PURDUE

COURSENAME/SECTIONNUMBER
EXAM TITLE

NAME ______________________________

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1. Make sure your name and PUID are clearly written at the top of every page, including any additional blank pages you use.
2. Write only on the front of the exam pages.
3. Add any additional pages used to the back of the exam before turning it in.
4. Ensure that all pages are facing the same direction.
5. Answer all questions in the area designated for that answer. Do not run over into the next question space.

Midterm #1 of ECE302, Section 3
8–9pm, Tuesday, September 12, 2023, SMTH 108.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**
2. This is a closed book exam.
3. This exam may contain some multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.
4. Use the back of each page for rough work.
5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Solution

Date: 9/12/2023

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Question 1: [16%, Work-out question] Consider a 2-dimensional continuous function $f(x, y)$.

$$f(x,y) = \begin{cases} \frac{1}{4} & \text{if } 1 \le x \le 5 \text{ and } 0 \le y \le 0.5(x-1) \\ 0 & \text{otherwise} \end{cases}$$

1. [12%] Compute the value of the following integral.

$$\int_{t=-\infty}^{\infty} \int_{s=-\infty}^{\infty} t \cdot f(s,t) ds dt. \tag{1}$$

2. [4%] Define the following expression:

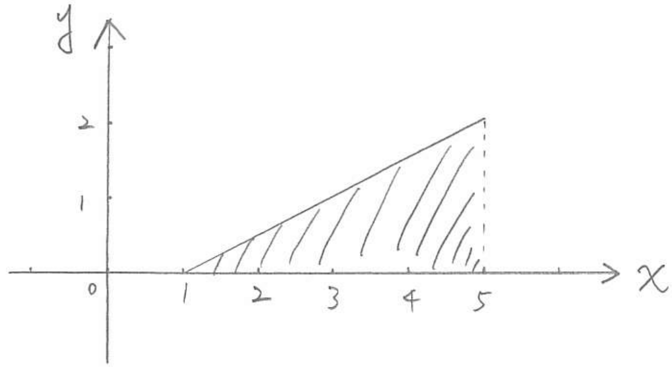
$$F(x,y) = \int_{t=-\infty}^{y} \int_{s=-\infty}^{x} f(s,t) ds dt. \tag{2}$$

Question: Which of the following statement is true? (a) $F(7,8) < F(9,10)$; (b) $F(7,8) = F(9,10)$; or (c) $F(7,8) > F(9,10)$.

Hint 1: This is not a multiple-choice question. You need to write down some justification. A correct answer without any justification will not earn any point.

Hint 2: There is no need to actually compute the values of $F(7,8)$ and/or $F(9,10)$.

1.



$$\int_{t=-\infty}^{\infty}\int_{s=-\infty}^{\infty} t\cdot f(s,t)\,ds\,dt = \int_{s=-\infty}^{\infty}\int_{t=-\infty}^{\infty} t\cdot f(s,t)\,dt\,ds$$

$$= \int_1^5 \int_0^{\frac{1}{2}(s-1)} t\,\frac{1}{4}\,dt\,ds = \int_1^5 \left[\frac{1}{8}t^2\right]_0^{\frac{1}{2}(s-1)} ds = \int_1^5 \frac{1}{8}\cdot\left(\frac{1}{2}(s-1)\right)^2 ds$$

$$= \int_1^5 \frac{1}{32}(s^2-2s+1)\,ds = \frac{1}{32}\left[\frac{1}{3}s^3 - s^2 + s\right]_1^5$$

$$= \frac{1}{32}\,\frac{64}{3} = \boxed{\frac{2}{3}}$$

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This sheet is for Question 1.

2. (b) $F(7,8) = F(9,10)$

$\because$ Both the range of the integration in $F(7,8)$ and $F(9,10)$ cover the whole area that $f(x,y)$ is nonzero. #

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This sheet is for Question 1.

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Question 2: [16%, Work-out question] Consider a discrete function $f(k)$ (k being integer) as follows.

$$f(k) = \begin{cases} \frac{1}{50} & \text{if } 26 \le k \le 50 \\ \frac{1}{100} & \text{if } 50 < k \le 100 \\ 0 & \text{otherwise} \end{cases}$$

Define a function

$$F(z) = \sum_{k=-\infty}^{\infty} f(k) \cdot z^k \tag{3}$$

1. [4%] Find the value of $F(1)$.
2. [12%] Find the expression of $F(z)$.

   Hint 1: The following formula may be useful.

$$\sum_{k=1}^{K} a \cdot r^{k-1} = \frac{a \cdot (1 - r^K)}{1 - r} \tag{4}$$

   if $r \neq 1$ and $K \geq 1$.

   Hint 2: Your answer can be something like $\frac{z^{12}(1-z^{-18})}{1+3z} + (z + z^5)$. There is no need to further simplify it.

1. $F(1) = \sum_{k=-\infty}^{\infty} f(k) \cdot 1^k = \sum_{k=-\infty}^{\infty} f(k) = \sum_{k=26}^{50} \frac{1}{50} + \sum_{k=51}^{100} \frac{1}{100}$

$= \frac{25}{50} + \frac{50}{100} = \boxed{1}$ #

2. $F(z) = \sum_{k=-\infty}^{\infty} f(k) \cdot z^k = \sum_{k=26}^{50} \frac{1}{50} \cdot z^k + \sum_{k=51}^{100} \frac{1}{100} \cdot z^k$

$= \frac{1}{50} \sum_{k=26}^{50} z^k + \frac{1}{100} \sum_{k=51}^{100} z^k$

$= \frac{1}{50} \sum_{k=1}^{25} z^{26} \cdot z^{k-1} + \frac{1}{100} \sum_{k=1}^{50} z^{51} z^{k-1}$

$= \frac{1}{50} \frac{z^{26}(1-z^{25})}{1-z} + \frac{1}{100} \frac{z^{51}(1-z^{50})}{1-z}$ if $z \neq 1$

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This sheet is for Question 2.

$$F(z) = \begin{cases} \dfrac{1}{50}\,\dfrac{z^{26}(1-z^{25})}{1-z} + \dfrac{1}{100}\,\dfrac{z^{51}(1-z^{50})}{1-z} & \text{if } z \neq 1 \\ 1 & \text{if } z = 1 \end{cases}$$

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Question 3: [16%, Work-out question] Consider a 1-D continuous function

$$f(x) = \begin{cases} \frac{1}{e-1}e^x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

We generate another function

$$M(s) = \int_{-\infty}^{\infty} e^{sx} f(x) dx. \quad (6)$$

1. [8%] Find the expression of $M(s)$.

2. [8%] Denote the first order derivative of $M(s)$ by $M'(s)$. Find the value of $M'(0)$.

1. $M(s) = \int_{-\infty}^{\infty} e^{sx} f(x) dx = \int_0^1 e^{sx} \frac{1}{e-1} e^x dx$

$= \frac{1}{e-1} \int_0^1 e^{(s+1)x} dx = \frac{1}{e-1} \left[\frac{1}{s+1} e^{(s+1)x} \right]_{x=0}^{1}$

$= \boxed{\frac{1}{e-1} \frac{1}{s+1} \left(e^{(s+1)} - 1 \right)}$ #

2. $M(s) = \frac{1}{e-1} (s+1)^{-1} \left(e^{(s+1)} - 1 \right)$

$M'(s) = \frac{1}{e-1} (-1)(s+1)^{-2} \left(e^{(s+1)} - 1 \right) + \frac{1}{e-1} (s+1)^{-1} e^{(s+1)}$

$M'(0) = -\frac{1}{e-1} (e-1) + \frac{1}{e-1} e^1$

$= \frac{-e+1}{e-1} + \frac{e}{e-1}$

$= \boxed{\frac{1}{e-1}}$ #

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Question 4: [16%, Work-out question] We consider a continuous random experiment X, which can take any real value. For example, we may have $X = 0.001$ or $X = \sqrt{2}$.

Suppose the *probability density function* of X is

$$f_X(x) = \begin{cases} c \cdot |x| & \text{if } -2 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

for some constant c. Answer the following questions.

1. [6%] What is the value of c?
2. [10%] What is the probability that "$(X-1)^2$ is less than 0.25"?

   Hint 1: If you do not know the answer of c, you can still write down your answer by assuming c is a constant. You will receive full credit if your answer is correct.

   Hint 2: Your answer can be something like $\sqrt{3} + 17^3 - 8^5$. There is no need to further simplify it.

   Hint 3: If you do not know how to solve this sub-question, you can find the value of $P(X > 1)$ instead. You will receive 6 points if your answer for the alternative sub-question is correct.

1. $$\int_{-2}^{2} c \cdot |x| \, dx = c \int_{-2}^{2} |x| \, dx = c \cdot \left\{ \int_{-2}^{0} -x \, dx + \int_{0}^{2} x \, dx \right\}$$

$$= c \cdot \left\{ \left[-\frac{1}{2} x^2 \right]_{x=-2}^{0} + \left[\frac{1}{2} x^2 \right]_{x=0}^{2} \right\} = c \cdot \{ 2 + 2 \} = 4c = 1$$

$$\Rightarrow \boxed{c = \frac{1}{4}} \ \#$$

2. $$P\left((X-1)^2 < 0.25 \right) = P(-0.5 < X - 1 < 0.5) = P(0.5 < X < 1.5)$$

$$= \int_{0.5}^{1.5} \frac{1}{4} |x| \, dx = \frac{1}{4} \int_{0.5}^{1.5} x \, dx = \frac{1}{4} \left[\frac{1}{2} x^2 \right]_{x=\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{1}{8} \left(\frac{9}{4} - \frac{1}{4} \right) = \frac{1}{8} \cdot \frac{8}{4} = \boxed{\frac{1}{4}} \ \#$$

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Question 5: [18%, Work-out question] Prof. Wang has four cards, with the face values being 1, 2, 3, and 4, respectively. He shuffled the deck very carefully so that each possible outcome is equally likely.

A student can come forward to draw two cards without replacement. We call the value of the first card he/she drew as X_1 and we call the value of the the second card he/she drew as X_2.

If $X_1 + X_2 \leq 5.18$, then the student does not earn any cash reward. If $X_1 + X_2 > 5.18$, then the student will earn exactly $(X_1 \times X_2)$ dollars.

For example, if the first card value is 1 and the second card value is 2, then the student does not earn any reward since $1 + 2 \leq 5.18$. If the first card value is 3 and the second card value is 4, then since $3 + 4 > 5.18$, the student will earn $3 \times 4 = 12$ dollars.

1. [4%] Write down the sample space in terms of (X_1, X_2) and the corresponding weight assignment.
2. [4%] What is the probability $P(\text{student earns} \geq 10 \text{ dollars})$?
3. [10%] What is the conditional probability that

   $$P(\text{student earns} \geq 10 \text{ dollars} \mid \text{student earns a non-zero amount of money})?$$

   Hint: Your answer can be something like $\frac{1/56}{1/19+1/29}$. There is no need to further simplify it.

1. $S = \{(1,2), (1,3), (1,4), (2,1), (2,3), (2,4)$
$(3,1), (3,2), (3,4), (4,1), (4,2), (4,3)\}$ #

Each outcome has probability $\frac{1}{12}$ #

2. The event that students earn money $(X_1+X_2) > 5.18$:
$X = \{(2,4), (3,4), (4,2), (4,3)\}$
$P(\text{student earns} \geq 10 \text{ dollars}) = P(\{(3,4), (4,3)\}) = \boxed{\frac{1}{6}}$ #

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This sheet is for Question 5.

3.

P(student earns ≥ 10 dollars | student earns a non-zero amount of money)

$$= \frac{P(\{(3,4),(4,3)\})}{P(X)} = \frac{\frac{1}{6}}{\frac{1}{3}} = \boxed{\frac{1}{2}}$$

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Question 6: [18%, Work-out question] Consider a continuous random variable X, and the pdf of X is $f_X(x) = 0.5e^{-|x|}$.

1. [8%] What is the probability $P(5 < |X| < 10)$?
2. [10%] What is the conditional probability $P(-2 < X < 8 \mid 5 < |X| < 10)$

Hint: Your answers can be something like $\frac{e^{\sqrt{2}}}{1-e^{-5}} + e^{\pi}$. There is no need to further simplify it.

1. $P(5<|X|<10) = P(-10<X<-5)+P(5<X<10)$

$$= \int_{-10}^{-5} \frac{1}{2} e^{x} dx + \int_{5}^{10} \frac{1}{2} e^{-x} dx$$

$$= \frac{1}{2}\left[e^{x}\right]_{-10}^{-5} + \frac{1}{2}\left[-e^{-x}\right]_{5}^{10} = \frac{1}{2}\left[e^{-5}-e^{-10}\right]+\frac{1}{2}\left[-e^{-10}+e^{-5}\right]$$

$$= \boxed{\left(e^{-5}-e^{-10}\right)}$$ #

2.

$$P(-2<X<8 \mid 5<|X|<10)$$

$$= \frac{P(-2<X<8 \text{ and } 5<|X|<10)}{P(5<|X|<10)}$$

$$= \frac{P(5<X<8)}{P(5<|X|<10)}$$

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This sheet is for Question 6.

$$P(5<X<8)=\int_5^8 \frac{1}{2}e^{-x}\,dx = \frac{1}{2}\left[-e^{-x}\right]_5^8 = \frac{1}{2}\left[e^{-5}-e^{-8}\right]$$

$$P(-2<X<8 \mid 5<|X|<10)$$

$$= \boxed{\frac{1}{2}\,\frac{e^{-5}-e^{-8}}{e^{-5}-e^{-10}}}$$

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This sheet is for Question 6.

Other Useful Formulas

Geometric series

$$\sum_{k=1}^{n} a \cdot r^{k-1} = \frac{a(1-r^n)}{1-r} \tag{1}$$

$$\sum_{k=1}^{\infty} a \cdot r^{k-1} = \frac{a}{1-r} \text{ if } |r| < 1 \tag{2}$$

$$\sum_{k=1}^{\infty} k \cdot a \cdot r^{k-1} = \frac{a}{(1-r)^2} \text{ if } |r| < 1 \tag{3}$$

Binomial expansion

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n \tag{4}$$

The bilateral Laplace transform of any function $f(x)$ is defined as

$$L_f(s) = \int_{-\infty}^{\infty} e^{-sx} f(x) dx.$$

Some summation formulas

$$\sum_{k=1}^{n} 1 = n \tag{5}$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{6}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{7}$$