On Finite-Dimensional Bounds for LDPC-like Codes with Iterative Decoding

Bounds on Binary and $\mathbb{Z}_m$ LDPC Codes

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- The support tree is a $\{0,1\} \mapsto \mathbf{Y}$ channel, where $\mathbf{Y} = \mathbb{R}^{\#\{\text{involved var. nodes}\}}$.

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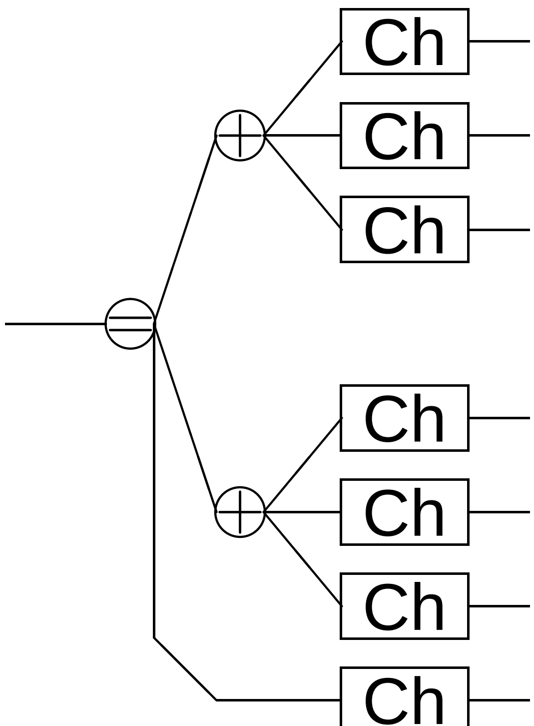

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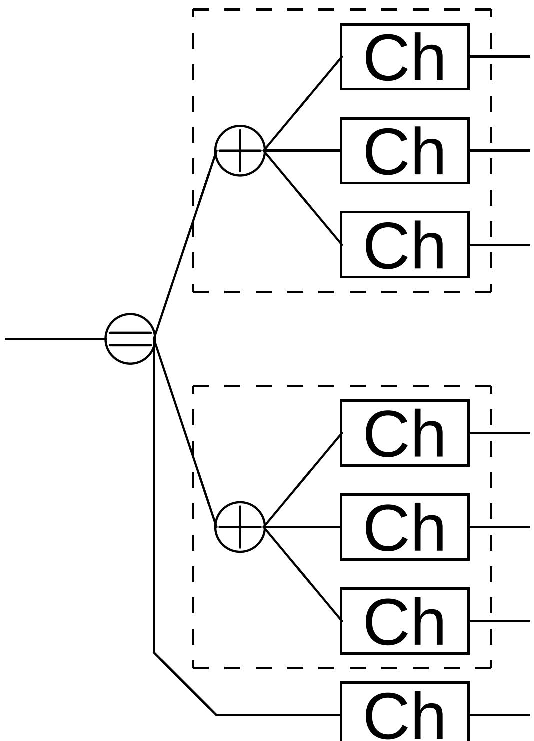

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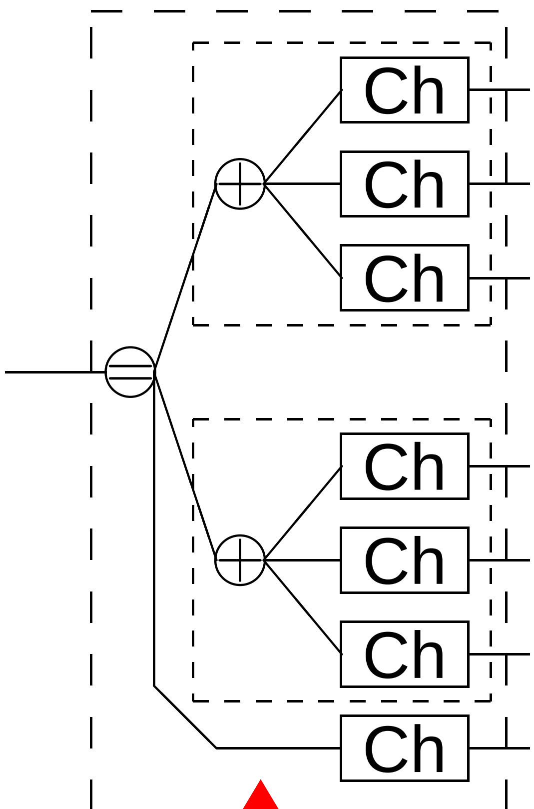

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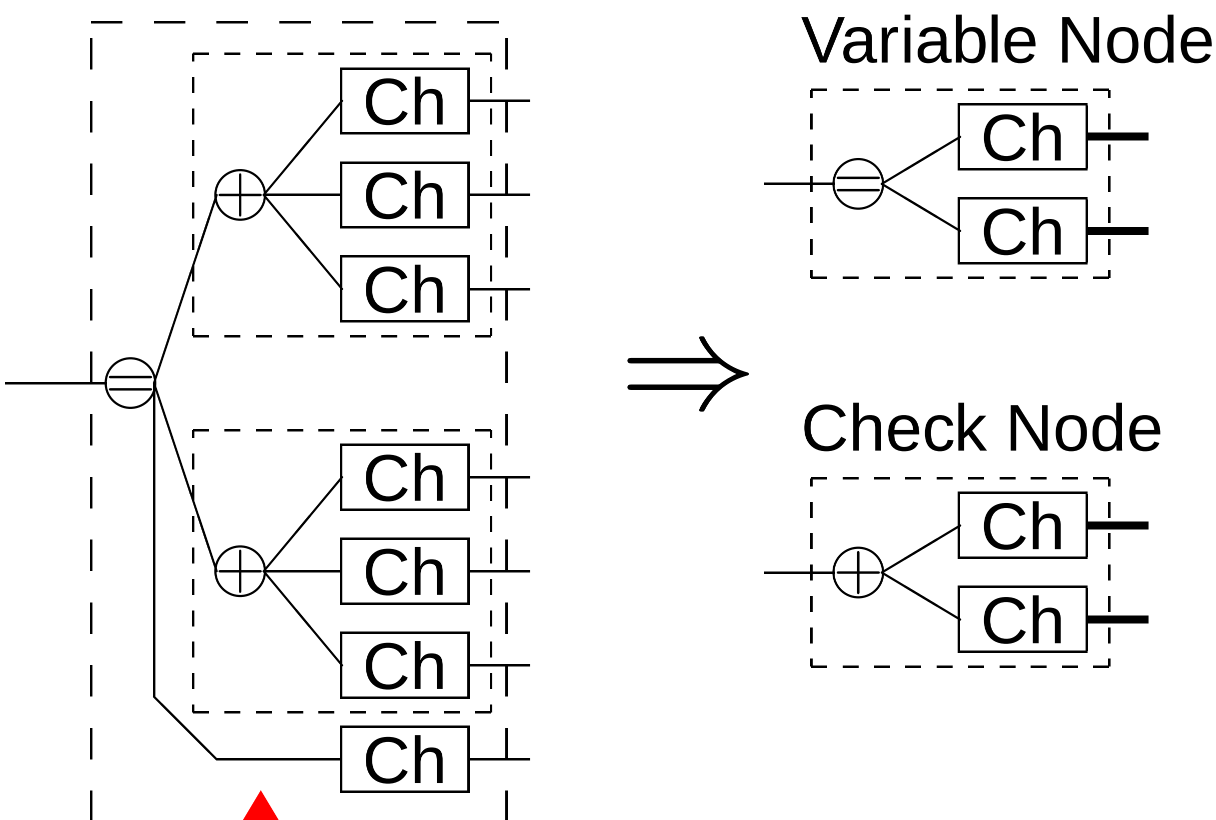

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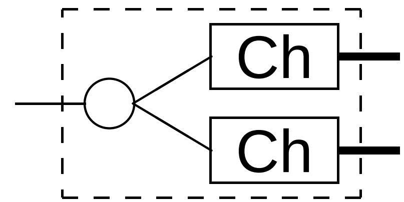

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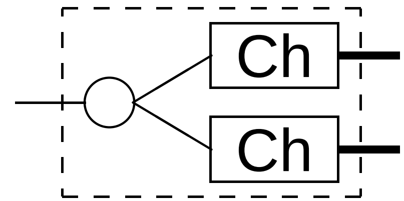

  - $p_e := \mathsf{P}\{X \neq \hat{X}_{ML}(Y)\}$.
  - [Burshtein and Miller 02]: $DCB := 2\mathsf{E}_{XY}\{\mathsf{P}(\bar{X}|Y)\}$.
    Note: $DCB \neq 2p_e$
  - [Khandekar and McEliece 01]: $CB := \mathsf{E}_{XY}\{\sqrt{\frac{\mathsf{P}(\bar{X}|Y)}{\mathsf{P}(X|Y)}}\}$.
  - [Land *et al.* 03], [Sutskover *et al.* 03]: Mutual information, or equivalently, conditional entropy $h := H_2(X|Y)$.

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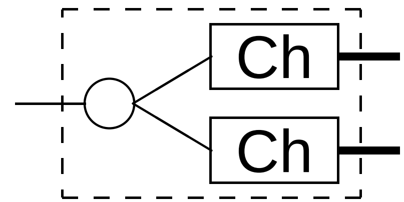

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  - [Land *et al.* 03], [Sutskover *et al.* 03]: Mutual information, or equivalently, conditional entropy $h := H_2(X|Y)$.
- Simpler iteration formulas admit more analytical results.

Symmetric Channels: Probabilistic Combination of BSC's

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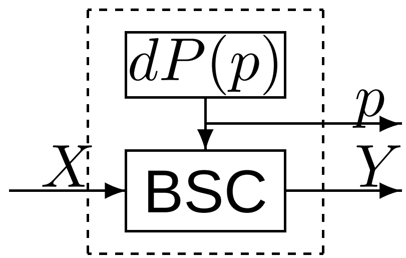

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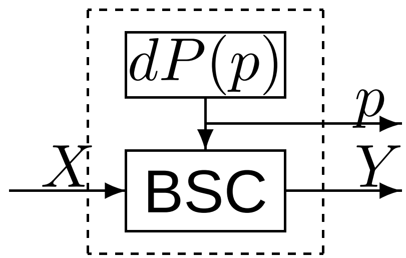
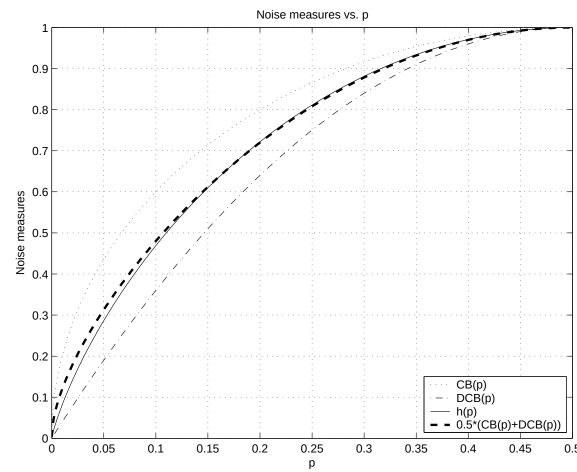
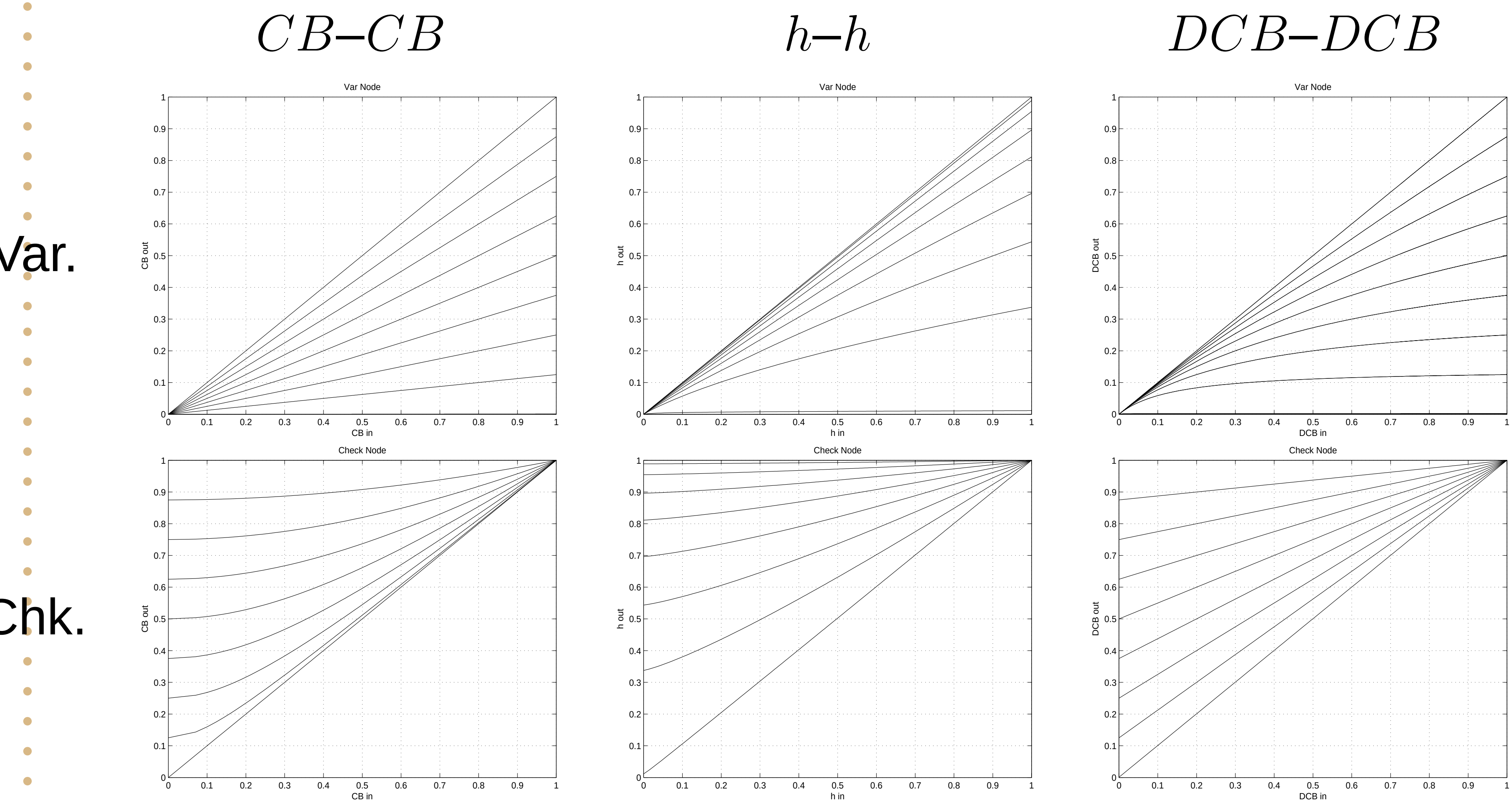




- Observation 3: For BSC's, any one of CB, DCB, h, and p_e can uniquely specified the channel.

Convexity & Concavity



Existing Results

- UB_{CB} and LB_{CB} [Khandekar *et al.* 01, Wang *et al.* 04], UB_{DCB} and LB_{DCB} [Burshtein *et al.* 02], UB_{info} and LB_{info} [Land *et al.* 03, Sutskover *et al.* 03].
- UB_{CB}/UB_{DCB}/UB_{info}:
  Assuming consituent channels are BEC's (of the same CB/DCB/h values) at the check nodes and are BSC's at the variable nodes.
- LB_{CB}/LB_{DCB}/LB_{info}:
  Assuming consituent channels are BSC's (of the same CB/DCB/h values) at the check nodes and are BEC's at the variable nodes.

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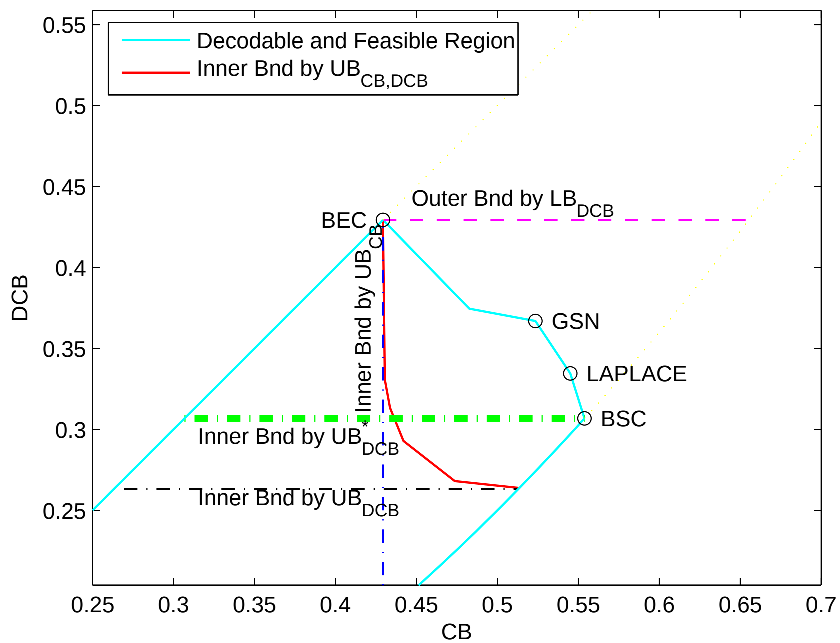
  Existing results:



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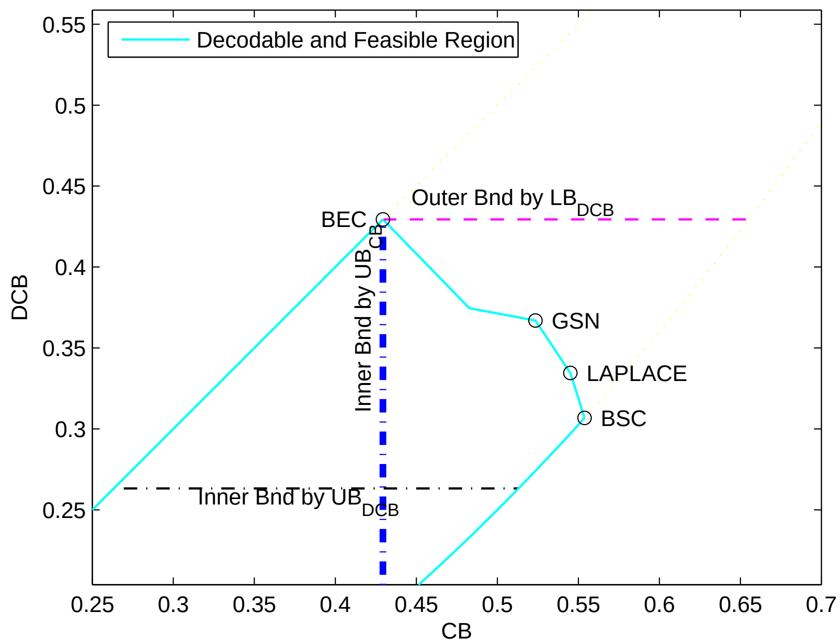

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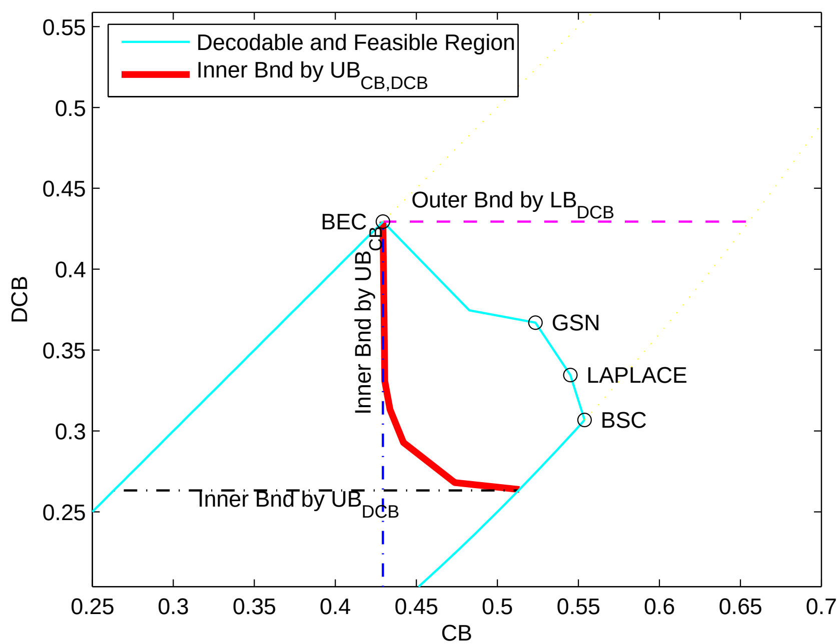

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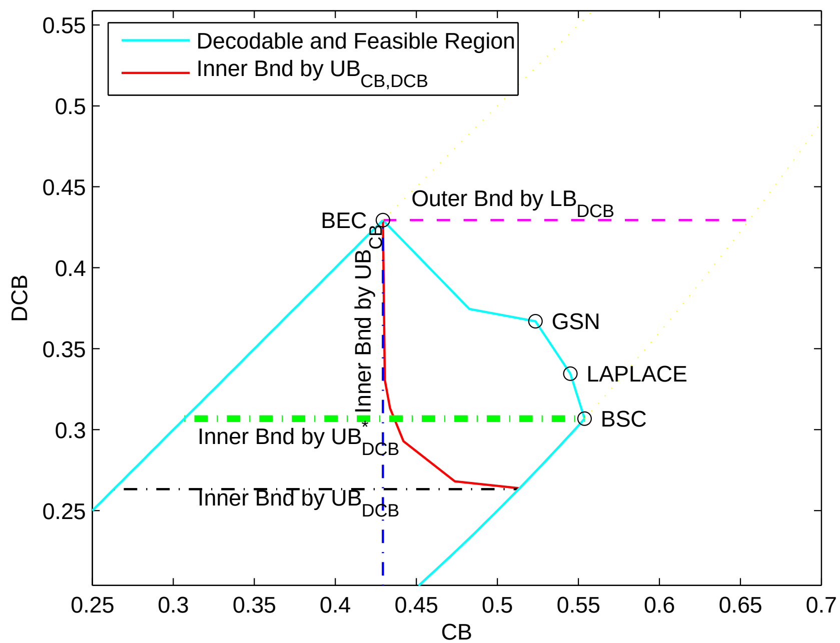

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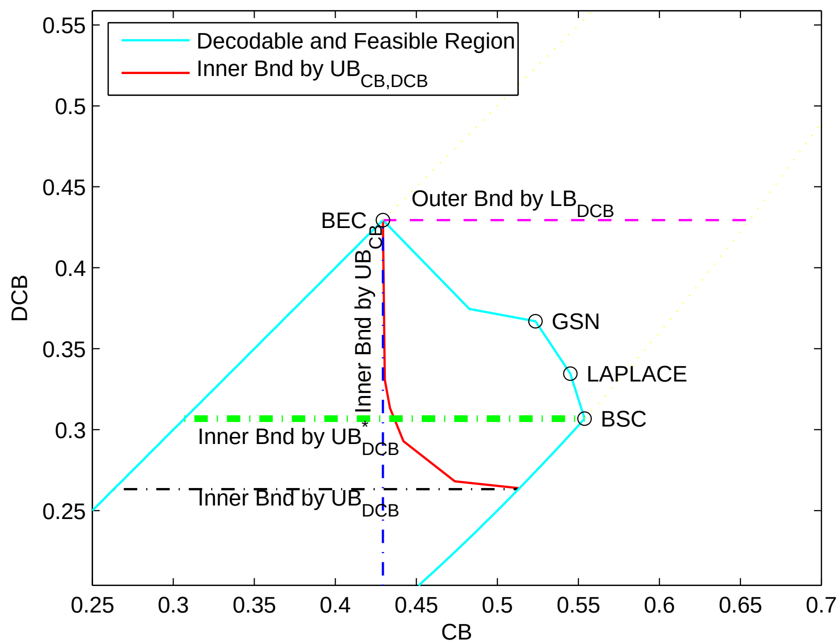
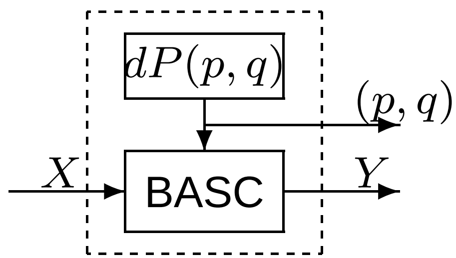
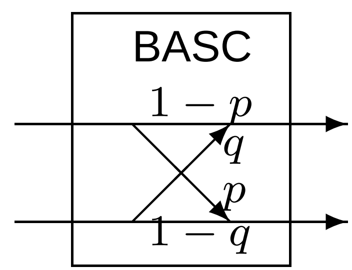


- $\mathbb{Z}_m$-based LDPC codes: A CB-based iterative bound, a tight pair of necessary and sufficient stability conditions.

Memoryless Non-symmetric Channels

- Assuming X is uniformly distributed on $\{0, 1\}$,
  $CB := \mathsf{E}_{X,Y}\left\{\sqrt{\frac{p(\overline{X}|Y)}{p(X|Y)}}\right\}$ is well-defined even for non-symmetric channels.
- Probabilistic combination of BASC's:





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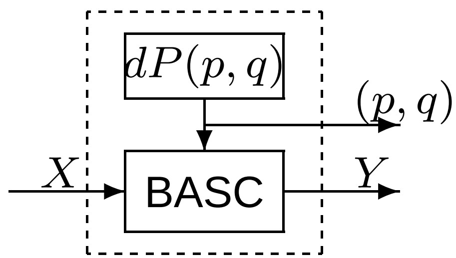
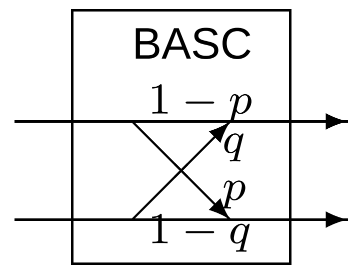




- Hardness:
  For BASC: $CB = \sqrt{p(1-q)} + \sqrt{q(1-p)}$ and (p,q) are not one-to-one, and neither the convexity nor concavity exists.

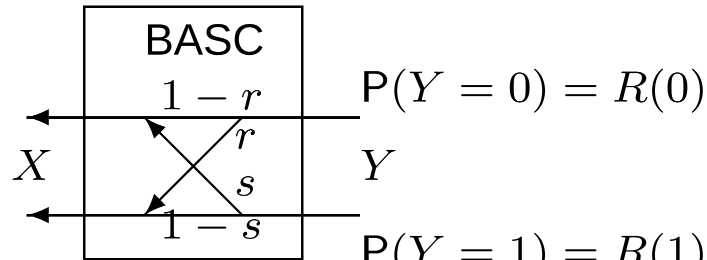
Reverse Channel Perspective

$$r = \frac{q}{1-p+q}$$

$$s = \frac{p}{1-q+p}$$

$$R(0) = \frac{1-p+q}{2}$$

$$R(1) = \frac{1-q+p}{2}$$

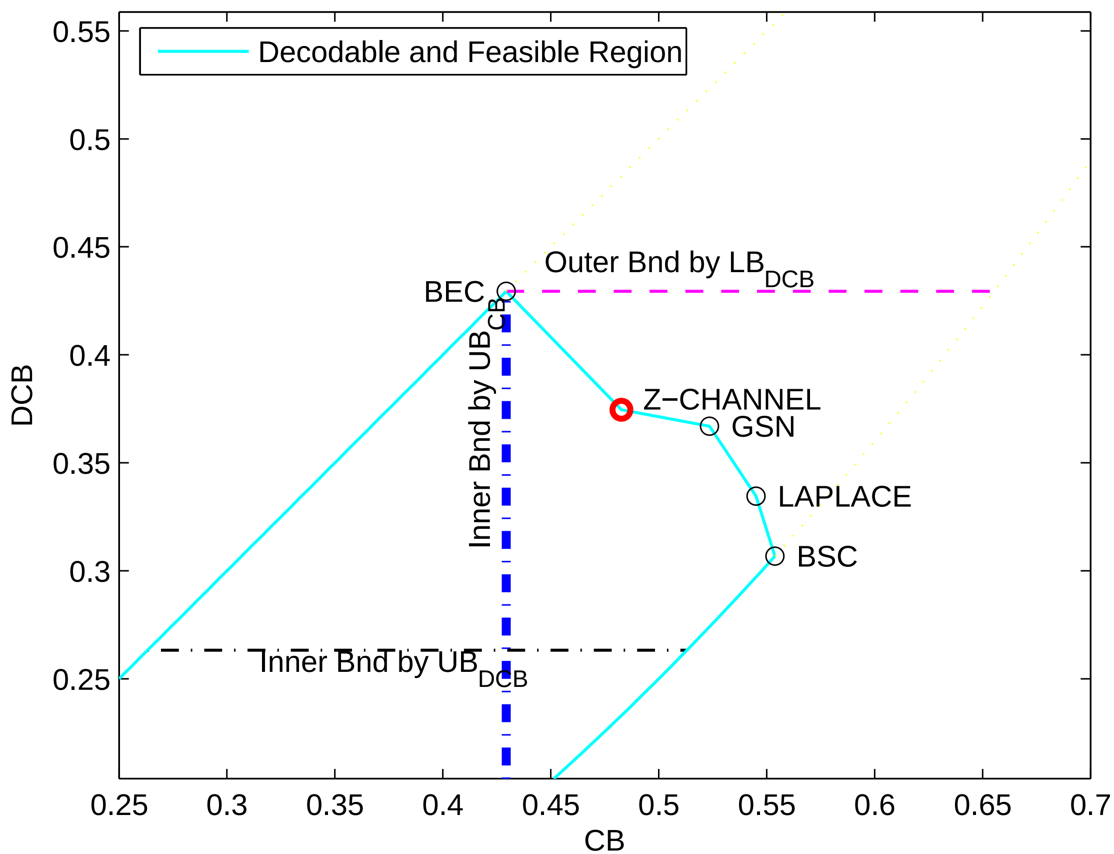


$$CB := \mathsf{E}_{X,Y}\left\{\sqrt{\frac{p(\overline{X}|Y)}{p(X|Y)}}\right\} = R(0)\sqrt{r(1-r)} + R(1)\sqrt{s(1-s)}.$$

The CB value is *seemingly* decoupled as a probabilistic combination of two BSC's with weights $R(0)$ and $R(1)$.

A CB Bound for Non-Symmetric Channels

$$CB^{(l+1)} \le CB^{(0)} \left(1 - \left(1 - CB^{(l)}\right)^{d_c - 1}\right)^{d_v - 1}$$



$UB_{CB,DCB}$ for Symmetric Channels

For both variable and check nodes, the (CB_{out}, DCB_{out})-maximizing constituent channel can be formally stated as a universal maximizer:

$$\forall p_2 \in [0, 1/2], \max \quad \int CB_{out}(p_1, p_2) dP_1(p_1)$$

$$\text{and} \int DCB_{out}(p_1, p_2) dP_1(p_1),$$

$$\text{subject to} \quad \int 2\sqrt{p_1(1-p_1)} dP_1(p_1) \leq CB_{in,1}$$

$$\int 4p_1(1-p_1) dP_1(p_1) \leq DCB_{in,1}.$$

$UB_{CB,DCB}$ at Check Nodes

- For check nodes, such a universal maximizer $dP_1^*(p_1)$ exists:

$$dP_1^*(p_1) = \begin{cases} 1 - \frac{CB_{in,1}}{t} & \text{if } p_1 = 0 \\ \frac{CB_{in,1}}{t} & \text{if } 2\sqrt{p_1(1-p_1)} = t \\ 0 & \text{otherwise} \end{cases},$$

where

$$t = \frac{DCB_{in,1}}{CB_{in,1}}.$$

$UB_{CB,DCB}$ at Variable Nodes

- For variable nodes, no such universal maximizer exists. Alternatively, we find a universal bound, $dP_1^{\dagger}(p_1)$, as a function of $CB_{in,1}$ and $DCB_{in,1}$.

$$\int CB_{out}(p_1, p_2)dP_1^{\dagger}(p_1) \geq \int CB_{out}(p_1, p_2)dP_1(p_1), \quad \forall p_2 \in [0, 1/2]$$

subject to $\int 2\sqrt{p_1(1-p_1)}dP_1(p_1) \leq CB_{in,1}$

$$\int 4p_1(1-p_1)dP_1(p_1) \leq DCB_{in,1}.$$

$dP_1^\dagger(p_1)$ for Variable Nodes

$$dP_1^\dagger(p_1) = \begin{cases} (1-f_{DCB})\frac{t}{t+CB_{in,1}} & \text{if } 2\sqrt{p_1(1-p_1)} = CB_{in,1} \\ f_{DCB} & \text{if } 2\sqrt{p_1(1-p_1)} = \sqrt{DCB_{in,1}} \\ (1-f_{DCB})\frac{CB_{in,1}}{t+CB_{in,1}} & \text{if } 2\sqrt{p_1(1-p_1)} = t \\ 0 & \text{otherwise} \end{cases},$$

$$t = \frac{DCB_{in,1}}{CB_{in,1}}$$

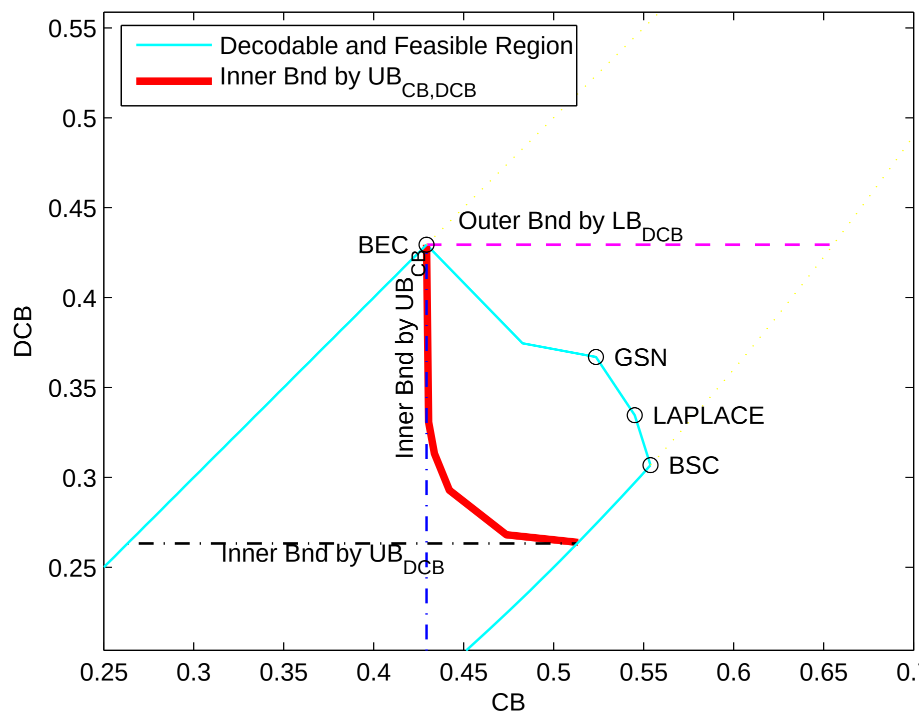
$$f_{DCB} = \begin{cases} 0 & \text{if } 2\sqrt{tCB_{in,1}} - t + \sqrt{CB_{in,1}(2t - CB_{in,1})} \geq 0 \\ \frac{\eta(w^*)}{2t(t-CB_{in,1})^2} & \text{otherwise} \end{cases}$$

$$\eta(w) = w^3 - 2tw^2 + (t - CB_{in,1})^2 w$$

$$w^* = \begin{cases} 2\sqrt{tCB_{in,1}} & \text{if } \eta'(2\sqrt{tCB_{in,1}}) \leq 0 \\ \frac{2t - \sqrt{4t^2 - 3(t-CB_{in,1})^2}}{3} & \text{otherwise.} \end{cases}$$

$UB_{CB,DCB}$

- Assuming consituent channels are $dP^*(p)$'s (of the same (CB, DCB) values) at the check nodes and are $dP^\dagger(p)$'s at the variable nodes.



A New Channel Degradation Argument

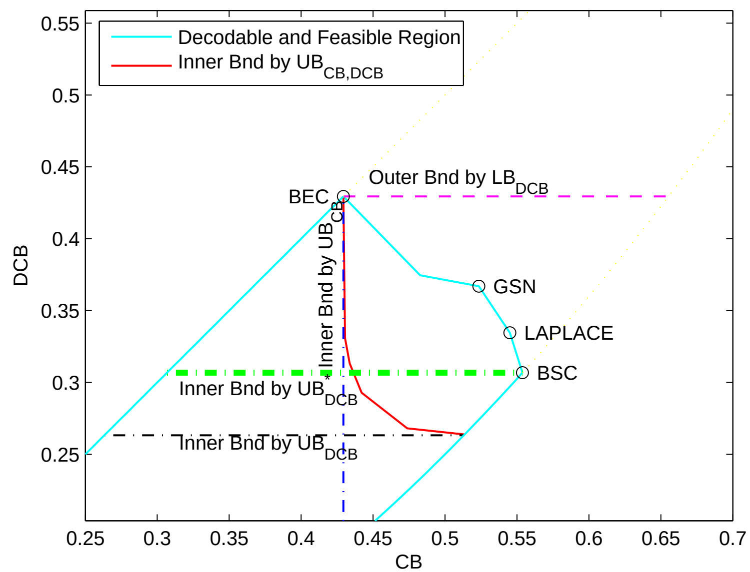
Stochastic Degraded Channel Arguments [Cover & Thomas 91]: if the conditional distribution $\mathsf{P}_{Y|X}(y|x)$ can be written as $\int \mathsf{P}_{Y|W}(y|w)\mathsf{P}_{(}w|x)dw$, then we have

- $\mathsf{P}\left(\hat{X}_{ML}(Y) \neq X\right) \geq \mathsf{P}\left(\hat{X}_{ML}(W) \neq X\right)$,
- $CB_{X\to Y} \geq CB_{X\to W}$,
- and $H(X|Y) \geq H(X|W)$.

Application: among all constituent channels of the same p_e value, BSC's will lead to the highest $\mathsf{P}\left(\hat{X}_{ML}(Y_1, \cdots, Y_n)\right)$.

Theorem 1 *Among all* symmetric-output *constituent channels of the same DCB values, BSC's will lead to the highest $CB := \mathsf{E}_{X\mathbf{Y}}\left\{\sqrt{\frac{\mathsf{P}(\bar{X}|\mathbf{Y})}{\mathsf{P}(X|\mathbf{Y})}}\right\}$.*

Comparisons



Consider regular (3,6) codes on Laplace channels with antipodal $\{+1,-1\}$ inputs:

	DE	UB_{DCB}	UB_{CB}	UB_{info}	$UB_{CB,DCB}$	$UB^*_{p_e}$	UB^*_{DCB}
Decodable Thresholds	—	$DCB^* \geq 0.2632$	$CB^* \geq 0.4294$	$h^* \geq 0.3644$	—	$p_e \geq 0.0837$	$DCB^* \geq 0.3068$
BEC (ϵ^*)	≈ 0.4294	≥ 0.2632	≥ 0.4294	≥ 0.3644	≥ 0.4294	≥ 0.0837	≥ 0.3068
Rayleigh (σ^*)		≥ 0.5191	≥ 0.6134	≥ 0.6088	≥ 0.6148	≥ 0.471	≥ 0.5804
Z-channels (p^*_{10})	$= 0.2304$	—	≥ 0.1844	—	—	≥ 0.1674	—
BiAWGNC (σ^*)	≈ 0.8790	≥ 0.7460	≥ 0.7690	≥ 0.8018	≥ 0.7826	≥ 0.7244	≥ 0.8001
BiLC(λ^*)	≈ 0.65	≥ 0.5610	≥ 0.5221	≥ 0.5864	≥ 0.5670	≥ 0.5595	≥ 0.6146
BSC (p^*)	≈ 0.0837	≥ 0.0708	≥ 0.0484	≥ 0.0696	≥ 0.0710	≥ 0.0837	≥ 0.0837

CB-Bounds on $\mathbb{Z}_m$-based LDPC Codes

- $\mathbb{Z}_m$-based LDPC Codes: The ensemble of the parity check matrices is the same as that of binary LDPC codes. Namely, the entries in the parity check matrices can only be 0 or 1. The party check equation $\mathbf{Hx} = \mathbf{0}$ is evaluated in $\mathbb{Z}_m$.
- $GF(q)$-based LDPC codes: Similar to $\mathbb{Z}_m$-based codes, but the non-zero entries in the parity check matrices are randomly chosen from $\{1, 2, \cdots, q-1\}$. Namely, there are “weights" assigned to the edges of the graph.

Results

Consider symmetric $\mathbb{Z}_m \mapsto \mathbb{Z}_m$ channels, e.g., $\mathsf{P}(Y = y|X = x) = \mathsf{P}(Y = y - x|X = 0)$. We can then defi ne the pairwise Chernoff bound values as $CB(0 \to x) = \mathsf{E}\left\{ \sqrt{\frac{\mathsf{P}(x|Y)}{\mathsf{P}(0|Y)}} \,\middle|\, X = 0 \right\}$, and a compact vector representation $\mathsf{CB} = (CB(0 \to x))_{x \in \mathbb{Z}_m}$. We have

Theorem 2 *For variable nodes, we have* $\mathsf{CB}_{out} = \mathsf{CB}_{in,1} \bullet \mathsf{CB}_{in,2}$, *where "$\bullet$" stands for the component-wise product. For check nodes, we have* $\mathsf{CB}_{out} \leq \mathsf{CB}_{in,1} \otimes \mathsf{CB}_{in,2}$, *where "$\otimes$" is the circular convolution. The combined result is thus*

$$\mathsf{CB}^{(l+1)} \leq \mathsf{CB}^{(0)} \prod_{j=1}^{d_v-1} \left(\bigotimes_{i=1}^{d_c-1} \mathsf{CB}^{(l)} \right).$$

Stability Condition for Symmetric Channels

Corollary 1 (Sufficient Stability Condition)

$$\forall x \in \mathbb{Z}_m \backslash \{0\},\ CB^{(0)}(0 \to x) < \frac{1}{\lambda_2 \rho'(1)},$$

Furthermore, the convergence speed (w.r.t. l) is either exponential or super exponential depending on whether $\lambda_2 \neq 0$.

Theorem 3 (Necessary Stability Condition)

If $\exists x_0 \neq 0$ such that $CB^{(0)}(0 \to x_0) > \dfrac{1}{\lambda_2 \rho'(1)}$,

$\Longrightarrow$ the system is not stable.

Corollaries

Since multiplication of random non-zero weights in GF(q) is equivalent to even shuffling among non-zero elements, for GF(q)-based codes, we can rederive some results in [Bennatan & Burshtein 03].

Corollary 2 (Suff. Stab. Cond. for $GF(q)$ Codes)

$$\frac{\sum_{x \in \mathsf{GF}(q)\backslash\{0\}} CB^{(0)}(0 \to x)}{q-1} < \frac{1}{\lambda_2 \rho'(1)},$$

Corollary 3 (Necc. Stab. Cond. for $GF(q)$ Codes)

If $$\frac{\sum_{x \in \mathsf{GF}(q)\backslash\{0\}} CB^{(0)}(0 \to x)}{q-1} > \frac{1}{\lambda_2 \rho'(1)},$$

$\Longrightarrow$ *the system is not stable.*

Conclusions

Summary

- A general framework is provided for fi nite-dimensional bounds on LDPC codes.
- An iterative CB-based bound for non-symmetric channels.
- An iterative (CB, DCB)-based two-dimensional bound.
- A non-iterative tight DCB-based bound.
- A CB-based bound for $\mathbb{Z}_m$ LDPC codes.
- A pair of necessary and suffi cient stability conditions for $\mathbb{Z}_m$ LDPC codes.

Future Direction