular geometries, the expression in (2) can be simplified to yield explicit formulas [4]. While they are simple to evaluate, these formulas are numerically ill-conditioned, consequently when the separation between the conductor surfaces is large (and yet realistic), the evaluation of the formulas, even with double-precision, leads to erroneous results (see Section III for numerical results).

The numerical ill conditioning in the formulas is caused by cancellation errors. The same issue arises in the calculation of inductance of rectangular conductors, where it has been observed that the numerical inaccuracies in the evaluation of mutual inductance formulas are much more severe than those with self inductance formulas [8]. Numerically stable formulas for self-inductances were given in [3] and [6]. The authors in [8] then derived numerically well conditioned formulas for mutual inductances of parallel conductors by expressing them in terms of self inductances. Our work in spirit takes a similar approach to derive stable expressions for mutual potential coefficients. However, the work here is more involved than deriving inductance formulas. First, in inductance extraction with parallel conductors, the basic problem has the filaments in a parallel orientation. However, with capacitance extraction, the plates can be aligned in three different configurations (see Section II). Second, the expressions for self-potential coefficients

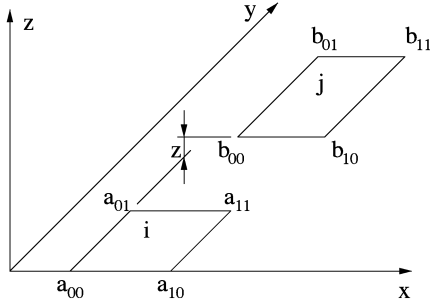
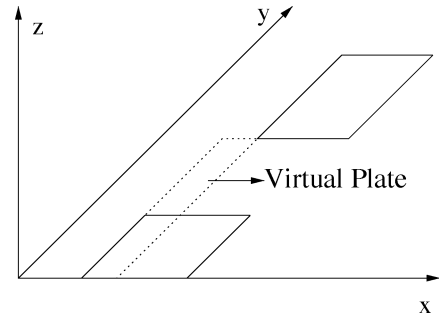


Fig. 1. Cells aligned in parallel direction.


 Fig. 2. Virtual plate for corners  $a_{00}$  and  $b_{00}$  of Fig. 1 when  $z = 0$ .

in [4] suffer from cancellation errors, and must be rewritten to improve their numerical stability.

In this brief, we derive the explicit formulas for the expression in (2) for rectangular conductors. We first present formulas for self potential coefficients that are theoretically equivalent to, but numerically more stable than, the ones derived in [4]. We then express the mutual potential between two plates as a weighted sum of self potentials. Numerical results show that our formulas are numerically more stable than those derived in [4].

## II. FORMULAS FOR POTENTIALS

### A. Parallel Plates

We first consider the case of two parallel plates, as shown in Fig. 1. Without loss of generality, we assume that plate  $i$  is in the  $xy$ -plane. Let  $a_{kl}$ ,  $k, l \in \{0, 1\}$  denote the four corners of plate  $i$ , with  $\{x_k, y_l, 0\}$  being the cartesian coordinates of corner  $a_{kl}$ . Let  $b_{kl}$ ,  $k, l \in \{0, 1\}$  denote the four corners of plate  $j$ , with  $\{u_k, v_l, z\}$  being the cartesian coordinates of corner  $b_{kl}$ . Let  $A_i$  and  $A_j$  be the areas of plates  $i$  and  $j$  respectively. For this case, (2) yields

$$P_{ij} = \frac{1}{4\pi\epsilon_0 A_i A_j} \int_{x_0}^{x_1} \int_{u_0}^{u_1} \int_{y_0}^{y_1} \int_{v_0}^{v_1} \frac{1}{r_{ij}} dx du dy dv \quad (4)$$

where  $r_{ij} = \sqrt{(x-u)^2 + (y-v)^2 + z^2}$ . The formulas in [4] are essentially simplified forms of this integral.

The approach that we employ relies on the following simple, yet key, observation from calculus [8].

*Lemma 1:* Let  $f : \mathbf{R}^2 \rightarrow \mathbf{R}$  satisfy  $f(x, y) = g(|x - y|)$  for some  $g : \mathbf{R} \rightarrow \mathbf{R}$  and for all  $x, y \in \mathbf{R}$ . Then

$$\begin{aligned} & \int_{x_0}^{x_1} \int_{u_0}^{u_1} f(x, u) dx du \\ &= \frac{1}{2} \left( \int_{x_0}^{u_1} \int_{x_0}^{u_1} f(x, u) dx du + \int_{x_1}^{u_0} \int_{x_1}^{u_0} f(x, u) dx du \right. \\ & \quad \left. - \int_{x_0}^{u_0} \int_{x_0}^{u_0} f(x, u) dx du - \int_{x_1}^{u_1} \int_{x_1}^{u_1} f(x, u) dx du \right) \\ &= \frac{1}{2} \sum_{i,j=0}^1 (-1)^{i+j+1} \int_{x_i}^{x_j} \int_{x_i}^{x_j} f(x, u) dx du. \end{aligned} \quad (5)$$

Lemma 1 states that the integral of a function of two variables  $x$  and  $u$ , that is symmetric about the line  $x = u$  over a *rectangle*, can be expressed as a combination of integrals over four *squares*.

Using (5), we may rewrite (4) as

$$\begin{aligned} P_{ij} &= \frac{1}{4\pi\epsilon_0 A_i A_j} \int_{x_0}^{x_1} \int_{u_0}^{u_1} \frac{1}{2} \sum_{k,l=0}^1 (-1)^{k+l+1} \int_{y_k}^{y_l} \int_{y_k}^{y_l} \frac{1}{r_{ij}} dx dy du dv \\ &= \frac{1}{16\pi\epsilon_0 A_i A_j} \sum_{k,l,m,n=0}^1 (-1)^{k+l+m+n} \\ & \quad \times \int_{x_m}^{u_n} \int_{x_m}^{u_n} \int_{y_k}^{v_l} \int_{y_k}^{v_l} \frac{1}{r_{ij}} dx du dy dv. \end{aligned} \quad (6)$$

Thus, the implication of Lemma 1 for the calculation of the potential coefficient  $P_{ij}$  is that  $P_{ij}$  can be expressed as a weighted sum of sixteen integrals. These integrals have interesting physical interpretations, and can be further simplified.

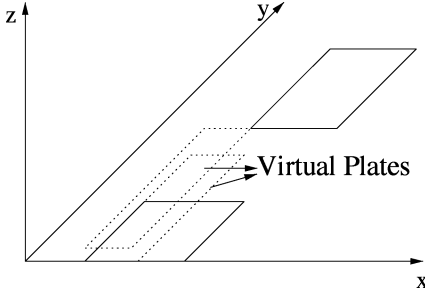
- Consider the case  $z = 0$ , i.e., when plates  $i$  and  $j$  both lie in the  $xy$ -plane. In this case every integral in the formula (6) can be interpreted as the *self-potential* of a “virtual” rectangular plate of dimensions  $|x_m - u_n| \times |y_k - v_l|$ . One such virtual plate is shown in Fig. 2.

It is readily shown that in this case, we may further simplify each integral

$$\begin{aligned} I(k, l, m, n) &= \int_{x_m}^{u_n} \int_{x_m}^{u_n} \int_{y_k}^{v_l} \int_{y_k}^{v_l} \frac{1}{r_{ij}} dx du dy dv \\ &= -\frac{2}{3} \ell^2 w^2 \left( \frac{1}{\ell + \sqrt{\ell^2 + w^2}} + \frac{1}{w + \sqrt{\ell^2 + w^2}} \right) \\ & \quad + 2\ell^2 w S\left(\frac{w}{\ell}\right) + 2w^2 \ell S\left(\frac{\ell}{w}\right) \end{aligned} \quad (7)$$

where  $\ell = |x_m - u_n|$ ,  $w = |y_k - v_l|$ , and  $S(x) = \sinh^{-1}(x)$ .

- Next, consider the case  $z \neq 0$ . In this case each integral can be interpreted as the potential coefficient between two parallel plates of dimension  $|x_m - u_n| \times |y_k - v_l|$ , with a separation of  $z$ . One such set of virtual plates is shown in Fig. 3.

Fig. 3. Virtual plate for corners  $a_{00}$  and  $b_{00}$  of Fig. 1 when  $z \neq 0$ .

It is readily shown that in this case, each integral can be further simplified

$$\begin{aligned}
 I(k, l, m, n) &= \int_{x_m}^{u_n} \int_{x_m}^{u_n} \int_{y_k}^{v_l} \int_{y_k}^{v_l} \frac{1}{r_{ij}} dx dy dz \\
 &= z^3 \left[ (p^2 - 1)qS\left(\frac{q}{\sqrt{p^2 + 1}}\right) \right. \\
 &\quad + 6(q^2 - 1)pS\left(\frac{p}{\sqrt{q^2 + 1}}\right) \\
 &\quad + 6pS(p) + 6qS(q) - 12pqT\left(\frac{pq}{\sqrt{p^2 + q^2 + 1}}\right) \\
 &\quad + 4p^2\left(\frac{1}{\sqrt{p^2 + q^2 + 1} + \sqrt{q^2 + 1}} - \frac{1}{1 + \sqrt{1 + p^2}}\right) \\
 &\quad + 2q^2\sqrt{q^2 + 1} + 2p^2\sqrt{p^2 + 1} \\
 &\quad \left. - 2(p^2 + q^2)\sqrt{p^2 + q^2 + 1} \right], \quad (8)
 \end{aligned}$$

where  $T(x) = \tan^{-1}(x)$ ,  $S(x) = \sinh^{-1}(x)$ ,  $p = (|x_m - u_n|/z)$ ,  $q = (|y_k - v_l|/z)$ ,  $r = \sqrt{p^2 + q^2}$ .

In summary, we have the following new formula:

$$P_{ij} = \frac{1}{16\pi\epsilon_0 A_i A_j} \sum_{k, l, m, n=0}^1 (-1)^{k+l+m+n} I(k, l, m, n). \quad (9)$$

We note that it is possible that some of the virtual plates have zero area, e.g., when  $x_1 = u_1$ . In such cases, the corresponding integrals simply drop out of the sum in (9) yielding expressions with fewer than sixteen terms. Finally, while the formula in (9) appear different from the ones in [4], they evaluate in theory the same quantity. However, we will demonstrate via examples in Section III that the formula in (9) is much better conditioned numerically than the formulas in [4].

## B. Perpendicular Plates

We next consider the second potential coefficient calculation encountered in conductors with rectangular geometries, that of two perpendicular plates, as shown in Fig. 4.

Without loss of generality, we assume that plate  $i$  is in the  $xy$ -plane. Let  $a_{kl}$ ,  $k, l \in \{0, 1\}$  denote the four corners of plate  $i$ , with  $\{x_k, y_l, 0\}$  being the cartesian coordinates of corner  $a_{kl}$ .

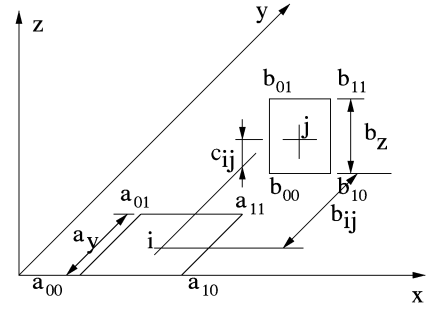
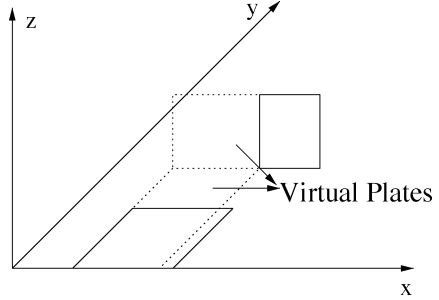


Fig. 4. Cells aligned in perpendicular direction.

Fig. 5. Virtual plate for corners  $a_{00}$  and  $b_{00}$  of Fig. 4.

Let  $b_{kl}$ ,  $k, l \in \{0, 1\}$  denote the four corners of plate  $j$ , with  $\{u_k, y_j, z_l\}$  being the Cartesian coordinates of corner  $b_{kl}$ . Let  $A_i$  and  $A_j$  be the areas of plates  $i$  and  $j$ , respectively.

Paralleling the development in Section II-A, the formula in (2) can be simplified to yield

$$P_{ij} = \frac{1}{4\pi\epsilon_0 A_i A_j} \int_{x_0}^{x_1} \int_{u_0}^{u_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \frac{1}{\hat{r}_{ij}} dx dy dz \quad (10)$$

where  $\hat{r}_{ij} = \sqrt{(x_0 - x_1)^2 + (y - y_j)^2 + z^2}$ .

Using (5), we can rewrite the above equation as

$$P_{ij} = \frac{1}{8\pi\epsilon_0 A_i A_j} \sum_{k, l=0}^1 (-1)^{k+l+1} \int_{x_k}^{u_k} \int_{x_k}^{u_k} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \frac{1}{\hat{r}_{ij}} dx dy dz. \quad (11)$$

Thus, as with the case of parallel plates, the potential coefficient  $P_{ij}$  can be expressed as a weighted sum of integrals of a special type, there being four integrals in the sum in this case. Each integral can be interpreted as the self potential of a virtual plate, an example of which is shown in Fig. 5. Moreover, each integral can be simplified to yield a closed-form formula

$$\begin{aligned}
 I(k, l) &= \int_{x_k}^{u_k} \int_{x_k}^{u_k} \int_{y_0}^{y_1} \int_{z_0}^{z_1} \frac{1}{\hat{r}_{ij}} dx dy dz \\
 &= \sum_{m=1}^2 \sum_{n=1}^2 (-1)^{m+n} \\
 &\quad \times \left[ x^2 c_n \ln(b_m + \rho_{mn}) - \frac{c_n^3}{3} \ln \frac{b_m + \rho_{mn}}{b_m + \tau_{mn}} \right. \\
 &\quad \left. + x^2 b_m \ln(c_n + \rho_{mn}) - \frac{b_m^3}{3} \ln \frac{c_n + \rho_{mn}}{c_n + \tau_{mn}} \right]
 \end{aligned}$$

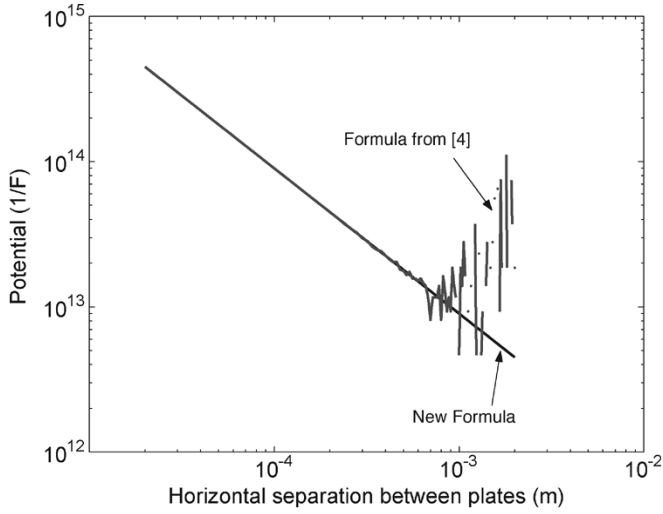


Fig. 6. Cells in the same plane.

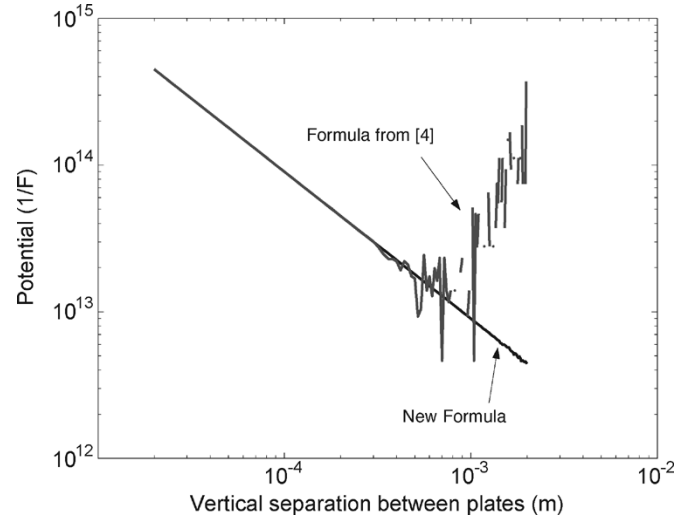


Fig. 8. Cells oriented in parallel direction.

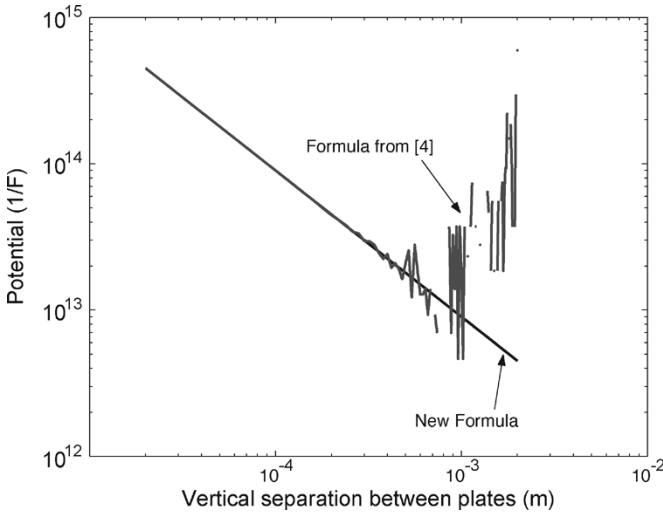


Fig. 7. Cells oriented in parallel direction.

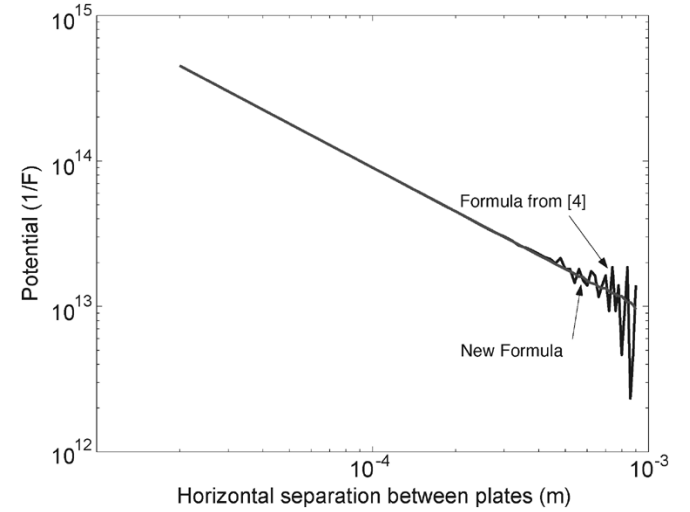


Fig. 9. Cells oriented in perpendicular direction.

$$\begin{aligned}
 & +xb_m c_n S\left(\frac{x}{\tau_{mn}}\right) - \frac{2b_m c_n}{3}(\rho_{mn} + \tau_{mn}) \\
 & - \frac{x^3}{3} T\left(\frac{b_m c_n}{x\rho_{mn}}\right) - b_m^2 x T\left(\frac{x c_n}{b_m \rho_{mn}}\right) \\
 & \left. - c_n^2 x T\left(\frac{x b_m}{c_n \rho_{mn}}\right)\right] \quad (12)
 \end{aligned}$$

where  $x = |x_k - u_l|$ ,  $\rho_{mn} = \sqrt{x^2 + b_m^2 + c_n^2}$ ,  $\tau_{mn} = \sqrt{b_m^2 + c_n^2}$ ,  $b_1 = b_{ij} + (a_y/2)$ ,  $b_2 = b_{ij} - (a_y/2)$ ,  $c_1 = c_{ij} + (b_z/2)$ ,  $c_2 = c_{ij} - (b_z/2)$ .

Thus

$$P_{ij} = \frac{1}{8\pi\epsilon_0 A_i A_j} \sum_{k,l=0}^1 (-1)^{k+l+1} I(k,l). \quad (13)$$

This formula is theoretically equivalent to those in [4]; however, it turns out to be much better conditioned.

### III. NUMERICAL RESULTS

In this section, we compare the results for numerical stability of potential coefficients as obtained by evaluating our expressions and as given by formulas in [4]. Results are shown for the

following four cases with parallel and perpendicular configurations. In each of the cases, the cross-section of the two plates are  $0.1 \mu\text{m} \times 0.1 \mu\text{m}$ . All the expressions have been implemented in MATLAB<sup>1</sup> with double precision.

- *Configuration 1:* Both plates are in the same plane. Potential coefficients are calculated as the horizontal separation between the plates is increased.
- *Configuration 2:* Plates are oriented in parallel direction in two different planes. Potential coefficients are calculated as the vertical separation between the planes is increased.
- *Configuration 3:* Plates are oriented in parallel direction in two different planes. Potential coefficients are calculated as the vertical as well as the horizontal separation between the plates is increased.
- *Configuration 4:* Plates are oriented in perpendicular directions in two different planes. Potential coefficients are calculated as the horizontal separation between the plates is increased.

<sup>1</sup>MATLAB is a registered trademark of Mathworks.

TABLE I  
CAPACITANCE VALUES ( $\times 10^{-18}$  F) AS OBTAINED BY OUR TOOL AND FASTCAP

L(in $\mu\text{m}$ )	Our tool	Fastcap	Standard
2	95.26	94.89	95.59
4	132.56	132.20	132.90
6	165.44	165.10	165.80
8	195.86	195.50	196.30
10	224.63	224.30	225.10
12	252.19	251.80	252.70
14	278.80	278.50	279.30
16	304.64	304.30	305.20

Results are shown in Figs. 6–9, respectively. As can be seen from the figures, the formulas from [4] tend to become unstable as the separation between the plates is increased. On the other hand, the new formula proposed are much more numerically stable over long distances. Note that some of the setup in these experiments may not reflect on-chip interconnect realistically. However, they do demonstrate the differences in the numerical robustness of the two formulas.

We have also implemented a simple capacitance extraction tool in MATLAB, using the new formulas, based on the Boundary Element Method (BEM) approach described in Section I. The purpose is to evaluate the accuracy of the newly derived potential coefficient expressions in the context of capacitance extraction. For this experiment we considered wires of cross-sectional area  $1 \mu\text{m} \times 1 \mu\text{m}$  and lengths varying from  $2 \mu\text{m}$  to  $16 \mu\text{m}$ . For each  $1 \mu\text{m} \times 1 \mu\text{m}$  surface, we used a uniform discretization of  $5 \times 5$  panels. The capacitance values extracted by our tool and Fastcap are listed in Table I, columns 2 and 3. Note that “L” denotes length of the conductor under consideration. To demonstrate the accuracy of our tool, we also ran Fastcap with very fine discretization ( $20 \times 20$  panels for each  $1 \mu\text{m} \times 1 \mu\text{m}$  surface), leading to more accurate values. These “Standard” results are reported in column 4 of Table I.

The table shows that the capacitance values obtained from our tool are at least as accurate as Fastcap with the same panel discretization.

#### IV. CONCLUSION

We have presented formulas for mutual and self potential coefficients of plates that are numerically more stable than the ones derived in literature [4]. The main reason behind the stability of our expressions is that all potential coefficients, mutual as well as self, have been expressed as a weighted sum of self potential coefficients of virtual plates, which are derived to be numerically stable than their counterparts in [4].

#### REFERENCES

- [1] R. Guerrieri and A. Sangiovanni-Vincentelli, “Three-dimensional capacitance evaluation on a connection machine,” *IEEE Trans. Comput.-Aided Des. Integr. Circuits Syst.*, vol. 7, no. 11, pp. 1125–1133, Nov. 1988.
- [2] K. Nabors and J. White, “Fastcap: A multipole accelerated 3-d capacitance extraction program,” *IEEE Trans. Comput.-Aided Des. Integr. Circuits Syst.*, vol. 11, no. 11, pp. 1447–1459, Nov. 1991.
- [3] A. E. Ruehli, “Inductance calculation in a complex integrated circuit environment,” *IBM J. Res. Dev.*, pp. 470–481, Sep. 1972.
- [4] A. E. Ruehli and P. A. Brennan, “Efficient capacitance calculations for three dimensional multiconductor systems,” *IEEE Trans. Microw. Theory Tech.*, vol. MTT-21, no. 2, pp. 76–82, Feb. 1973.
- [5] W. Shi, J. Liu, N. Kakani, and T. Yu, “A fast hierarchical algorithm for three dimensional capacitance extraction,” *IEEE Trans. Comput.-Aided Des. Integr. Circuits Syst.*, vol. 21, no. 4, pp. 330–336, Mar. 2002.
- [6] R.-B. Wu, C.-N. Kuo, and K. C. Kwei, “Inductance and resistance computations for three-dimensional multiconductor interconnect structures,” *IEEE Trans. Microw. Theory Tech.*, vol. 40, no. 2, pp. 263–270, Feb. 1992.
- [7] A. H. Zemanian, R. P. Tewarson, C. P. Ju, and J. F. Jen, “Three dimensional capacitance computations for vlsi/ulsi interconnections,” *IEEE Trans. Comput.-Aided Des. Integr. Circuits Syst.*, vol. 8, no. 12, pp. 1319–1326, Dec. 1989.
- [8] G. Zhong and C.-K. Koh, “Exact closed-form formula for partial mutual inductances of rectangular conductors,” *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 51, no. 10, pp. 1349–1352, Oct. 2003.