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On Finite-Dimensional Bounds for LDPC-like Codes with Iterative Decoding

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Abstract

This paper focuses on finite-dimensional upper and lower bounds on decodable thresholds assuming iterative decoding. Two noise measures will be considered: the Bhattacharyya noise parameter, and the expected soft bit of the considered channel. An iterative m-dimensional lower bound is derived for $\mathbb{Z}_m$ LDPC codes in symmetric channels, which gives a sufficient stability condition for $\mathbb{Z}_m$ LDPC codes and will be complemented by a new necessary stability condition introduced herein. Two new lower bounds are provided for binary-input/symmetric-output channels, including a two-dimensional iterative bound and a one-dimensional non-iterative bound, the latter of which is the best known bound that is tight for BSCs. A reverse channel perspective is then used to generalize existing bounds from binary-input/symmetric-output channels to binary-input/non-symmetric-output channels.

1. INTRODUCTION

Density evolution [1] explicitly computes the behavior of iterative decoders. The iterations of density evolution are based on the density of the log likelihood ratio (LLR), which is of infinite dimension and is a sufficient statistic describing any specific channel. In contrast with the accuracy of the infinite-dimensional LLR iterations, the simple structure of finite dimensional bounds fascilitates more analysis and sheds light on decoding behavior for general channels [2].

A number of such results for binary-input/symmetric-output (BI-SO) channels have been discovered, including a one-dimensional upper and lower bound pair on the expected soft bit [3], which suggests good performance of iterative decoding when applied to a variety of channels. Similar bounds on the Bhattacharyya noise parameter and the mutual information can be found in [4, 5, 6] respectively. The benefits of one-dimensional iterative bounds/approximations have culminated in the advent of the EXIT chart [7], the widely accepted design/analysis tool for iterative decoding. Finite-dimensional bounds have also been used to derive tight sufficient stability conditions for LDPC codes, including binary-input/non-symmetric-output (BI-NSO) channels [8] and $\mathsf{GF}(q)$-based LDPC codes [9] on q-ary-input/symmetric-output channels.

This paper is organized as follows. The channel model and finite-dimensional noise measures of interest will be defined in Section 2. Section 3 is devoted to iteration-based bounds for symmetric channels, including an m-dimensional bound for the quotient-ring-based $\mathbb{Z}_m$ LDPC codes and a two-dimensional bound for binary LDPC codes. A pair of matched stability conditions is then provided for $\mathbb{Z}_m$ LDPC codes, which generalizes the results for $\mathsf{GF}(q)$ LDPC codes with a prime[1] q [9]. Section 4 focuses on a non-iterative bound on the expected soft bits, which is tight for BSCs, while Section 5 gives a new pair of bounds for BI-NSO channels based on the Bhattacharyya noise parameter. Section 6 will provide some comparisons between existing bounds before concluding this paper.

2. FORMULATION

2.1. Finite-dimensional Noise Measures

In this paper, we focus on *memoryless* channels with binary/m-ary input. For a binary channel $\mathbf{X} = \{0,1\} \mapsto \mathbf{Y}$, we use $p(x|y)$ to denote the *a posteriori* probability $\mathsf{P}(X = x|Y = y)$, and consider the following two noise measures:

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[1] The parameter m in our derivation does not need to be prime.

[Bhattacharyya Noise Parameter]

$$\begin{aligned} CB &:= \mathsf{E}_{X,Y}\left\{\sqrt{\frac{p(\bar{X}|Y)}{p(X|Y)}}\right\} \\ &= \sum_{x\in\{0,1\}} \mathsf{P}(X=x)\mathsf{E}\left\{\sqrt{\frac{p(\bar{x}|Y)}{p(x|Y)}}\middle| X=x\right\}, \end{aligned}$$

[Expected Soft Bit]

$$\begin{aligned} DCB &:= 2\mathsf{E}_{X,Y}\left\{p(\bar{X}|Y)\right\} \\ &= 2\sum_{x\in\{0,1\}} \mathsf{P}(X=x)\mathsf{E}\left\{p(\bar{x}|Y)|X=x\right\}, \end{aligned}$$

the latter of which was also used in the bounds of [3]. Each of these noise measures has the property that $CB = 0$, or $DCB = 0$, represents the noise-free channel, while $CB = 1$, or $DCB = 1$, implies a completely random channel. For m-ary-input channels, we define the pair-wise Bhattacharyya noise parameter from x to x' as follows:

$$CB(x\to x') := \mathsf{E}\left\{\sqrt{\frac{\mathsf{P}(x'|Y)}{\mathsf{P}(x|Y)}}\middle| X=x\right\}.$$

Remark 1: These noise measures for binary/m-ary channels always exist even when the channels are non-symmetric and the *a priori* P_X is not uniform.

Remark 2: For binary-input/symmetric-output channels, these two quantities are directly related to the density evolution [1] by

$$\begin{aligned} CB &= \int e^{-\frac{m}{2}} dP(m) \\ DCB &= \int \frac{2}{e^m+1} dP(m), \end{aligned} \tag{1}$$

where $m = \log\frac{\mathsf{P}(Y=y|X=0)}{\mathsf{P}(Y=y|X=1)}$ is the LLR message, and $dP(m)$ is the evolved density of m.

2.2. Symmetric Channels

In terms of the CB and DCB, three new bounds will be presented for binary-input/symmetric-output (BI-SO) or m-ary-input/symmetric-output channels (MI-SO), in which the channel symmetry is defined as follows.

Definition 1 *An m-ary-input channel $\mathbb{Z}_m \mapsto \mathbf{Y}$ is (circularly) symmetric if there exists a bijective transform $\mathcal{T}: \mathbf{Y} \mapsto \mathbf{Y}$ such that*

$$\forall x \in \mathbb{Z}_m, F(dy|0) = F(\mathcal{T}^x(dy)|x),$$

where $F(dy|x)$ is the conditional distribution of Y given $X = x$. When $m = 2$, this definition collapses to that of the conventional BI-SO channels. Note: there is no constraint on $\mathbf{Y}$, the range of the channel output. For example, in PSK or QAM scenarios, $\mathbf{Y} = \mathbb{R}^2$.

One of the simplest MI-SO channels is the m-ary Symmetric Channel (MSC): $\mathbb{Z}_m \mapsto \mathbb{Z}_m$, which can be fully specified by a parameter vector $\mathbf{p} = (p_0, p_1, \cdots, p_{m-1})$ such that the conditional probability $\mathsf{P}(Y = x+i|X = x) = p_i$, $\forall x, i \in \mathbb{Z}_m$.

2.3. Error Probability v.s. CB v.s. DCB

Let $p_e = \mathsf{P}\left(X \neq \hat{X}_{\text{MAP}}(Y)\right)$ denote the error probability of the maximum *a posteriori* detector. The relationship between p_e and the above noise measures CB and DCB are stated by the following lemmas.

Lemma 1 *For general binary-input/non-symmetric-output (BI-NSO) channels and arbitrary a priori distribution P_X, we have*

$$\begin{aligned} 2p_e \leq CB \leq 2\sqrt{p_e(1-p_e)} \\ 2p_e \leq DCB \leq 4p_e(1-p_e). \end{aligned}$$

Lemma 2 *For general BI-NSO channels and arbitrary a priori distribution P_X, we have*

$$DCB \leq CB \leq \sqrt{DCB}. \tag{2}$$

Lemma 3 *For memoryless MI-SO channels and uniform a priori distribution P_X, we have*

$$\begin{aligned} \max_{x\in\mathbb{Z}_m} CB(0\to x) &\leq 2\sqrt{p_e(1-p_e)} \\ 2p_e &\leq \sum_{x=1,\cdots,m-1} CB(0\to x). \end{aligned}$$

3. Iterative Bounds for Symmetric Channels

We first note that all MI-SO channels can be represented as a probabilistic combination of several MSC's with the parameter vector $\mathbf{p}$ as side information observed by the receiver. Different channels will then have different "distributions" on the side information $\mathbf{p}$. All our results are obtained by considering fixed $\mathbf{p}$ first and then taking probabilistic average over possible $\mathbf{p}$'s.

Because the variable (check) nodes of degree > 3 can be obtained from concatenating several degree 3 variable (check) nodes, we focus only on degree 3 nodes as in Fig. 1, where $dP_{in,1}$ and $dP_{in,2}$ are the independent probabilistic weights of different MSC's. The vari-

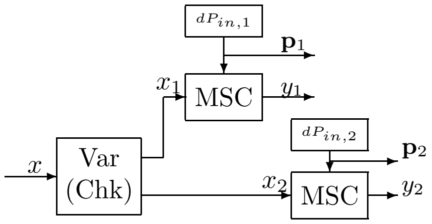

Figure 1: The channel model for the variable (check) nodes.

able (check) node thus becomes a $\mathbf{X} \mapsto \mathbf{Y}_1 \times \mathbf{Y}_2$ channel. By fixing the CB or DCB values of the two constituent $\mathbf{X}_i \mapsto \mathbf{Y}_i$, we are then able to bound the CB or DCB values of the $\mathbf{X} \mapsto \mathbf{Y}_1 \times \mathbf{Y}_2$ channel. Due to the nature of LDPC codes [1], this procedure can be iteratively applied, and we obtain the following two iterative bounds.

Remark: Without modifications, these results are applicable to any LDPC-like codes, e.g., irregular repeat accumulate codes [10], tree codes [11], etc.

3.1. An m-ary Bound for $\mathbb{Z}_m$ LDPC Codes

In this paper, a $\mathbb{Z}_m$ LDPC code is defined as a parity check code over the quotient ring $\mathbb{Z}_m$, such that there exists a sparse parity check matrix A, and $\underline{x} \in \mathbb{Z}_m^n$ is a valid codeword iff $A\underline{x} = 0$ in $\mathbb{Z}_m$. We further limit the entries of A to be either 0 or 1. In other words, A is the incidence matrix of the corresponding graph, with the weights of all edges being 1. For the concerned code ensemble, the distribution of the non-zero entries in A is as the conventional graph-based ensemble defined in [1].

For any MI-SO channels, we immediately have

$$\begin{aligned} &\text{[Symmetry:]} && CB(x \to x') = CB(x' \to x) \\ &\text{[Stationarity:]} && CB(x \to x') = CB(0 \to x' - x). \end{aligned}$$

By the stationarity, we can use $\mathsf{CB} := \{CB(0 \to x')\}_{x' \in \mathbb{Z}_m}$ as the representing vector for all $CB(x \to x')$. For the variable node of degree d_v, we are able to show that

$$\mathsf{CB}_{out} = \prod_{j=1}^{d_v-1} \mathsf{CB}_{in,j}, \qquad (3)$$

where CB_{out} is the CB vector of the equivalent channel: $\mathbf{X} \mapsto \mathbf{Y}_1 \times \cdots \times \mathbf{Y}_{d_v-1}$; $\mathsf{CB}_{in,j}$ is the CB vector of the constituent channels: $\mathbf{X}_j \mapsto \mathbf{Y}_j$; and $\prod$ is the component-wise product.

For the check node of degree d_c, we are able to show that

$$\mathsf{CB}_{out} \leq \bigotimes_{j=1}^{d_c-1} \mathsf{CB}_{in,j}, \qquad (4)$$

where $\bigotimes$ is the circular convolution. Let $\lambda := \sum \lambda_k z^{k-1}$ and $\rho := \sum \rho_k z^{k-1}$ be the variable and check node degree polynomials for irregular LDPC codes. By (3) and (4), we obtain the following theorem.

Theorem 1 *Assuming a MI-SO channel, let $\mathsf{CB}^{(l)}$ denote the representing CB vector of the belief propagation decoder on a $\mathbb{Z}_m$ LDPC code after l iterations. We have*

$$\mathsf{CB}^{(l+1)} \leq \mathsf{CB}^{(0)} \lambda \left(\rho \left(\mathsf{CB}^{(l)} \right) \right),$$

where the multiplication operations in λ are replaced by component-wise products, and those in ρ are replaced by circular convolutions.

By combining Theorem 1 and Lemma 1, we have

Corollary 1 (Sufficient Stability Condition) *For $\mathbb{Z}_m$ LDPC codes on MI-SO channels, if $\forall x \neq 0$, $CB^{(0)}(0 \to x) < \frac{1}{\lambda_2 \rho'(1)}$, then the belief propagation decoding is stable. Namely, there exists $\epsilon_0 > 0$ such that if the error probability after some l_0 iterations satisfies $p_e^{(l_0)} < \epsilon_0$ then $\lim_{l \to \infty} p_e^{(l)} = 0$. Furthermore, the convergence speed (w.r.t. l) is either exponential or super exponential depending on whether $\lambda_2 \neq 0$.*

This result can be viewed as a generalized version of the stability condition first proposed in [1].

A matched necessary stability condition can be proved as follows.

Theorem 2 *Assuming a MI-SO channel, if there exists $x_0 \neq 0$ such that $CB^{(0)}(0 \to x_0) > \frac{1}{\lambda_2 \rho'(1)}$, then the system is not stable and $\lim_{l \to \infty} p_e^{(l)}$ is strictly larger than 0.*

We conclude this subsection by showing that the results in [9] can be viewed as corollaries of Corollary 1 and Theorem 2.

Consider only prime q's for the following discussions. A $\mathsf{GF}(q)$-based LDPC code ensemble (formally defined in [9]) can be obtained from our $\mathbb{Z}_m$-based LDPC code ensemble by setting $m = q$ and associating each edge with some "weight," uniformly distributed between $\{1, 2, \cdots, q-1\}$. Namely, each non-zero entry in the incidence matrix A can now take values other than 1. A codeword $\underline{x}$ is valid iff $A\underline{x} = 0$ in $\mathsf{GF}(q)$.

Let $CB(0 \to x)$ denote the pairwise Bhattacharyya noise parameter for $\mathsf{GF}(q)$ LDPC codes. Since the multiplication in $\mathsf{GF}(q)$ is simply the permutation of different symbols, the $\mathsf{GF}(q)$-based code ensemble with uniform edge weight distributions is equivalent to a $\mathbb{Z}_q$-based code ensemble with new $CB'(0 \to z) = \frac{1}{q-1} \sum_{x=1}^{q-1} CB(0 \to x)$ for all $z \neq 0$. As direct results from Corollary 1 and Theorem 2, we have

Corollary 2 (Suff. Stab. Cond. for $\mathsf{GF}(q)$ Codes [9]) *For $\mathsf{GF}(q)$ LDPC codes on MI-SO channels, if $\frac{1}{q-1}\sum_{x=1}^{q-1} CB^{(0)}(0 \to x) < \frac{1}{\lambda_2 \rho'(1)}$, then the belief propagation decoding is stable. Similar to Corollary 1, the convergence speed depends on whether or not $\lambda_2 = 0$.*

Corollary 3 (Necc. Stab. Cond. for $\mathsf{GF}(q)$ Codes [9]) *Assuming a MI-SO channel, if $\frac{1}{q-1}\sum_{x=1}^{q-1} CB^{(0)}(0 \to x) > \frac{1}{\lambda_2 \rho'(1)}$, then the system is not stable and $\lim_{l\to\infty} p_e^{(l)}$ is strictly larger than 0.*

3.2. A Two-Dimensional Bound

For BI-SO channels, instead of focusing on CB or DCB separately as in [4] and [12], a new iterative upper bound is obtained by considering the two-dimensional noise measure (CB, DCB). The iteration-based inequalities are stated as follows.

Theorem 3 ($UB_{CB,DCB}$ for Check Nodes) *For a check node with degree distribution polynomial ρ and constituent channels of (CB_{in}, DCB_{in}) (as illustrated in Fig. 1), the equivalent channel of the check node has*

$$DCB_{out} \leq 1 - \rho\left(1 - DCB_{in}\right)$$

$$CB_{out} \leq \sum_k \rho_k \sum_{i=1}^{k-1} \binom{k-1}{i} \left(1 - \frac{CB_{in}^2}{DCB_{in}}\right)^{k-1-i} \left(\frac{CB_{in}^2}{DCB_{in}}\right)^i \sqrt{1 - \left(1 - \left(\frac{DCB_{in}}{CB_{in}}\right)^2\right)^i}.$$

Theorem 4 *For a variable node with degree distribution polynomial λ, we have*

$$\begin{aligned} CB_{out} &\leq CB_0 \lambda(CB_{in}) \\ DCB_{out} &\leq \sum_k \lambda_k \Phi_{k-1}\left((CB_0, DCB_0), (CB_{in}, DCB_{in})\right), \end{aligned}$$

where $\Phi_{k-1}\left((CB_0, DCB_0), (CB_{in}, DCB_{in})\right)$ computes the DCB value of a variable node channel with one maximizing constituent channel specified by (CB_0, DCB_0) — corresponding to initial message m_0; and $(k-1)$ maximizing constituent channels specified by (CB_{in}, DCB_{in}) — corresponding to incoming messages $m_1, \cdots, m_{d_v - 1}$.

Each maximizing channel is the probabilistic combination of 3 BSCs with crossover probabilities $\frac{1-\sqrt{1-CB^2}}{2}$, $\frac{1-\sqrt{1-DCB}}{2}$, and $\frac{1-\sqrt{1-(CB/DCB)^2}}{2}$. The probabilistic weights of these BSCs are

$$dP(p) = \begin{cases} (1 - f_{DCB})\frac{t}{t+CB} & \text{if } 2\sqrt{p(1-p)} = CB \\ f_{DCB} & \text{if } 2\sqrt{p(1-p)} = \sqrt{DCB} \\ (1 - f_{DCB})\frac{CB}{t+CB} & \text{if } 2\sqrt{p(1-p)} = t \\ 0 & \text{otherwise} \end{cases},$$

where

$$\begin{aligned} t &= \frac{DCB}{CB} \\ f_{DCB} &= \begin{cases} 0 & \text{if } 2\sqrt{tCB} - t + \sqrt{CB(2t - CB)} \geq 0 \\ \frac{\eta(w^*)}{2t(t-CB)^2} & \text{otherwise} \end{cases} \\ \eta(w) &= w^3 - 2tw^2 + (t - CB)^2 w \\ w^* &= \begin{cases} 2\sqrt{tCB} & \text{if } \left.\frac{d\eta}{dw}\right|_{w=2\sqrt{tCB}} \leq 0 \\ \frac{2t - \sqrt{4t^2 - 3(t-CB)^2}}{3} & \text{otherwise.} \end{cases}, \end{aligned}$$

The closed form representation of the function Φ_{k-1} is omitted here for the sake of brevity.

By iteratively substituting $(CB^{(l)}, DCB^{(l)})$ into the inequalities of Theorems 3 and 4, and checking whether $(CB^{(l)}, DCB^{(l)})$ converges to $(0, 0)$, we obtain a two-dimensional lower bound on the decodable threshold for the BI-SO channels.

Remark: Though the closed form of Φ_{k-1} is not given explicitly here, by specifying the probabilistic weights of its maximizing constituent channels, Φ_{k-1} can be easily evaluated by Eq. (1) and the density evolution method [1].

4. A Non-Iterative One-Dimensional Bound

One existing lower bound is based on the error probability of the ML detector, $p_e := \mathsf{P}(X \neq \hat{X}_{ML}(Y))$, and follows from degraded channel arguments. That is, if an irregular (λ, ρ) LDPC code is decodable when applied to a BSC with crossover probability p, then all BI-SO channels with $p_e \leq p$ are also decodable. For example, the BSC threshold of the regular (3,6) code is $p^* = 0.0837$, then all channels with $p_e \leq 0.0837$ are decodable, including BECs with $\epsilon \leq 0.1674$ and AWGN channels with $\sigma \leq 0.7246$. This bound, however, is not tight enough considering the exact thresholds for BECs and AWGN channels are $\epsilon^* = 0.4294$ and $\sigma^* = 0.8790$.

A new non-iterative bound on DCB rather than p_e is provided as follows.

Theorem 5 (UB_{DCB^*}) *For any BI-SO channel F_C, let F_{BSC} be a BSC such that these two channels have the same DCB value: $DCB(F_C) = DCB(F_{BSC})$. Then for any irregular (λ, ρ) LDPC code,*

$$CB^{(l)}(F_{BSC}) \geq CB^{(l)}(F_C),$$

where $CB^{(l)}(F_C)$ denotes the CB value after l iterations, assuming all channels are F_C, and similarly does $CB^{(l)}(F_{BSC})$.

Corollary 4 *If a (λ, ρ) LDPC code is decodable for a BSC with crossover probability p^*, then all BI-SO channels F_C with $DCB(F_C) \leq 4p^*(1 - p^*)$ are decodable under the same (λ, ρ) code.*

For example, the DCB value of a BEC equals the erasure probability ϵ. Corollary 4 shows that $\epsilon = 0.3068 = 4 * 0.0837 * (1 - 0.0837)$ is decodable, which is better than the $\epsilon = 0.1674$ obtained from the degraded channel argument. And it is easy to check that a BEC with $\epsilon = 0.3068$ is not physically degraded w.r.t. a BSC with $p^* = 0.0837$.

5. Bounds for BI-NSO Channels

Most of the existing bounds and the bounds in the previous sections are for BI-SO or MI-SO channels, while finite-dimensional bounds for non-symmetric channels remain absent. Khandekar *et al.* provided one iterative upper bound on the Bhattacharyya CB for BI-SO channels [4], such that

$$CB^{(l+1)} \leq CB^{(0)} \lambda \left(1 - \rho(1 - CB^{(l)})\right). \qquad (5)$$

By the reverse channel perspective and the convexity analysis of the variable/check node iterations, we have

Theorem 6 *Ineq. (5) holds even for BI-NSO channels.*

Since Ineq. (5) is tight for BECs and the CB value of a BEC equals its erasure probability ϵ, we have another threshold lower bound as follows.

Corollary 5 *If a (λ, ρ) irregular LDPC code is decodable for a BEC with erasure probability ϵ^*, then all BI-NSO channels with $CB^{(0)} \leq \epsilon^*$ are decodable under the same (λ, ρ) LDPC code.*

Also by the reverse channel perspective, we provide an iterative lower bound on $CB^{(l)}$ for BI-NSO channels.

Theorem 7 *For BI-NSO channels, we have*

$$CB^{(l+1)} \geq CB^{(0)} \lambda \left(\sum \rho_k \sqrt{1 - (1 - (CB^{(l)})^2)^{k-1}} \right).$$

By iteratively applying these inequalities and checking whether the sequence converges to 0, we thus obtain a pair of lower and upper bounds on the docodable threshold for BI-NSO. One important applications of this bound is to z-channels, where only bits with value 1 may be flipped/corrupted and bits with value 0 will be perfectly received. Our results show that a z-channel, the simple model for optical channels, is decodable under regular (3,6) code when $p_{10} \geq 0.1844$. The exact threshold found by density evolution is $p_{10}^* = 0.2304$ [8].

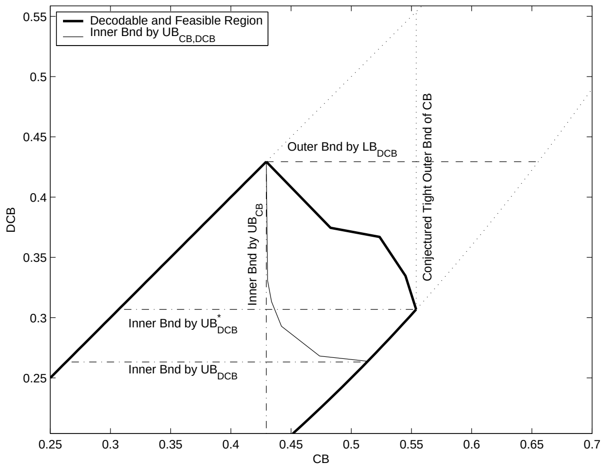

Figure 2: The decodable region with respect to the (CB, DCB) domain.

6. Performance Comparison & Conclusions

For binary input channels, Table 1 compares our two new finite-dimensional lower bounds on the decodable thresholds (obtained from $UB_{CB,DCB}$ and UB^*_{DCB}) with existing results, where UB_{CB} is the Bhattacharyya-bound from [4], UB_{DCB} is the expected-soft-bit-bound from [3], and UB_{info} is the capacity-bound from [6]. All these threshold lower bounds build their foundations on upper bounding the evolved noise measures.

It can be easily seen from Table 1 that that UB_{CB} performs better when the channels are more BEC-like, while UB_{DCB} is suitable for the BSC-like channels. UB_{info} falls right in the middle, and is tighter when the channels are neither BSC-like nor BEC-like. This phenomenon can be proved by the inherent convexity/concavity while bounding the target noise measures iteration by iteration. It is worth noting that our two-dimensional bound $UB_{CB,DCB}$ completely covers the existing UB_{CB} and UB_{DCB} bounds and thus has the best predictability in both BSC-like and BEC-like scenarios. This is at the expense of a more complicated iteration formula taking into account both CB and DCB.

Fig. 2 illustrates the decodable region when plotted in the (CB, DCB) domain. The thick border indicates the decodable region obtained from the density evolution and feasibility condition in Eq. (2). Our results, $UB_{CB,DCB}$ and UB_{DCB^*}, further push the inner bound of the decodable region from the existing results, UB_{CB} and UB_{DCB}. Besides the better performance of the proposed new bounds for most types of channels, UB_{CB} has now been generalized as the first

	Dens. Evo.	UB_{CB}	$UB_{CB,DCB}$	UB^*_{DCB}	UB_{DCB}	UB_{info}
Decodable Thresholds	—	$CB^* \geq 0.4294$	—	$DCB^* \geq 0.3068$	$DCB^* \geq 0.2632$	$h^* \geq 0.3644$
BEC (ϵ^*)	≈ 0.4294	≥ 0.4294	$\geq$ **0.4294**	$\geq$ **0.3068**	≥ 0.2632	≥ 0.3644
Rayleigh (σ^*)	≈ 0.645	≥ 0.6134	$\geq$ **0.6148**	$\geq$ **0.5804**	≥ 0.5191	≥ 0.6088
Z-channels (p^*_{10})	≈ 0.2304	$\geq$ **0.1844**	—	—	—	—
BiAWGNC (σ^*)	≈ 0.8790	≥ 0.7690	$\geq$ **0.7826**	$\geq$ **0.8001**	≥ 0.7460	≥ 0.8018
BiLC(λ^*)	≈ 0.65	≥ 0.5221	$\geq$ **0.5670**	$\geq$ **0.6146**	≥ 0.5610	≥ 0.5864
BSC (p^*)	≈ 0.0837	≥ 0.0484	$\geq$ **0.0710**	$\geq$ **0.0837**	≥ 0.0708	≥ 0.0696

Table 1: Comparision between different lower bounds on regular (3,6) codes.

finite-dimensional bound for BI-NSO channels.

7. Conclusions

Finite dimensional bounds on the decodable thresholds find applications in both theoretical analysis and practical approximations. In this paper, we have provided a new iterative upper bound for $\mathbb{Z}_m$-based LDPC codes, which leads to a sufficient stability condition and sheds more insight on the analytical structure of general LDPC codes. Combining with a matched necessary stability condition stated herein, our stability condition pair can be used to derive easily the existing stability conditions for $\mathsf{GF}(q)$ bounds.

Two new bounds for binary codes have been provided, which further pushes the inner bound of the decodable thresholds on two types of noise measures, CB and DCB.

An iteration-based bound for general non-symmetric memoryless channels has been derived, which finds applications in optical channels or magnetic storage channels.

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