PURDUE

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Tips for making sure GradeScope can read your exam:

1. Make sure your name and PUID are clearly written at the top of every page, including any additional blank pages you use.
2. Write only on the front of the exam pages.
3. Add any additional pages used to the back of the exam before turning it in.
4. Ensure that all pages are facing the same direction.
5. Answer all questions in the area designated for that answer. Do not run over into the next question space.

1. Please make sure that it is your name printed on the exam booklet. Enter your student ID number, and signature in the space provided on this page, **NOW!**

2. This is a closed book exam.

3. This exam may contain some multiple choice questions and work-out questions. For multiple choice questions, there is no need to justify your answers. You have one hour to complete it. The students are suggested not spending too much time on a single question, and working on those that you know how to solve.

4. Please do not write on the back of each page. Any work on the back of the pages will not be scanned and thus will not be graded.

5. Neither calculators nor help sheets are allowed.

Name:

Student ID:

As a Boiler Maker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together — We are Purdue.

Signature: Solution   Date: 11/16/2023

Question 1: [23%, Work-out question] Consider a random variable X with the corresponding generalized pdf being

$$f(x) = 0.5 \cdot \delta(x-2) + \begin{cases} 0.25 \cdot e^{-0.5x} & \text{if } 0 \leq x \\ 0 & \text{otherwise} \end{cases}$$

1. [6%] We use $F_X(x)$ to denote the cdf of X. What is the value of $F_X(\pi)$? Hint: $\pi \approx 3.14159$.

2. [10%] Find the expression of the characteristic function $\Phi_X(\omega) = E(e^{j\omega X})$.

3. [7%] Find the value of $E(X)$ using the *moment theorem*.

   Hint: if you do not know how to use the moment theorem to solve Q1.3, you can use any other method to solve Q1.3. You will receive 5 points if your answer is correct.

1. $F_X(\pi) = P(X \leq \pi) = \int_{-\infty}^{\pi} f(x)\,dx$

$$= 0.5 + \int_0^\pi \frac{1}{4} e^{-\frac{1}{2}x}\,dx$$

$$= 0.5 + \frac{1}{4}\left[-2 e^{-\frac{1}{2}x} \right]_0^\pi$$

$$= \frac{1}{2} + \frac{1}{2}\left(1 - e^{-\frac{\pi}{2}} \right)$$

$$= \boxed{1 - \frac{1}{2} e^{-\frac{\pi}{2}}} \quad \#$$

2. $\Phi_X(\omega) = E(e^{j\omega X}) = \int_{-\infty}^{\infty} e^{j\omega X} f(x)\,dx$

$$= \frac{1}{2} e^{j2\omega} + \int_0^\infty e^{j\omega X} \cdot \frac{1}{4} e^{-\frac{1}{2}x}\,dx$$

$$\overline{\Phi}_X(\omega) = \frac{1}{2} e^{j2\omega} + \frac{1}{4} \int_0^\infty e^{(-\frac{1}{2}+j\omega)x} \, dx$$

$$= \frac{1}{2} e^{j2\omega} + \frac{1}{4} \frac{1}{-\frac{1}{2}+j\omega} e^{(-\frac{1}{2}+j\omega)x} \Big|_{x=0}^{\infty}$$

$$= \frac{1}{2} e^{j2\omega} + \frac{1}{2} \frac{1}{1-j2\omega}$$

$$= \boxed{\frac{1}{2} \left(e^{j2\omega} + \frac{1}{1-j2\omega} \right)} \; \#$$

3.

$$E(X) = \frac{1}{j^1} \frac{d}{d\omega} \overline{\Phi}_X(\omega) \Big|_{\omega=0}$$

$$\frac{d}{d\omega} \overline{\Phi}_X(\omega) = \frac{1}{2} \left(2j \, e^{j2\omega} + \frac{2j}{(1-j2\omega)^2} \right)$$

$$\therefore E(X) = \left[e^{j2\omega} + \frac{1}{(1-j2\omega)^2} \right]_{\omega=0}$$

$$= \boxed{2} \; \#$$

4

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This sheet is for Question 1.

Question 2: [13%, Work-out question] Suppose Prof. Wang likes to broadcast a bit value b, which can be either $b = 0$ or $b = 1$. Instead of sending the signal b value directly, he decides to send $X = 4 \cdot (-1)^b$. Namely, if $b = 0$, the value $X = 4$ will be sent. If $b = 1$, the value $X = -4$ will be sent.

A student uses an antenna to "receive" the broadcast value $X = 4 \cdot (-1)^b$. Because of the noise, the student will receive $Y = X + W$ instead, where W is the noise, which is assumed to be Gaussian distributed with $\mu = 0$ and $\sigma^2 = 9$. That is, the X value will be "corrupted" by the noise W in the sense that the received value Y is not identical to X. Instead, the received value is $Y = X + W$.

Suppose Prof. Wang has decided to send $b = 1$. Answer the following questions

1. [1%] Assuming $b = 1$ is fixed, is X a "random number" or is X a "constant"? No need to justify your answer for this sub-question.

2. [1%] Assuming $b = 1$ is fixed, is Y a "random number" or is Y a "constant"? No need to justify your answer for this sub-question.

3. [3%] Assuming $b = 1$ is fixed, what is the distribution of Y? Please write down (i) the pdf $f_Y(y)$ of Y; and (ii) please write down one to three short sentences to justify your answer. An answer without justification will receive zero point for this question.

   Hint: If you know what kind of distribution Y has, then you can write down the pdf with the help of the table.

4. [8%] Assuming $b = 1$ is fixed, what is the probability that "the magnitude of the received signal is no less than 6"? Namely, what is the value of $P(|Y| \geq 6)$ assuming $b = 1$?

   Hint 1: The following values may be useful.

$$Q(0) = 0.5 \tag{1}$$
$$Q(2/3) = 0.2525 \tag{2}$$
$$Q(4/3) = 0.0912 \tag{3}$$
$$Q(2) = 0.0228 \tag{4}$$
$$Q(8/3) = 0.00383 \tag{5}$$
$$Q(10/3) = 0.00043 \tag{6}$$

   Hint 2: Your answer can be something like $\frac{1-2*0.729+0.883}{0.958+3*0.729+0.883}$. There is no need to further simplify it.

6

1. constant

2. random number

3. Since $b = 1$, we have $X = -4$.

Since W is Gaussian distributed with $\mu = 0$ and $\sigma^2 = 9$,

$Y = X + W = -4 + W$ is also Gaussian distributed with

$$\begin{cases} \mu_Y = \mu - 4 = -4 \\ \sigma_Y^2 = 1^2 \cdot \sigma^2 = 9 \end{cases}$$

Thus,

$$f_Y(y) = \frac{1}{\sqrt{2\pi \cdot 9}} e^{-\frac{(y+4)^2}{18}} \quad \text{for } -\infty < y < \infty$$

\#

4. Let Z be standard Gaussian distributed.

$Y = \sigma_Y Z + \mu_Y = 3Z - 4$

$P(|Y| \geq 6) = P(Y \geq 6) + P(Y \leq -6)$

$\qquad\qquad = P(3Z - 4 \geq 6) + P(3Z - 4 \leq -6)$

$\qquad\qquad = P(Z \geq \frac{10}{3}) + P(Z \leq -\frac{2}{3})$
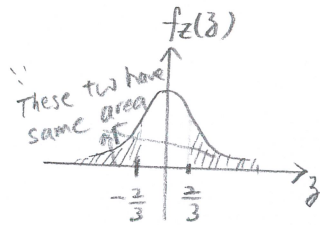
7

$$P(|Y| \geq 6) = P\left(Z \geq \frac{10}{3}\right) + P\left(Z \leq -\frac{2}{3}\right)$$

$$= P\left(Z \geq \frac{10}{3}\right) + P\left(Z \geq \frac{2}{3}\right)$$

$$= Q\left(\frac{10}{3}\right) + Q\left(\frac{2}{3}\right)$$

$$= 0.00043 + 0.2525$$

$$= \boxed{0.25293} \, \#$$

These two have same area

$f_Z(\delta)$

$-\frac{2}{3}$ $\frac{2}{3}$

Question 3: [24%, Work-out question] Consider a binomial random variable X with parameter $n = 20$ and $p = \frac{1}{3}$. Given $X = x_0$, the random variable Y is a binomial distribution with $n = x_0$ and $p = 0.5$. Answer the following questions:

1. [3%] What is the probability that $P(X = 5$ and $Y = 9)$?

2. [5%] Write down the expression of the joint 2-dimensional pmf of $p_{k,h} = P(X = k, Y = h)$.

3. [7%] Define the marginal pmf of Y by $p_h = P(Y = h)$. Find the value of p_{19} (i.e., the value of p_h when $h = 19$). Hint: your answer can be something like $\frac{4!5^{15}}{2!7!} - \frac{8!2^5}{3!6!}$. There is no need to further simplify it.

   Hint: If you don't know how to find the value of p_{19}, you can write down the value of p_{19} in terms of the joint pmf $p_{k,h}$. You will receive 4 points for this sub-question if your answer is correct.

4. [9%] Find the value of $E(Y)$.

   Hint 1: You may like to write $E(Y) = E(f(X, Y))$ for the function $f(x, y) = y$ as discussed in the lecture note p. 143. Namely, instead of finding the expectation of a single random variable Y, you can find the expectation of the 2-dimensional function $E(f(X, Y))$.

   Hint 2: The formula table may be useful when solving this question.

   Hint 3: This sub-question may be a bit challenging. If you do not know the answer to this question, please find the conditional expectation of $E(X|Y = 19)$. You will receive **8 points** if your answer is correct. Your answer can be something like $\frac{\binom{8}{3}\pi^2 4^{0.5}}{\binom{5}{3}0.3^7 0.4^5 + \binom{3}{1}e^8}$. There is no need to further simplify it.

4. Alternative:

$$P_{19,19} = \frac{40}{3^{20} \cdot 2^{19}} \quad, \quad P_{20,19} = \frac{20}{3^{20} \cdot 2^{20}}$$

$$P(X = 19 \mid Y = 19) = \frac{P_{19,19}}{P(Y = 19)} = \frac{80}{100} = \frac{4}{5}$$

$$P(X = 20 \mid Y = 19) = \frac{P_{20,19}}{P(Y = 19)} = \frac{1}{5}$$

$$E(X \mid Y = 19) = 19 \cdot \frac{4}{5} + 20 \cdot \frac{1}{5} = \boxed{\frac{96}{5}}$$

___

$$P_k = P(X=k) = \binom{20}{k}\left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{20-k}, \quad k=0,1,\ldots,20$$

$$P(Y=h \mid X=k) = \binom{k}{h}\left(\frac{1}{2}\right)^h \left(\frac{1}{2}\right)^{k-h}, \quad 0 \le h \le k.$$

1 $P(X=5 \text{ and } Y=9) = P(X=5) \cdot P(Y=9 \mid X=5)$

$= P(X=5) \cdot 0 = \boxed{0}$ #

2.

$$P_{k,h} = P(X=k, Y=h) = P(X=k) \cdot P(Y=h \mid X=k)$$

$$= \begin{cases} \binom{20}{k}\left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{20-k}\binom{k}{h}\left(\frac{1}{2}\right)^k & \text{for } 0 \le h \le k \le 20 \\ \\ 0 & \text{, otherwise.} \end{cases}$$

#

3. Method 1:

$$P_h = P(Y=h) = \sum_{k=0}^{20} P(X=k)P(Y=h \mid X=k)$$

$$P_{19} = \sum_{k=0}^{20} \binom{20}{k}\left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{20-k}\binom{k}{19}\left(\frac{1}{2}\right)^k$$

$$= \binom{20}{19}\left(\frac{1}{3}\right)^{19}\left(\frac{2}{3}\right)\binom{19}{19}\left(\frac{1}{2}\right)^{19} + \binom{20}{20}\left(\frac{1}{3}\right)^{20}\binom{20}{19}\left(\frac{1}{2}\right)^{20}$$

$$= \frac{20!}{19!}\frac{2}{3^{20}} \cdot \frac{1}{2^{19}} + \frac{1}{3^{20}}\frac{20!}{19!}\frac{1}{2^{20}}$$

$$= \frac{40}{3^{20} \cdot 2^{19}} + \frac{20}{3^{20} \cdot 2^{20}} \qquad 10$$

$$= \boxed{\frac{100}{3^{20} \cdot 2^{20}}}$$ #

Method 2:

The question can be characterized as an experiment that we first flip a coin $n = 20$ times and $P_1(head) = \frac{1}{3}$. Denote X as the number of heads. Then, for those heads, we flip it again with $P_2(head) = \frac{1}{2}$.

Denote Y as the final number of heads.

Thus, overall Y is binomial distributed with $n = 20$

$$P(head) = P_1(head) \cdot P_2(head) = \frac{1}{6}$$

$$\therefore P_{19} = P(Y = 19) = \binom{20}{19}\left(\frac{1}{6}\right)^{19}\left(\frac{5}{6}\right)$$

$$= \frac{20!}{19!} \cdot \frac{5}{6^{20}} = \boxed{\frac{100}{6^{20}}} \quad \#$$

4.
$$E(Y) = E_X(E_Y(Y|X)) = E_X\left(\frac{1}{2}X\right) = \frac{1}{2}E(X)$$

$$= \frac{1}{2} \cdot 20 \cdot \frac{1}{3} = \boxed{\frac{10}{3}} \quad \#$$

11
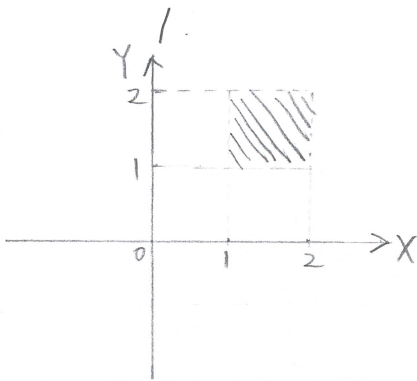
Question 4: [20%, Work-out question] Consider two continuous random variables X and Y. Suppose their joint pdf is

$$f_{XY}(x,y) = \begin{cases} 0.25x & \text{if } 0 < x < 2 \text{ and } 0 < y < 2 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

1. [10%] What is the probability $P(\min(X,Y) > 1)$?

2. [10%] Prof. Wang defines a new random variable $W = X + Y$. Find the probability $P(W < 1)$.

$$P(\min(X,Y) > 1)$$

$$= \int_1^2 \int_1^2 \frac{1}{4} x \, dx \, dy = \frac{1}{4} \int_1^2 \left[\frac{1}{2} x^2\right]_1^2 dy$$

$$= \frac{1}{8} \int_1^2 3 \, dy = \frac{1}{8} [3y]_1^2$$

$$= \boxed{\frac{3}{8}}$$

#

2.

$$P(W < 1) = P(X+Y < 1)$$

$$= \int_0^1 \int_0^{1-y} \frac{1}{4} x \, dx \, dy = \frac{1}{4} \int_0^1 \left[\frac{1}{2} x^2\right]_0^{1-y} dy$$

$$= \frac{1}{8} \int_0^1 (1-y)^2 dy = \frac{1}{8} \int_0^1 1 - 2y + y^2 \, dy = \frac{1}{8}\left[y - y^2 + \frac{1}{3}y^3\right]_0^1$$

$$= \boxed{\frac{1}{24}}$$

#

12

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This sheet is for Question 4.

Question 5: [20%, Multiple choice question. There is no need to justify your answers]

Each of these 9 sub-questions are completely separate. There is no connection between them in any way.

True

1. [3%] X has some unknown distribution P_X, and Y is standard Gaussian distribution. X and Y are independent. Please answer whether the following statement is true or false: "We always have $P(XY > 0) = 0.5(P(X > 0) + P(X < 0))$".

False

2. [2%] Suppose X is standard Gaussian distribution and $P(Y = 1) = 0.5$ and $P(Y = -1) = 0.5$. Please answer whether the following statement is true or false: "$W = X \cdot Y$ must be a Gaussian distribution."

True

3. [2%] Suppose X is standard Gaussian distribution and $P(Y = 1) = 0.0$ and $P(Y = -1) = 1.0$. Please answer whether the following statement is true or false: "$W = X \cdot Y$ must be a Gaussian distribution."

False

4. [2%] If X is an exponential distribution with parameter $\lambda = 1$. Please answer whether the following statement is true or false: "The equality $P(X < 3 | X < 5) = P(X < 2)$ holds."

False

5. [2%] X is Bernoulli distributed with parameter $p = 0.2$, Y is Bernoulli distributed with parameter $p = 0.3$, and X and Y are *independent*. Define $W = X \oplus Y$ being the binary exclusive-or value of X and Y. I.e., $0 \oplus 0 = 0$; $0 \oplus 1 = 1$; and $1 \oplus 1 = 0$. Please answer whether the following statement is true or false: "W is Bernoulli distributed with parameter $p = 0.2 + 0.3 = 0.5$."

True

6. [3%] X has is the outcome of a fair 6-faced die, and Y is Bernoulli distribution with $p = 0.5$. Define $W = X + Y$. Please answer whether the following statement is true or false: "It is possible to design a joint pmf of (X, Y) such that $P(W$ is an even number$) = 1$".

True

7. [2%] Suppose the number of customer arrivals is Poisson distributed with parameter $\alpha = 30$. Please answer whether the following statement is true or false: "By the Markov inequality, we have $P(X \geq 60) \leq 0.5$."

False

8. [2%] Suppose X has mean $m_X = 10$ and variance $\sigma_X^2 = 100$. Please answer whether the following statement is true or false: "By the Chebyshev's inequality, we always have $P(X \geq 40) \leq \frac{100}{40^2} = \frac{1}{16}$."

True

9. [2%] For two random variables X and Y, consider their joint pmf $p_{k,h}$ and marginal pmfs p_k and p_h. Is the following statement always true? "We have $p_{k,h} \leq p_k$ for all possible k and h values."

1. $P(XY>0) = P(X>0 \text{ and } Y>0) + P(X<0 \text{ and } Y<0)$

$= P(X>0) \cdot P(Y>0) + P(X<0) \cdot P(Y<0)$ ∵ X and Y are indep.

$= P(X>0) \cdot 0.5 + P(X<0) \cdot 0.5$

2. Counter example: Define $Y = -1$ if $X<0$ and $Y = 1$ if $X>0$.
Clearly, $P(Y=1) = P(Y=-1) = 0.5$ as specified by the question. However,
$W = X \cdot Y = |X|$ is not a Gaussian anymore since the sample space of W is always Positive.

3. ∵ $P(W = -X) = 1$ ∴ W is Gaussian distributed.

4. $P(X < x) = \int_0^x e^{-u} du = 1 - e^{-x}$

$P(X<3 | X<5) = \dfrac{1-e^{-3}}{1-e^{-5}} \neq 1 - e^{-2} = P(X<2)$

5.

$P(W=1) = P(X=0)P(Y=1) + P(X=1)P(Y=0) = 0.8 \cdot 0.3 + 0.2 \cdot 0.7 = 0.38$

$P(W=0) = P(X=0)P(Y=0) + P(X=1)P(Y=1) = 0.8 \cdot 0.7 + 0.2 \cdot 0.3 = 0.62$

6.

Y \ X	1	2	3	4	5	6
0	0	$\frac{1}{6}$	0	$\frac{1}{6}$	0	$\frac{1}{6}$
1	$\frac{1}{6}$	0	$\frac{1}{6}$	0	$\frac{1}{6}$	0

7. 1° $P(X<0) = 0$ 2° $E(X) = 30$

By Markov inequality $P(X \geq a) \leq \dfrac{E(X)}{a}$, we have $P(X \geq 60) \leq 0.5$

8. Chebyshev's inequality: $P(|X - m_X| \geq a) \leq \dfrac{\sigma_X^2}{a^2}$

$P(X \geq 10+a) + P(X \leq 10-a) \leq_{16} \dfrac{100}{a^2}$

9. $P_{k,h} = P_k \cdot P_{h|k}$,   ∵ $0 \leq P_{h|k} \leq 1$ for all possible (k, h).

∴ $P_{k,h} = P_k \cdot P_{h|k} \leq P_k$