Density Evolution for Asymmetric Memoryless Channels

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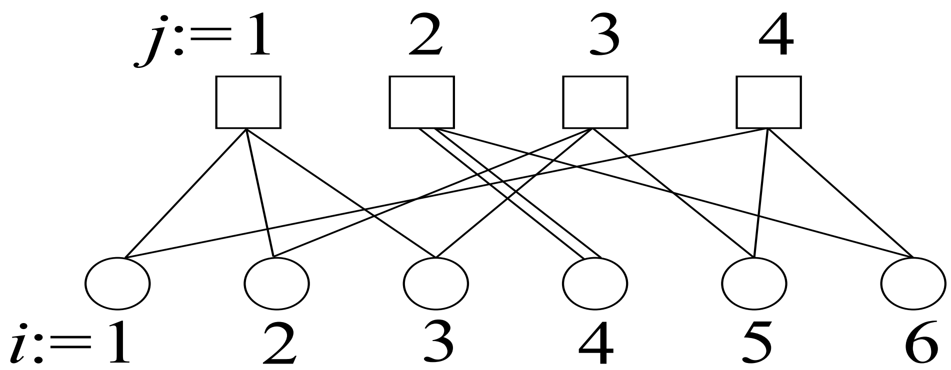
Graph Ensemble & Belief Propagation



$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\mathbf{Ax} = \mathbf{0}$$

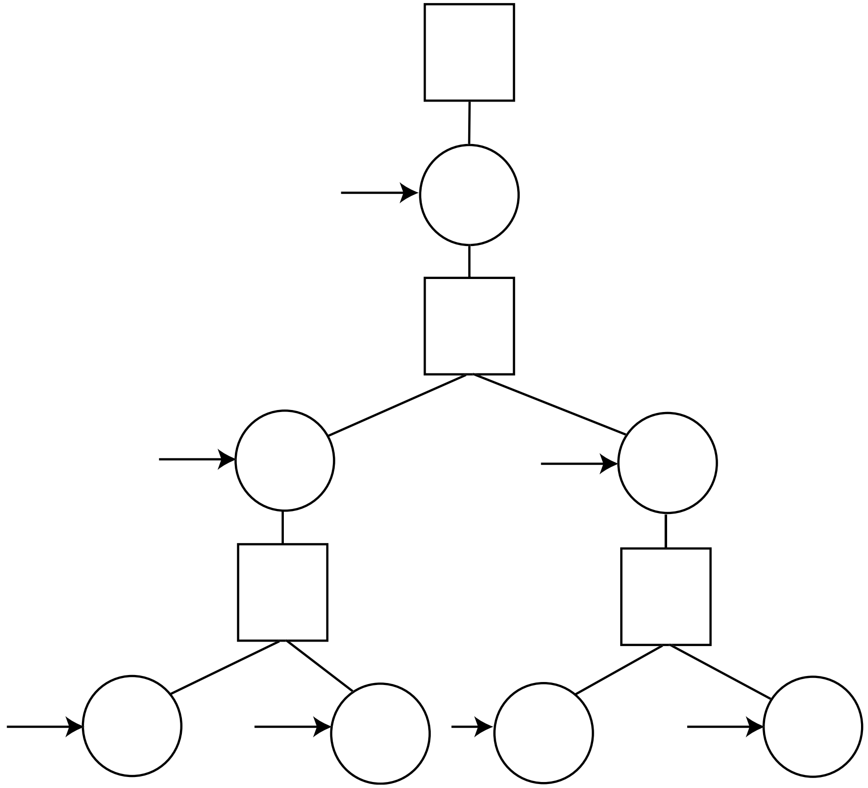
Figure 1: $\mathcal{C}^6(d_v, d_c)$ and $d_v = 2$, $d_c = 3$.

Belief Propagation:

(i) m_0, (ii) $\Psi_v(m_0, m_1, \cdots, m_{d_v-1})$, (iii) $\Psi_c(m_1, \cdots, m_{d_c-1})$

Note: the belief propagation is an inference algorithm, which is *independent of channel models.*

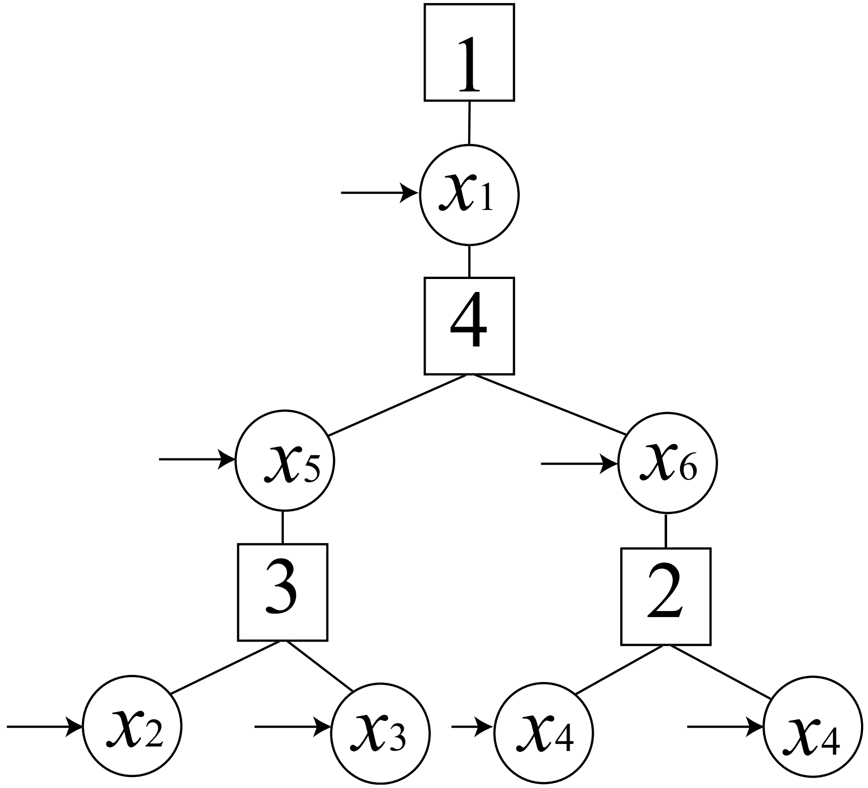
Supporting Tree and the Density Evolution



$$P^{(l)} = P^{(0)} \otimes \left(Q^{(l-1)}\right)^{\otimes(d_v-1)}$$

$$Q^{(l-1)} = \Gamma^{-1}\left(\left(\Gamma\left(P^{(l-1)}\right)\right)^{\otimes(d_c-1)}\right),$$

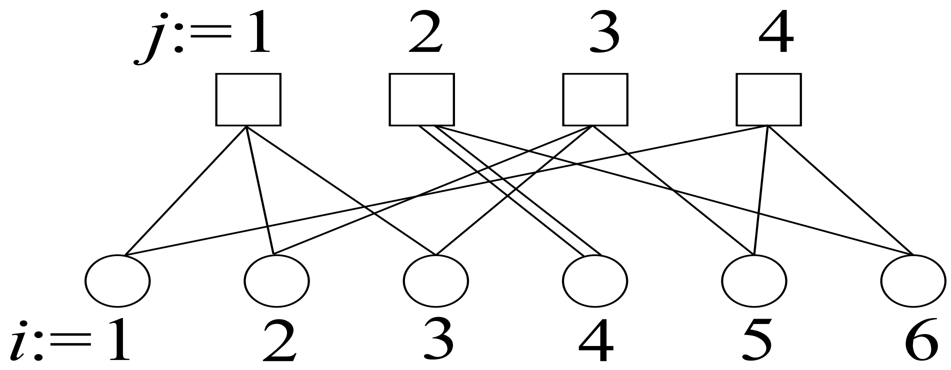
Codeword Dependence



$$P^{(l)}(\mathbf{x}) \quad = \quad P^{(0)}(\mathbf{x}) \otimes \left(Q^{(l-1)}(\mathbf{x})\right)^{\otimes(d_v-1)}$$

$$Q^{(l-1)}(\mathbf{x}) \quad = \quad \Gamma^{-1}\left(\left(\Gamma\left(P^{(l-1)}(\mathbf{x})\right)\right)^{\otimes(d_c-1)}\right),$$

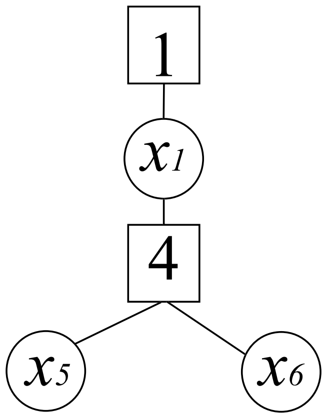
Difficulty of Codeword Averaging



$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\mathbf{Ax} = \mathbf{0}$$

$$x_6 = 0$$



$$\mathbf{X}^1_{(1,1)} := \{x_1x_5x_6 : x_1x_5x_6 = 000, 011, 101, 110\}$$

Averaging of the trimmed **tree code** is not equivalent to averaging the **original code**.

Full Rank Condition

Definition 1 (Full Rank of $\mathcal{N}^{2l}$) *The supporting tree $\mathcal{N}^{2l}$ is of full rank, if for any codeword $\mathbf{x}_t$ of the tree code $\mathbf{X}_t$,*

$$\frac{|\{\mathbf{x} \in \mathbf{X} : \mathbf{x}|_{tree} = \mathbf{x}_t\}|}{|\mathbf{X}|} = \frac{1}{|\mathbf{X}_t|}.$$

averaging of the trimmed **tree code** = averaging over the **original code**.

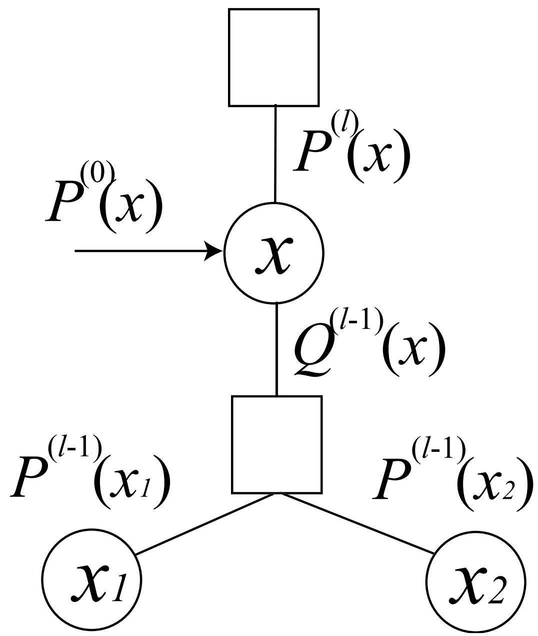
Suppose the supporting tree $\mathcal{N}^{2l}$ is of full rank. We have

$$P^{(l)}(x) := \left\langle P^{(l)}(\mathbf{x}) \right\rangle_{\{\mathbf{x}\in\mathbf{X}:\mathbf{x}|_0=x\}} = \left\langle P^{(l)}(\mathbf{x}_t) \right\rangle_{\{\mathbf{x}_t\in\mathbf{X}_t:\mathbf{x}_t|_0=x\}},$$

and

$$p_e^{(l)} = \frac{1}{2}\left(\int_0^\infty d\left(P^{(l)}(0)\right) + \int_{-\infty}^0 d\left(P^{(l)}(1)\right)\right).$$

New Iterative Formula



$x = 0, \quad x_1x_2 = 00, 11$

$x = 1, \quad x_1x_2 = 01, 10$

$$P^{(l)}(x) = P^{(0)}(x) \otimes \left(Q^{(l-1)}(x)\right)^{\otimes(d_v-1)}$$

$$Q^{(l-1)}(x) = \Gamma^{-1}\left(\frac{1}{2^{d_c-2}} \sum_{\mathbf{x}^1 \in \mathbf{X}^1(x)} \bigotimes_{v=1}^{d_c-1} \Gamma\left(P^{(l-1)}(x_v)\right)\right)$$

New Iterative Formula

$$
\begin{aligned}
P^{(l)}(x) &= P^{(0)}(x) \otimes \left(Q^{(l-1)}(x)\right)^{\otimes(d_v-1)} \\
Q^{(l-1)}(x) &= \Gamma^{-1}\left(\left(\Gamma\left(\frac{P^{(l-1)}(0)+P^{(l-1)}(1)}{2}\right)\right)^{\otimes(d_c-1)}\right. \\
&\qquad \left. +(-1)^x\left(\Gamma\left(\frac{P^{(l-1)}(0)-P^{(l-1)}(1)}{2}\right)\right)^{\otimes(d_c-1)}\right).
\end{aligned}
$$

Convergence to Full Rank

Theorem 1 (Convergence to Full Rank in Probability) *For any* $\mathcal{C}^n(d_v, d_c)$, *with fixed* l, *we have*

$$\mathsf{P}\left(\mathcal{N}^{2l} \textit{ is of full rank}\right) = 1 - \mathcal{O}(n^{-0.1}).$$

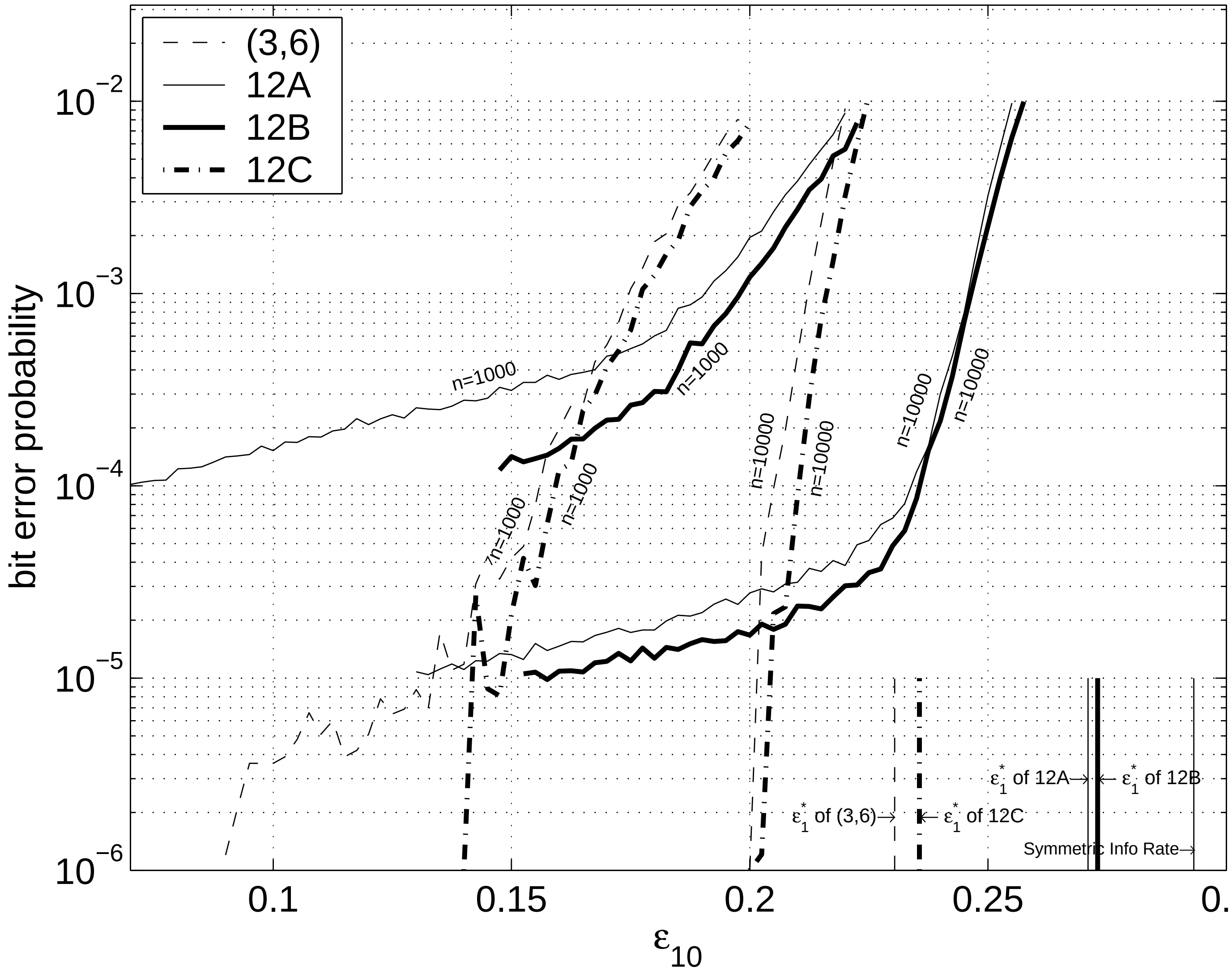
Theoretical foundations for the codeword averaging approach:

Cycle-free convergence.

Performance concentration. ∼ [Richardson *et al.* 01]

Convergence to full rank.

Finite Codeword Length Simulations

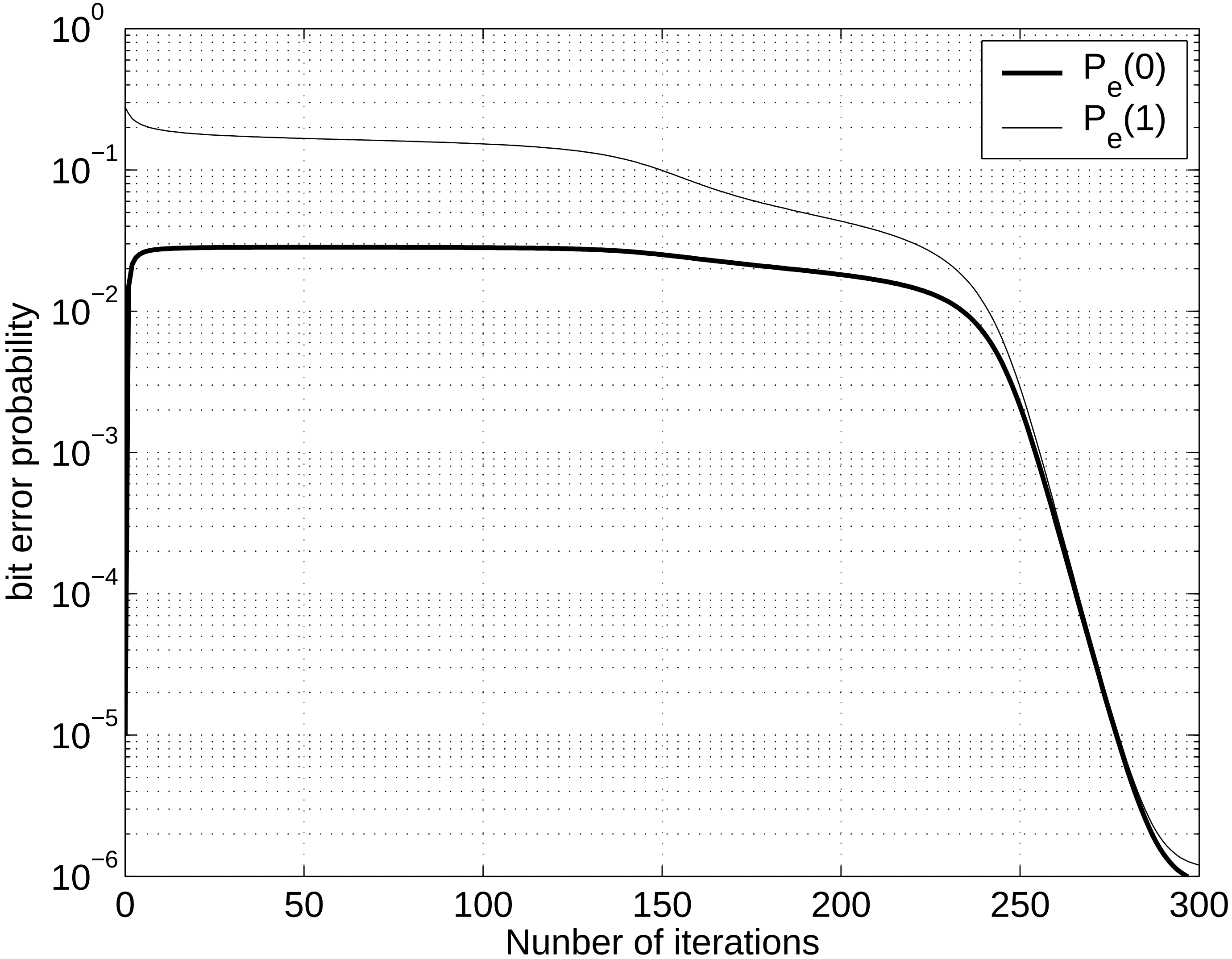


12A: [Richardson01], optimized for BiAWGNC, maxDv=12.

12B: Optimized for z-channels, maxDv=12.

12C: Optimized for z-channels, maxDv=12, $\lambda_2 = 0$.

Density Evolution



Monotonicity & Symmetry

Monotonicity of $p_e^{(l)} = (p_e^{(l)}(0) + p_e^{(l)}(1))/2$ with respect to the number of iterations and to physically degraded channels.

Symmetry:

- $P^{(l)}(x)$ is the distribution of $m := \log \frac{\mathsf{P}(x=1|y)}{\mathsf{P}(x=0|y)}$.
- $I(\omega) := -\omega$.
- $\langle P^{(l)} \rangle := \frac{P^{(l)}(1)+P^{(l)}(0)\circ I^{-1}}{2}$ is symmetric. I.e. it is as if being obtained from a symmetric channel.
- $\langle CBP^{(l)} \rangle := \int_{\mathbf{R}} e^{-\omega/2} \langle P^{(l)} \rangle (d\omega)$, is the Chernoff version of $\langle P^{(l)} \rangle$.
- $r := \langle CBP^{(0)} \rangle = \int_{\mathbf{R}} e^{-\omega/2} \langle P^{(0)} \rangle (d\omega)$ is the Bhattacharyya noise parameter

Sufficient Stability Conditions

Proposition: $2p_e^{(l)} \leq \langle CBP^{(l)} \rangle \leq 2\sqrt{p_e^{(l)}(1-p_e^{(l)})}.$

Let $\lambda(x) := \sum \lambda_k x^{k-1}$ and $\rho(x) := \sum \rho_k x^{k-1}$.

Theorem 2 (Sufficient Stability Condition) *Suppose $\lambda_2 \rho'(1) r < 1$, and let ϵ^* be the smallest strictly positive root of the following equation.*

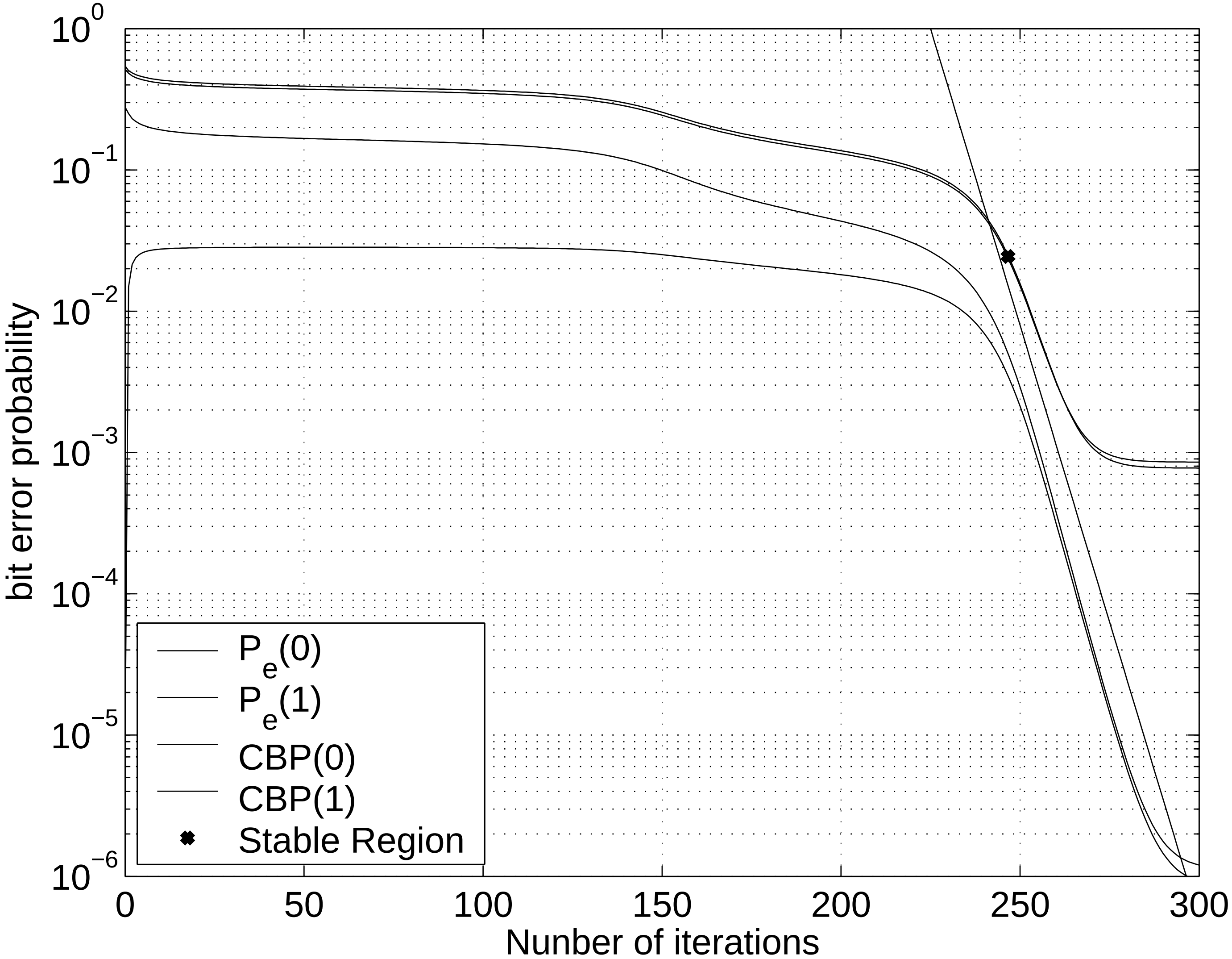
$$\lambda(\rho'(1)\epsilon) r = \epsilon.$$

If for certain l_0, $\langle CBP^{(l_0)} \rangle < \epsilon^*$, then

$$\langle CBP^{(l)} \rangle = \begin{cases} \mathcal{O}\left(\left(\lambda_2 \rho'(1) r\right)^l \right) & \text{if } \lambda_2 > 0 \\ \mathcal{O}\left(e^{-\mathcal{O}\left((k_\lambda - 1)^l \right)} \right) & \text{if } \lambda_2 = 0, \text{ where } k_\lambda = \min\{k : \lambda_k > 0\} \end{cases},$$

which immediately implies $\lim_{l \to \infty} \langle CBP^{(l)} \rangle = 0$.

Sufficient Stability Conditions

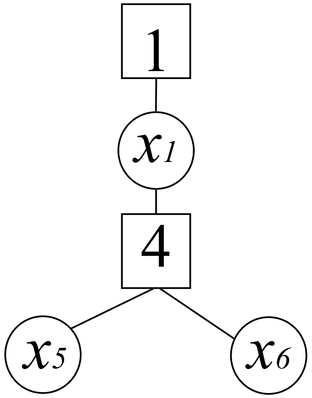


Necessary Stability Condition

Theorem 3 (Necessary Stability Condition) $r := \langle CBP^{(0)} \rangle$. *If* $\lambda_2 \rho'(1) r > 1$, *then* $\lim_{l \to \infty} p_e^{(l)} > 0$.

Note: It is first stated in [Richardson *et al.* 01] for symmetric channels.

Sketches of the Sufficient Stability Condition



$$CBQ := \int e^{-m/2} dQ = \int \sqrt{\frac{e^{m_1} + e^{m_2}}{1 + e^{m_1} e^{m_2}}} dP_1 \times dP_2$$

$$\leq \int \left(e^{-m_1/2} + e^{-m_2/2} \right) dP_1 \times dP_2 = CBP_1 + CBP_2.$$

Regular codes:

$$\langle CBP^{(l+1)} \rangle \quad \leq \quad \langle CBP^{(0)} \rangle \left((d_c - 1) \langle CBP^{(l)} \rangle \right)^{d_v - 1}.$$

Irregular codes:

$$\langle CBP^{(l+1)} \rangle \quad \leq \quad \langle CBP^{(0)} \rangle \lambda \left(\rho'(1) \langle CBP^{(l)} \rangle \right).$$

Sketches of the Sufficient Stability Condition

Stronger version:

$$\langle CBP^{(l+1)} \rangle \leq \langle CBP^{(0)} \rangle \lambda \left(4 \left[1 - \rho \left(\frac{1 - \langle CBP^{(l)} \rangle}{4} \right) \right] \right).$$

Comparison between BEC:

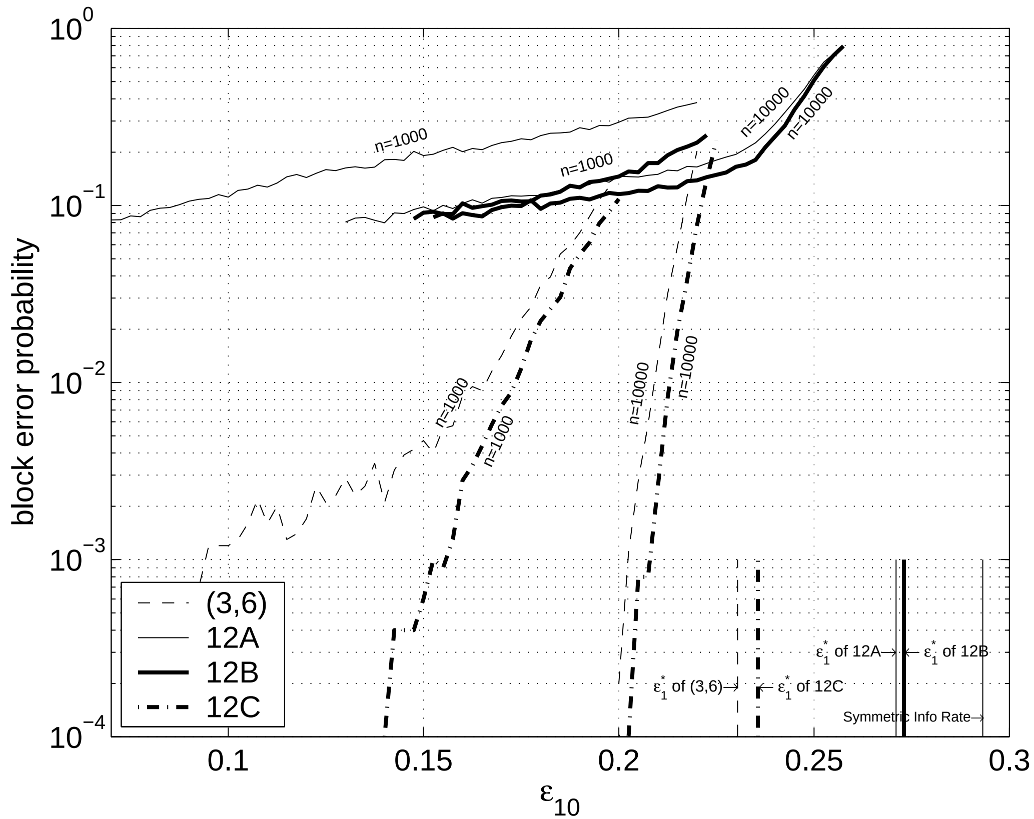
$$\langle CBP^{(l+1)} \rangle = \langle CBP^{(0)} \rangle \lambda \left(1 - \rho \left(1 - \langle CBP^{(l)} \rangle \right) \right).$$

Applications of the sufficient stability condition:

- Stopping time for the density evolution
- Finite dimensional iterative formula.
- Analytical upper threshold computation: BEC, Gallager's decoding algorithm A [Bazzi and Richardson].

Convergence Rate and Block Error Probability

By the convergence rate results and the union bound, we prove that if $\lambda_2 = 0$, $\lim_{n,l\to\infty} p_{e,B}^{(l)} = 0$.



Thresholds for Different Channels

Our best codes for z-channels: $\max d_v = 20$ and $\max d_c = 10$.

- Hitting ϵ^* within 100 iterations: 0.2741.
- Hitting ϵ^* within 500 iterations: 0.2796.
- Symmetric Information Rate bound: 0.2932.
- Capacity bound: 0.3035.

Conclusions

- A **new iterative formula** of density evolution for asymmetric memoryless channels with the same computational comlexity.
- **Good codes** for z-channels have been found. Rate 1/2 code: 0.2796 vs. symmetric mutual info. rate: 0.2932.
- **LDPC codes** plus **belief propagation algorithms** work well even in asymmetric memoryless channels as expected and theoretically shown by the new density evolution.
- The **stability region**/stopping criterion and asymptotic **convergence rates** have been found and are fully specified by λ_2, ρ, and the Bhattacharyya noise parameter

$$r := \langle CBP^{(0)} \rangle = \int_{\mathbf{R}} e^{-\omega/2} \langle P^{(0)} \rangle (d\omega).$$

Conclusions

- By focusing on its Chernoff domain, a **one-dimensional universal upper bound** is provided and with similar form to BEC thresholds.
- The convergence behavior of the **block error probability** is discussed.
- Applications beside z-channels and other asymmetric channels:
  - The analysis of artificial channels. Example: from q-ary to binary.