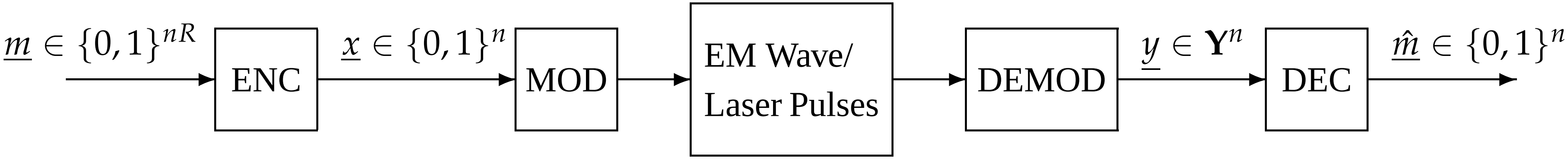
On the Typicality of the Linear Code Among the LDPC Coset Code Ensemble

Chih-Chun Wang, Prof. S.R. Kulkarni, and Prof. H.V. Poor

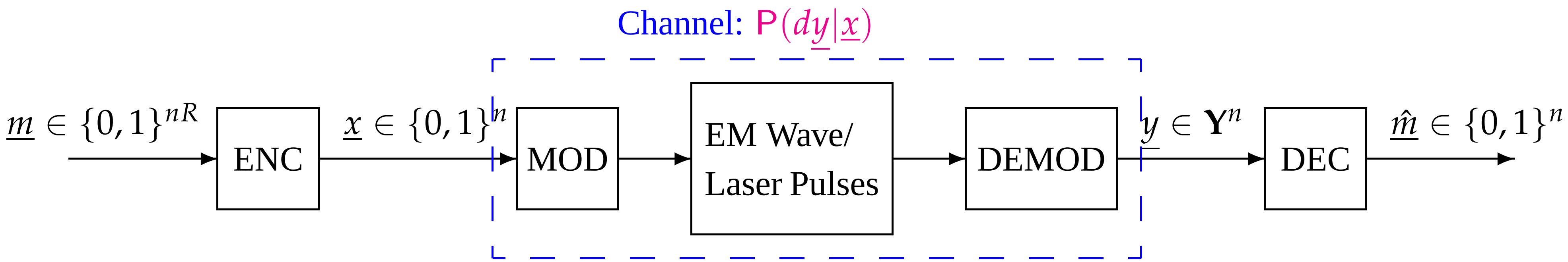
{chihw, kulkarni, poor}@princeton.edu.

Princeton University

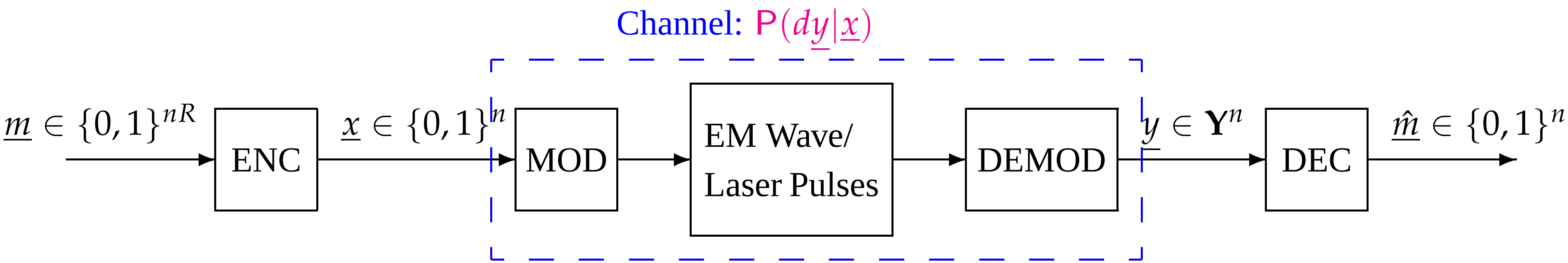
Error Correcting Codes



Error Correcting Codes

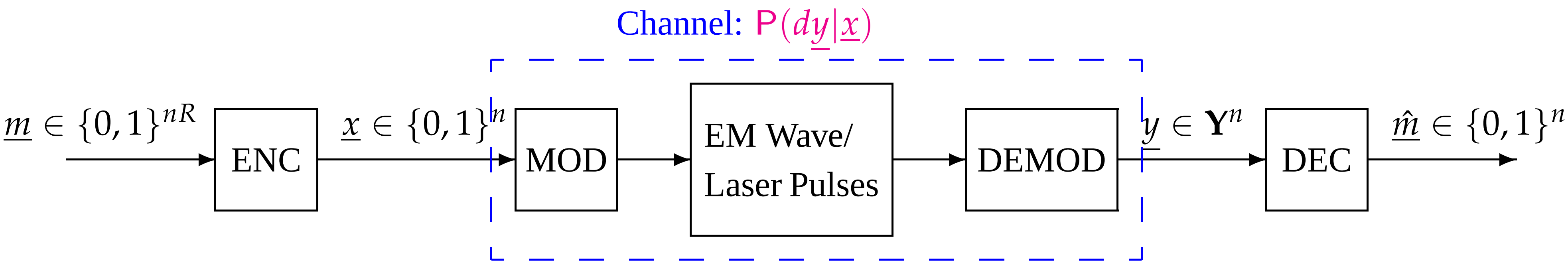


Error Correcting Codes



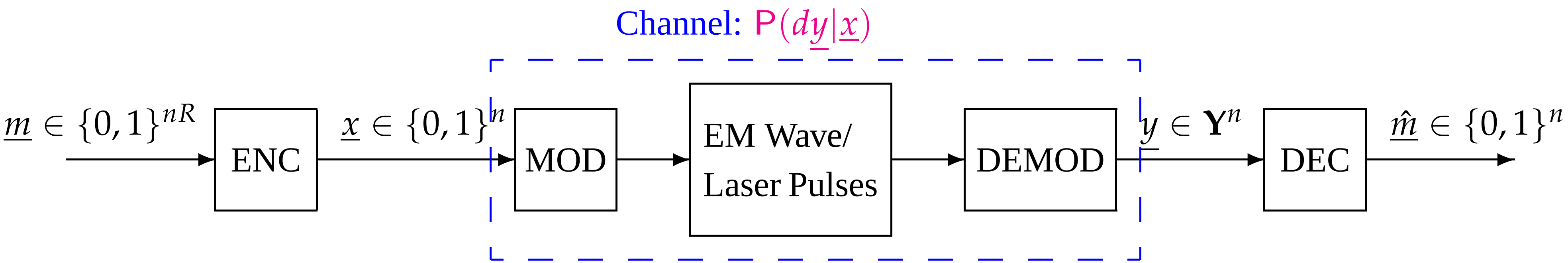
- Memoryless channels: $\mathsf{P}(d\underline{y}|\underline{x}) = \prod_{i=1}^{n} \mathsf{P}(dy_i|x_i)$

Error Correcting Codes



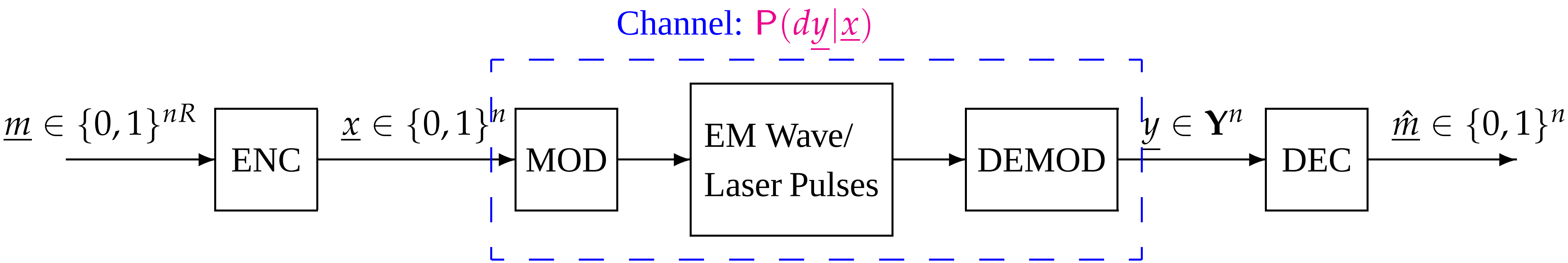
- Memoryless channels: $\mathsf{P}(d\underline{y}|\underline{x}) = \prod_{i=1}^{n} \mathsf{P}(dy_i|x_i)$
- Symmetric channels: $\exists \mathcal{T} : \mathbf{Y} \mapsto \mathbf{Y}$ s.t. $\mathcal{T}^2(y) = y, \forall y \in \mathbf{Y}$, and $\mathsf{P}(dy|x=0) = \mathsf{P}(\mathcal{T}(dy)|x=1)$.

Error Correcting Codes



- Memoryless channels: $\mathsf{P}(d\underline{y}|\underline{x}) = \prod_{i=1}^{n} \mathsf{P}(dy_i|x_i)$
- Symmetric channels: $\exists \mathcal{T} : \mathbf{Y} \mapsto \mathbf{Y}$ s.t. $\mathcal{T}^2(y) = y, \forall y \in \mathbf{Y}$, and $\mathsf{P}(dy|x=0) = \mathsf{P}(\mathcal{T}(dy)|x=1)$.
- Shannon's channel coding theorem: Let $C = \max_{P_X} \mathsf{E}_{XY} \log\left(\frac{P_{XY}(X,Y)}{P_X(X)P_Y(Y)}\right)$. Reliable communication requires $R < C$.

Error Correcting Codes



- Memoryless channels: $\mathsf{P}(d\underline{y}|\underline{x}) = \prod_{i=1}^{n} \mathsf{P}(dy_i|x_i)$
- Symmetric channels: $\exists \mathcal{T} : \mathbf{Y} \mapsto \mathbf{Y}$ s.t. $\mathcal{T}^2(y) = y, \forall y \in \mathbf{Y}$, and $\mathsf{P}(dy|x=0) = \mathsf{P}(\mathcal{T}(dy)|x=1)$.
- Shannon's channel coding theorem: Let $C = \max_{P_X} \mathsf{E}_{XY} \log\left(\frac{P_{XY}(X,Y)}{P_X(X)P_Y(Y)}\right)$. Reliable communication requires $R < C$.
- Memoryless symmetric channels: Capacity-approaching error correcting codes have been constructed, including turbo codes, low-density parity-check (LDPC) codes, irregular RA codes, LT codes, concatenated tree codes, etc.
- Performance: 0.1∼1.5dB away from capacity.

Ultra high performance on almost all symmetric channels.

Ultra high performance on almost all symmetric channels.

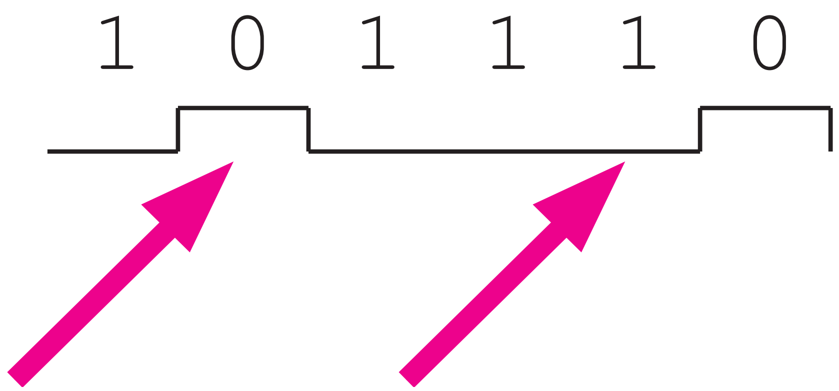
Question: how about non-symmetric memoryless channels?

Ultra high performance on almost all symmetric channels.

Question: how about non-symmetric memoryless channels?

Examples:

Z-Channels

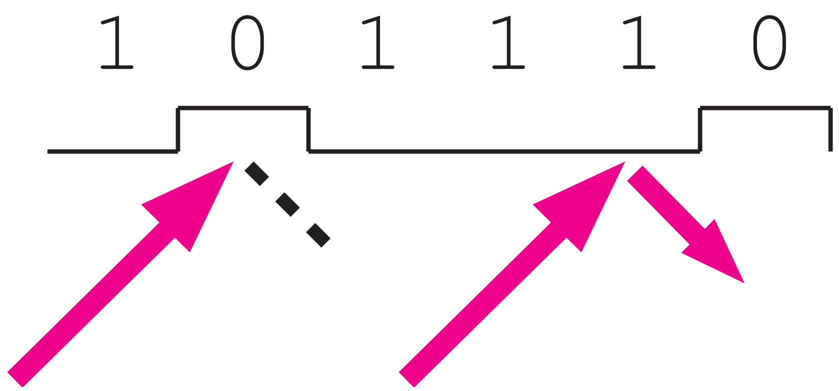


Ultra high performance on almost all symmetric channels.

Question: how about non-symmetric memoryless channels?

Examples:

Z-Channels

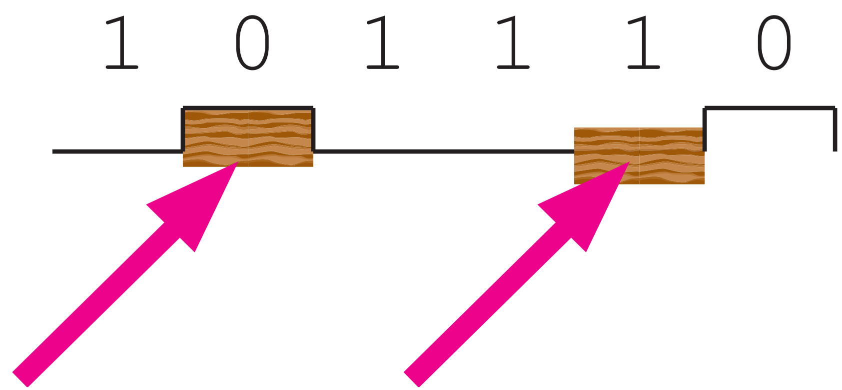


Ultra high performance on almost all symmetric channels.

Question: how about non-symmetric memoryless channels?

Examples:

Z-Channels

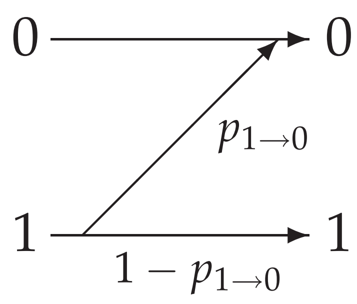


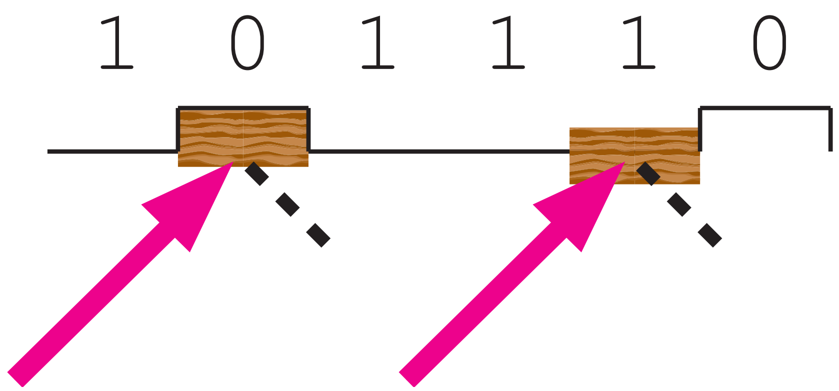
Ultra high performance on almost all symmetric channels.

Question: how about non-symmetric memoryless channels?

Examples:

Z-Channels



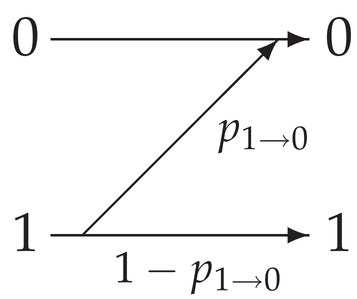


Ultra high performance on almost all symmetric channels.

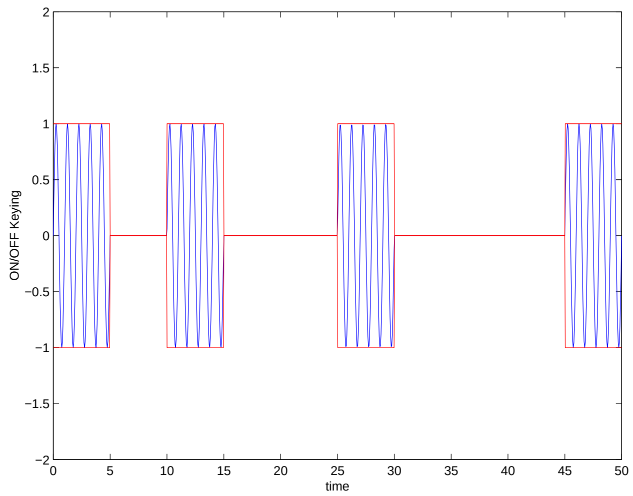
Question: how about non-symmetric memoryless channels?

Examples:

Z-Channels



On/Off Keying w. Rayleigh Fading

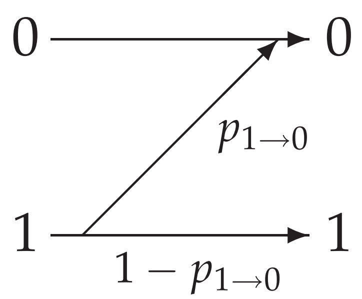


Ultra high performance on almost all symmetric channels.

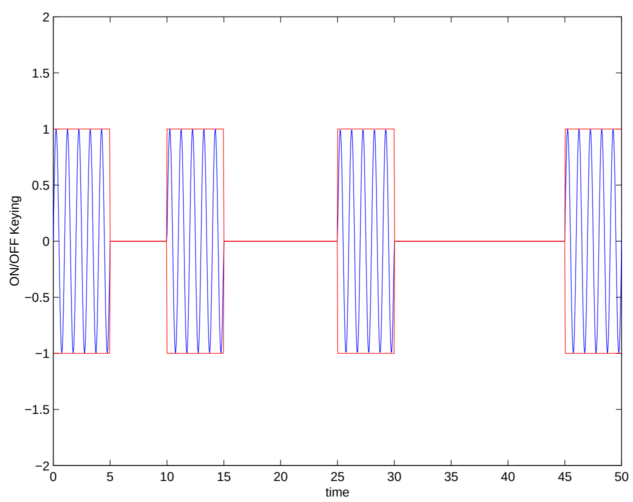
Question: how about non-symmetric memoryless channels?

Examples:

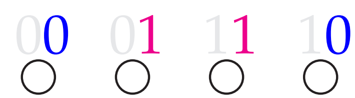
Z-Channels



On/Off Keying w. Rayleigh Fading



BICM



Ultra high performance on almost all symmetric channels.

Question: how about non-symmetric memoryless channels?

Exan

Z-Ch

[Majani & Rumsey 91] showed that the ratio between the symmetric mutual information rate and the capacity is lower bounded by $\frac{e \ln 2}{2} \approx 0.942$.

[Shulman & Feder 04] further proved that the absolute difference is upper bounded by 0.011 bit/sym.

Content

- Low-density parity-check codes and the challenges of non-symmetric memoryless channels
  - Codeword-dependent error resiliency

Content

- Low-density parity-check codes and the challenges of non-symmetric memoryless channels
  - Codeword-dependent error resiliency
- Two approaches:
  - The LDPC coset code ensemble
  - Linear LDPC codes with generalized density evolution

Content

- Low-density parity-check codes and the challenges of non-symmetric memoryless channels
  - Codeword-dependent error resiliency
- Two approaches:
  - The LDPC coset code ensemble
  - Linear LDPC codes with generalized density evolution
- The typicality of linear LDPC codes among the coset code ensemble

Content

- Low-density parity-check codes and the challenges of non-symmetric memoryless channels
  - Codeword-dependent error resiliency
- Two approaches:
  - The LDPC coset code ensemble
  - Linear LDPC codes with generalized density evolution
- The typicality of linear LDPC codes among the coset code ensemble
- The convergence rate consideration

Content

- Low-density parity-check codes and the challenges of non-symmetric memoryless channels
  - Codeword-dependent error resiliency
- Two approaches:
  - The LDPC coset code ensemble
  - Linear LDPC codes with generalized density evolution
- The typicality of linear LDPC codes among the coset code ensemble
- The convergence rate consideration
- Applications

LDPC Codes & the Graph-based Code Ensemble

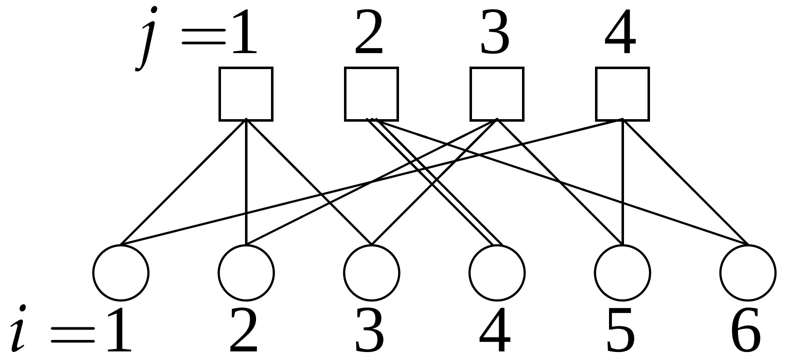
$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\boxed{\mathbf{Ax} = \mathbf{0}}$$

LDPC Codes & the Graph-based Code Ensemble

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\boxed{\mathbf{Ax} = \mathbf{0}}$$

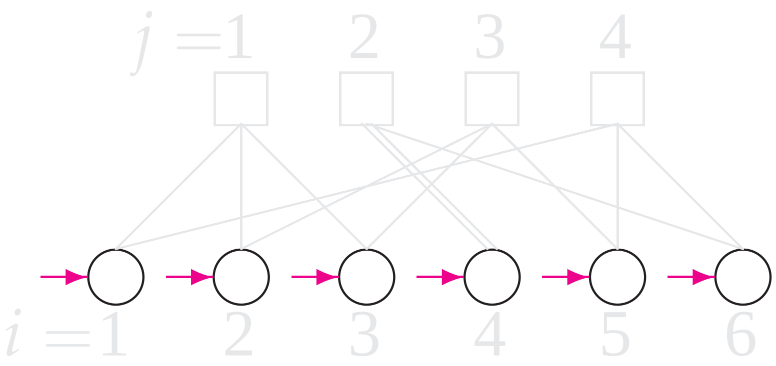


$\mathcal{C}^6(d_v, d_c)$, $d_v = 2$, $d_c = 3$

LDPC Codes & the Graph-based Code Ensemble

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\boxed{\mathbf{Ax} = \mathbf{0}}$$



$\mathcal{C}^6(d_v, d_c)$, $d_v = 2$, $d_c = 3$

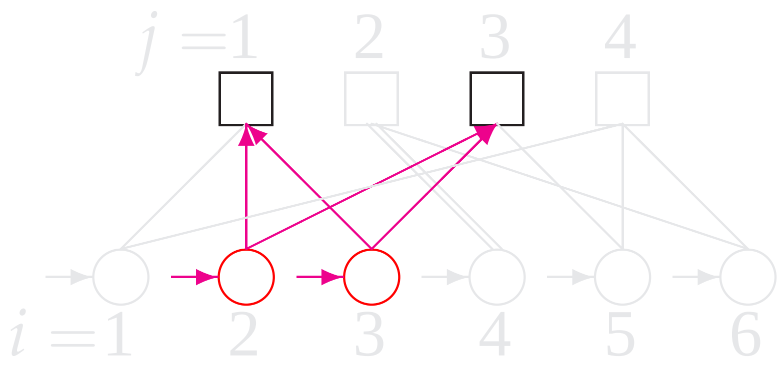
Belief Propagation:

(i) m_0, (ii) $\Psi_v(m_0, m_1, \cdots, m_{d_v-1})$, (iii) $\Psi_c(m_1, \cdots, m_{d_c-1})$

LDPC Codes & the Graph-based Code Ensemble

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\mathbf{Ax} = \mathbf{0}$$



$\mathcal{C}^6(d_v, d_c)$, $d_v = 2$, $d_c = 3$

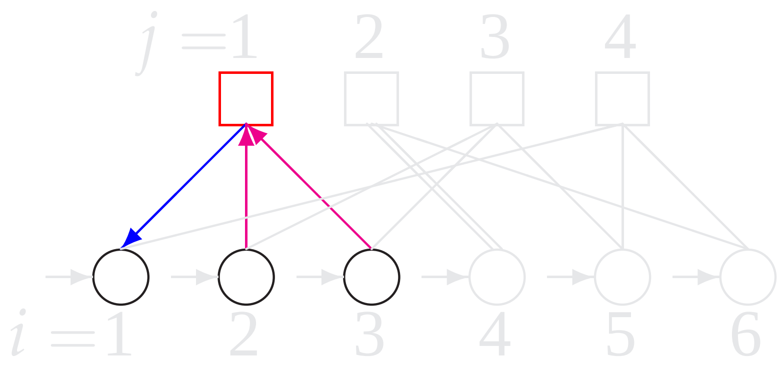
Belief Propagation:

(i) m_0, (ii) $\Psi_v(m_0, m_1, \cdots, m_{d_v-1})$, (iii) $\Psi_c(m_1, \cdots, m_{d_c-1})$

LDPC Codes & the Graph-based Code Ensemble

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\mathbf{Ax} = \mathbf{0}$$



$\mathcal{C}^6(d_v, d_c)$, $d_v = 2$, $d_c = 3$

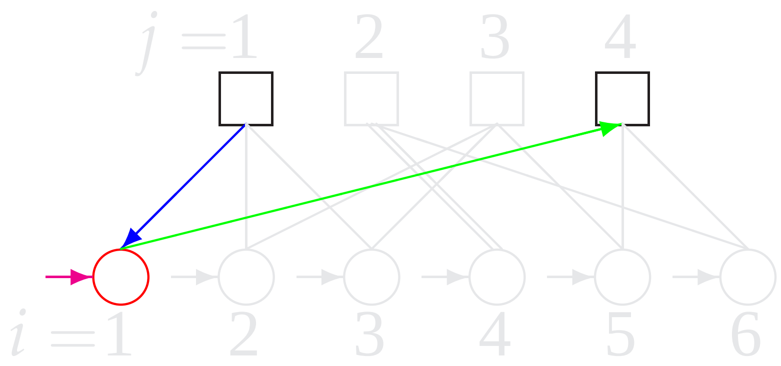
Belief Propagation:

(i) m_0, (ii) $\Psi_v(m_0, m_1, \cdots, m_{d_v-1})$, (iii) $\Psi_c(m_1, \cdots, m_{d_c-1})$

LDPC Codes & the Graph-based Code Ensemble

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\mathbf{Ax} = \mathbf{0}$$



$\mathcal{C}^6(d_v, d_c)$, $d_v = 2$, $d_c = 3$

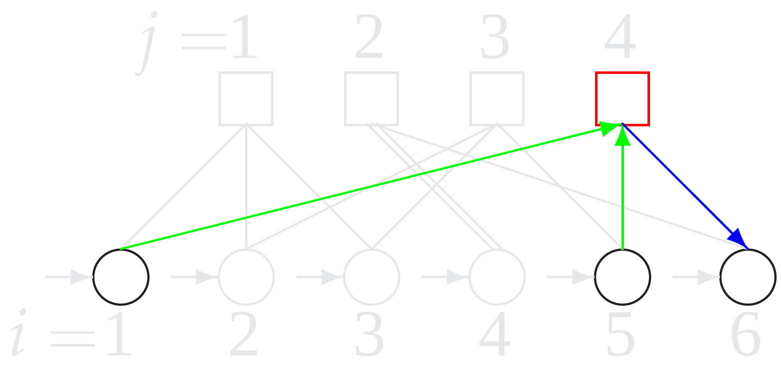
Belief Propagation:

(i) m_0, (ii) $\Psi_v(m_0, m_1, \cdots, m_{d_v-1})$, (iii) $\Psi_c(m_1, \cdots, m_{d_c-1})$

LDPC Codes & the Graph-based Code Ensemble

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\boxed{\mathbf{A}\mathbf{x} = \mathbf{0}}$$



$\mathcal{C}^6(d_v, d_c)$, $d_v = 2$, $d_c = 3$

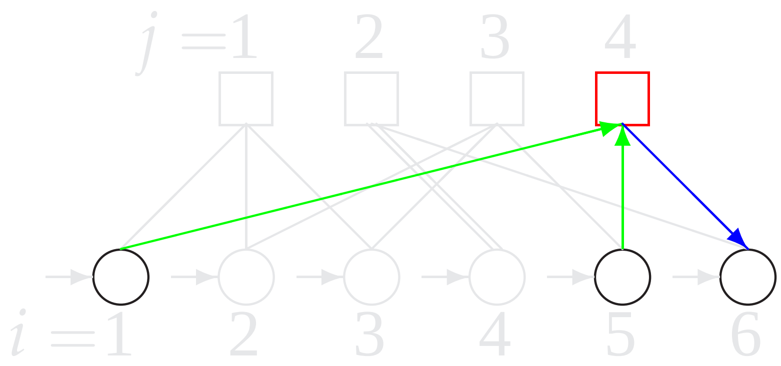
Belief Propagation:

(i) m_0, (ii) $\Psi_v(m_0, m_1, \cdots, m_{d_v-1})$, (iii) $\Psi_c(m_1, \cdots, m_{d_c-1})$

LDPC Codes & the Graph-based Code Ensemble

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\boxed{\mathbf{Ax} = \mathbf{0}}$$



$\mathcal{C}^6(d_v, d_c)$, $d_v = 2$, $d_c = 3$

Belief Propagation:

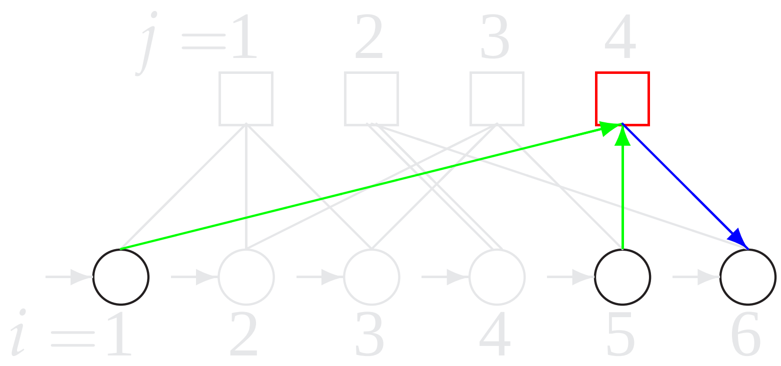
(i) m_0, (ii) $\Psi_v(m_0, m_1, \cdots, m_{d_v-1})$, (iii) $\Psi_c(m_1, \cdots, m_{d_c-1})$

- For cycle-free networks, the belief propagation works for non-symmetric channels as well.

LDPC Codes & the Graph-based Code Ensemble

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\boxed{\mathbf{Ax} = \mathbf{0}}$$



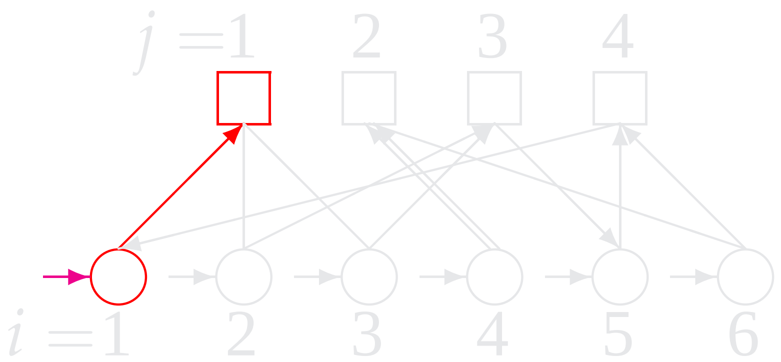
$\mathcal{C}^6(d_v, d_c)$, $d_v = 2$, $d_c = 3$

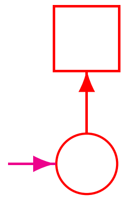
Belief Propagation:

(i) m_0, (ii) $\Psi_v(m_0, m_1, \cdots, m_{d_v-1})$, (iii) $\Psi_c(m_1, \cdots, m_{d_c-1})$

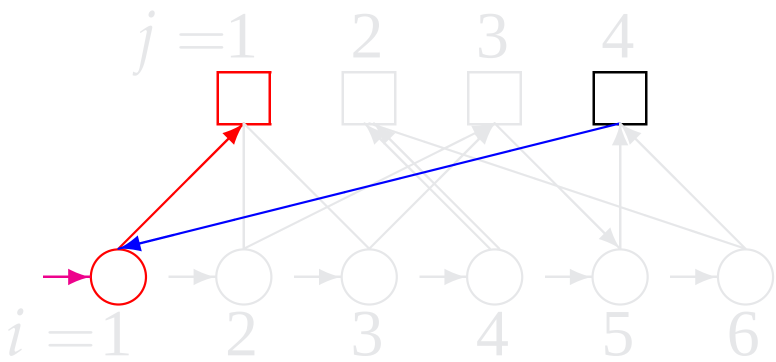
- For cycle-free networks, the belief propagation works for non-symmetric channels as well.
- By simulation, belief propagation + LDPC codes also have outstanding performance for non-symmetric channels.

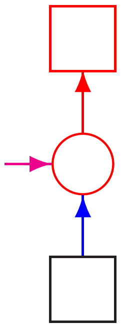
The Density Evolution



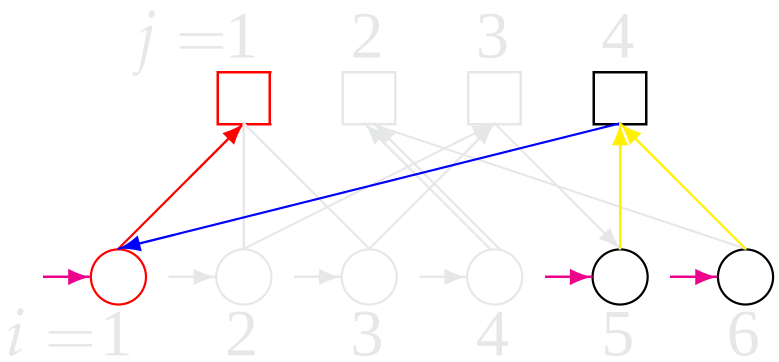


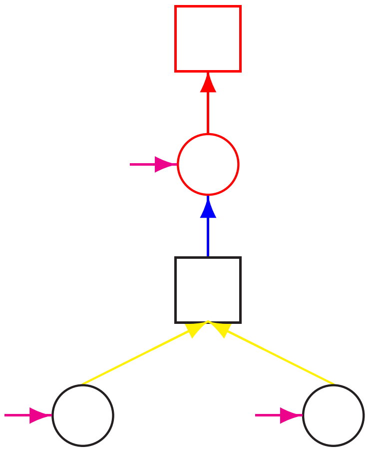
The Density Evolution



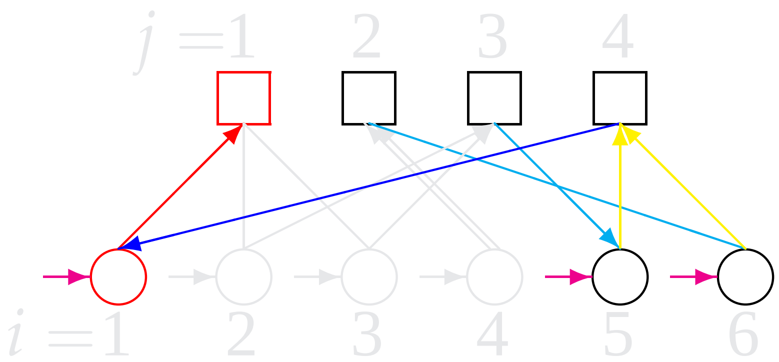


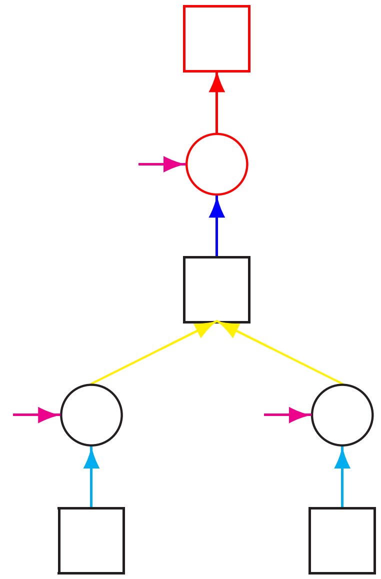
The Density Evolution



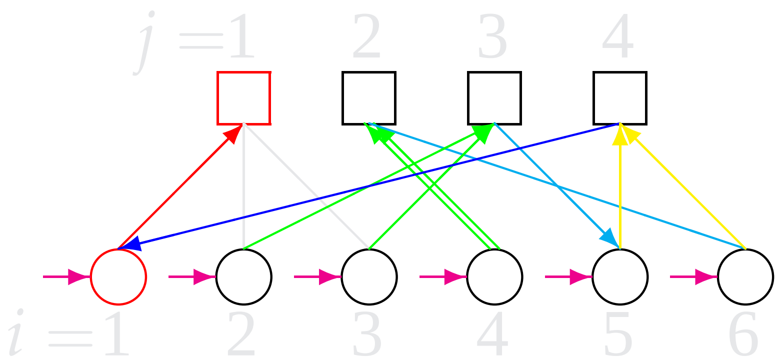


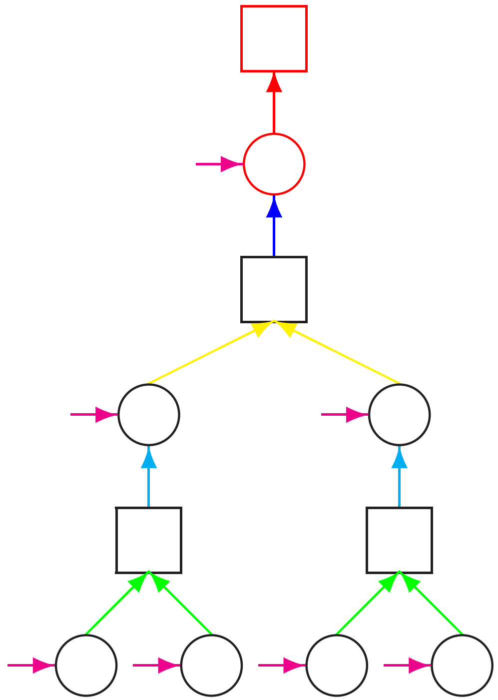
The Density Evolution





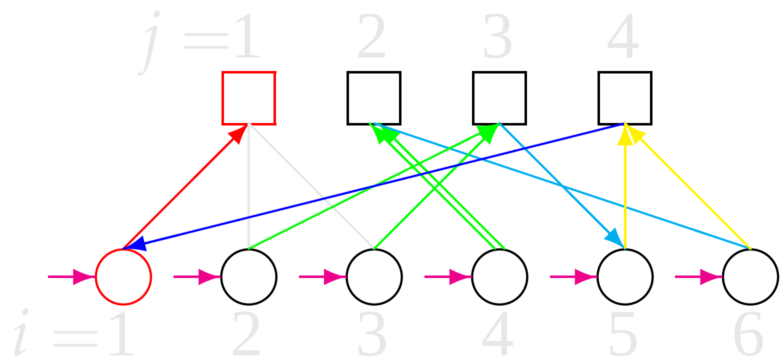
The Density Evolution

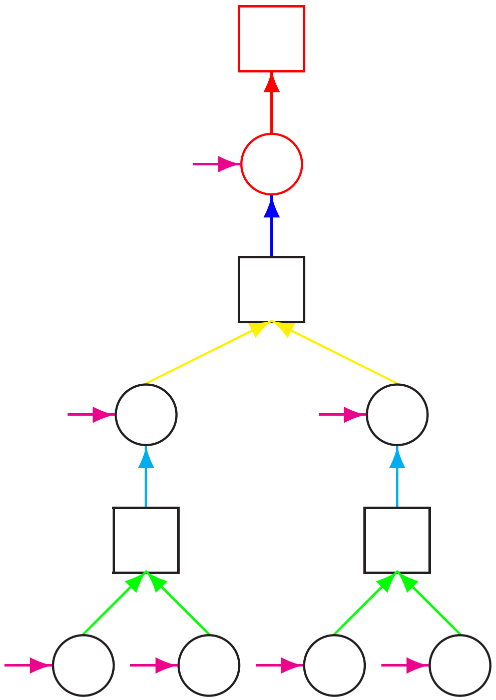




The Density Evolution

Sym. Chs: Assuming $\mathbf{x} = \mathbf{0}$.



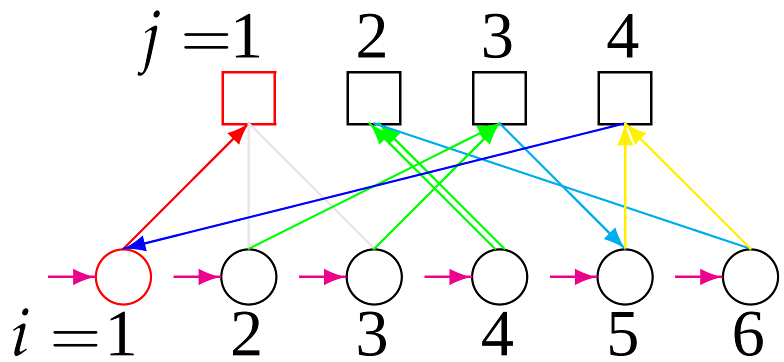


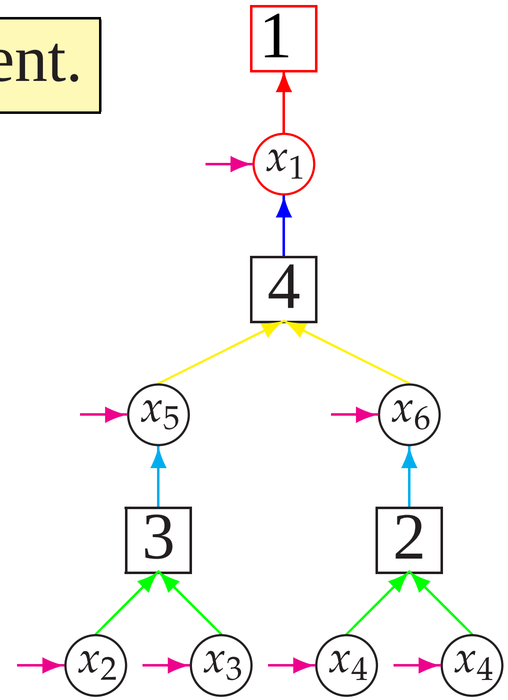
$$P^{(l)} = P^{(0)} \otimes \left(Q^{(l-1)}\right)^{\otimes(d_v-1)}$$

$$Q^{(l-1)} = \Gamma^{-1}\left(\left(\Gamma\left(P^{(l-1)}\right)\right)^{\otimes(d_c-1)}\right),$$

The Density Evolution

Non-sym. Chs: Codeword-dependent.





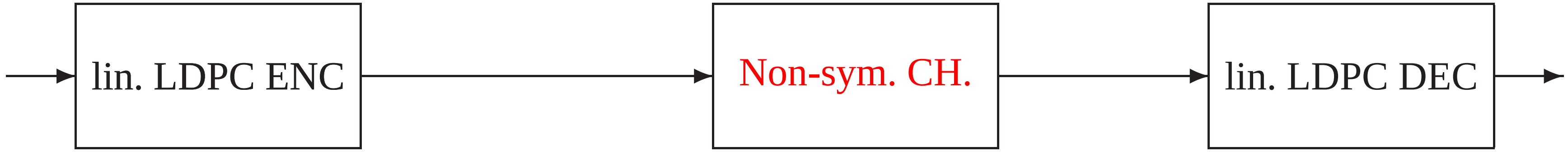
$$
\begin{aligned}
P^{(l)}(\mathbf{x}) &= P^{(0)}(\mathbf{x}) \otimes \left(Q^{(l-1)}(\mathbf{x})\right)^{\otimes(d_v-1)} \\
Q^{(l-1)}(\mathbf{x}) &= \Gamma^{-1}\left(\left(\Gamma\left(P^{(l-1)}(\mathbf{x})\right)\right)^{\otimes(d_c-1)}\right),
\end{aligned}
$$

First Approach: the LDPC Coset Ensemble

- **Coset codes**: Any valid codeword $\mathbf{x}$ satisfies $\mathbf{A}\mathbf{x} = \mathbf{s}$, where $\mathbf{A}$ is from the same equiprobable bipartite graph ensemble, and the coset-defining syndrome $\mathbf{s}$ is uniformly drawn from $\{0, 1\}^{n(1-R)}$.

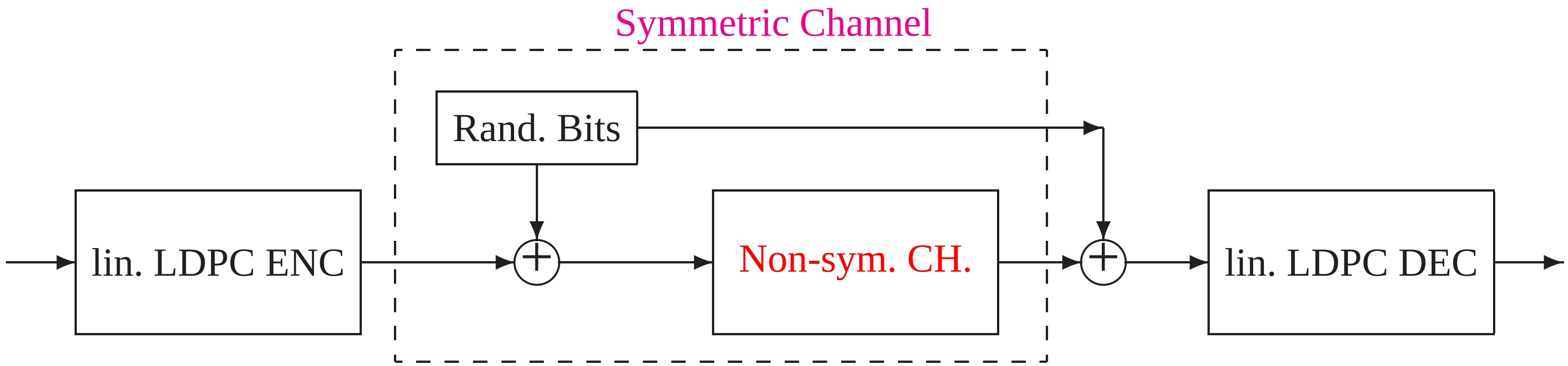
First Approach: the LDPC Coset Ensemble

- **Coset codes**: Any valid codeword $\mathbf{x}$ satisfies $\mathbf{Ax} = \mathbf{s}$, where $\mathbf{A}$ is from the same equiprobable bipartite graph ensemble, and the coset-defining syndrome $\mathbf{s}$ is uniformly drawn from $\{0, 1\}^{n(1-R)}$.



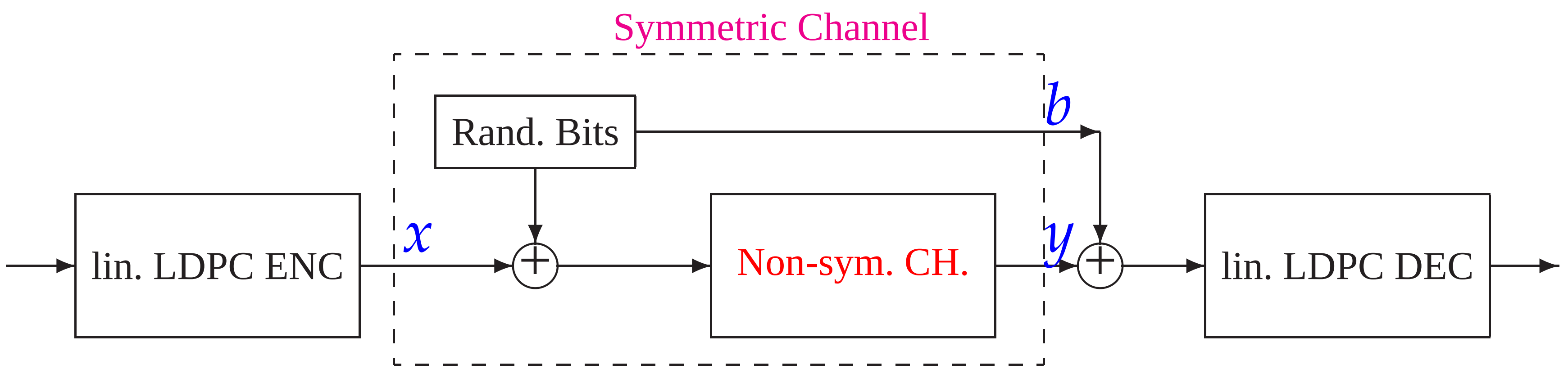
First Approach: the LDPC Coset Ensemble

- **Coset codes**: Any valid codeword $\mathbf{x}$ satisfies $\mathbf{Ax} = \mathbf{s}$, where $\mathbf{A}$ is from the same equiprobable bipartite graph ensemble, and the coset-defining syndrome $\mathbf{s}$ is uniformly drawn from $\{0,1\}^{n(1-R)}$.
- Symmetrizing the non-symmetric channel:



First Approach: the LDPC Coset Ensemble

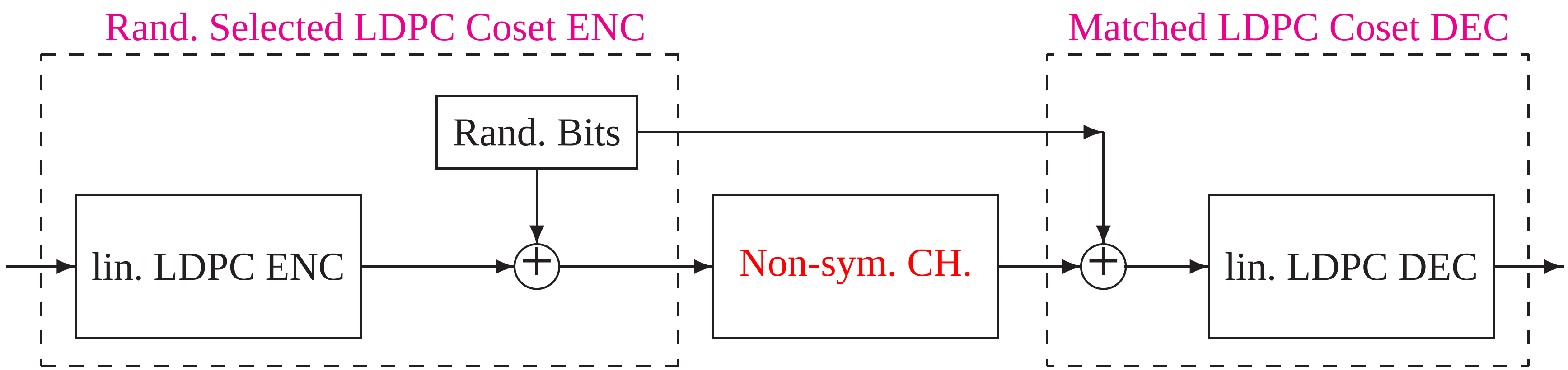
- **Coset codes**: Any valid codeword $\mathbf{x}$ satisfies $\mathbf{Ax} = \mathbf{s}$, where $\mathbf{A}$ is from the same equiprobable bipartite graph ensemble, and the coset-defining syndrome $\mathbf{s}$ is uniformly drawn from $\{0,1\}^{n(1-R)}$.
- Symmetrizing the non-symmetric channel:



$$\mathsf{P}(b, y|x = 0) = \mathsf{P}(b - 1, y|x = 1) = \mathsf{P}(\mathcal{T}(b, y)|x = 1).$$

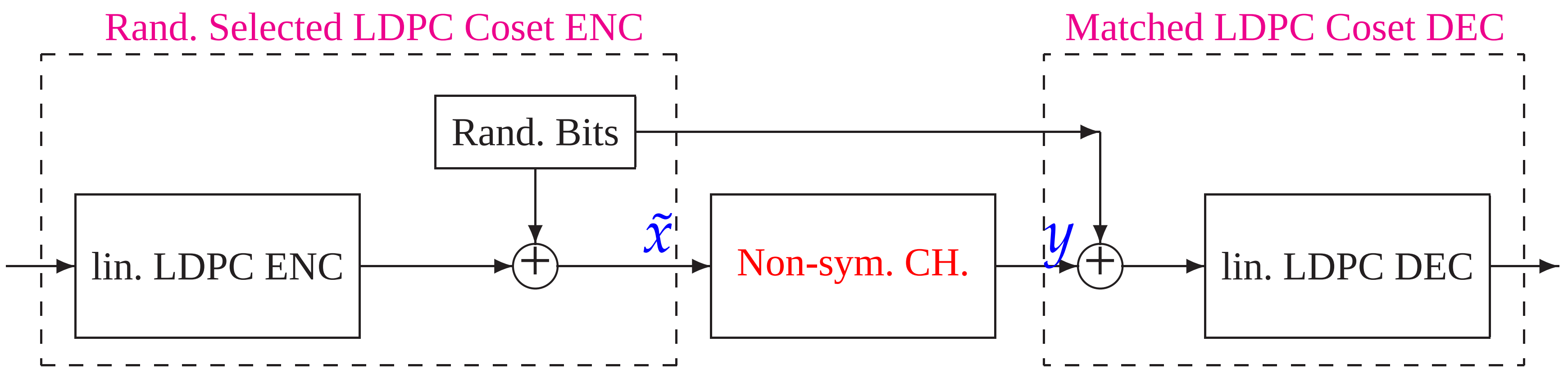
First Approach: the LDPC Coset Ensemble

- **Coset codes**: Any valid codeword $\mathbf{x}$ satisfies $\mathbf{A}\mathbf{x} = \mathbf{s}$, where $\mathbf{A}$ is from the same equiprobable bipartite graph ensemble, and the coset-defining syndrome $\mathbf{s}$ is uniformly drawn from $\{0,1\}^{n(1-R)}$.
- Symmetrizing the non-symmetric channel:



First Approach: the LDPC Coset Ensemble

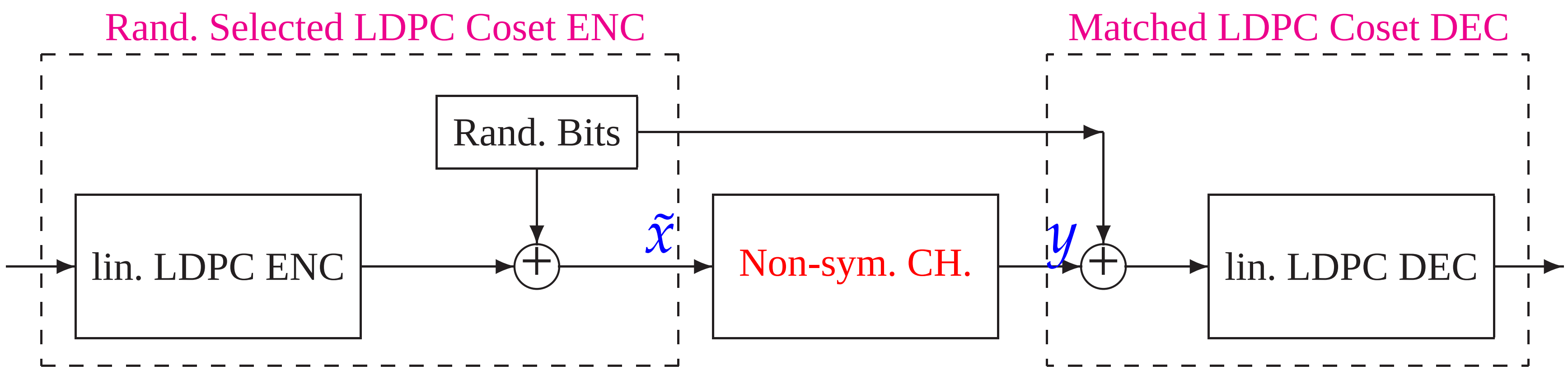
- **Coset codes**: Any valid codeword $\mathbf{x}$ satisfies $\mathbf{Ax} = \mathbf{s}$, where $\mathbf{A}$ is from the same equiprobable bipartite graph ensemble, and the coset-defining syndrome $\mathbf{s}$ is uniformly drawn from $\{0,1\}^{n(1-R)}$.
- Symmetrizing the non-symmetric channel:



$$\mathbf{A}\tilde{\mathbf{x}} = \mathbf{A}(\mathbf{x} + \mathbf{b}) = \mathbf{Ax} + \mathbf{Ab} = \mathbf{Ab} = \mathbf{s}$$

First Approach: the LDPC Coset Ensemble

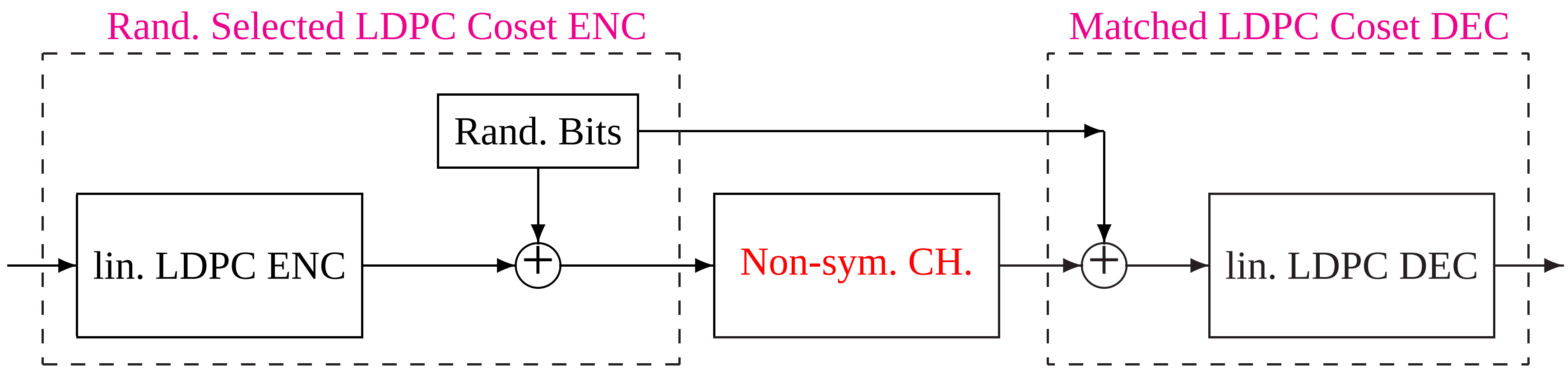
- **Coset codes**: Any valid codeword $\mathbf{x}$ satisfies $\mathbf{A}\mathbf{x} = \mathbf{s}$, where $\mathbf{A}$ is from the same equiprobable bipartite graph ensemble, and the coset-defining syndrome $\mathbf{s}$ is uniformly drawn from $\{0,1\}^{n(1-R)}$.
- Symmetrizing the non-symmetric channel:



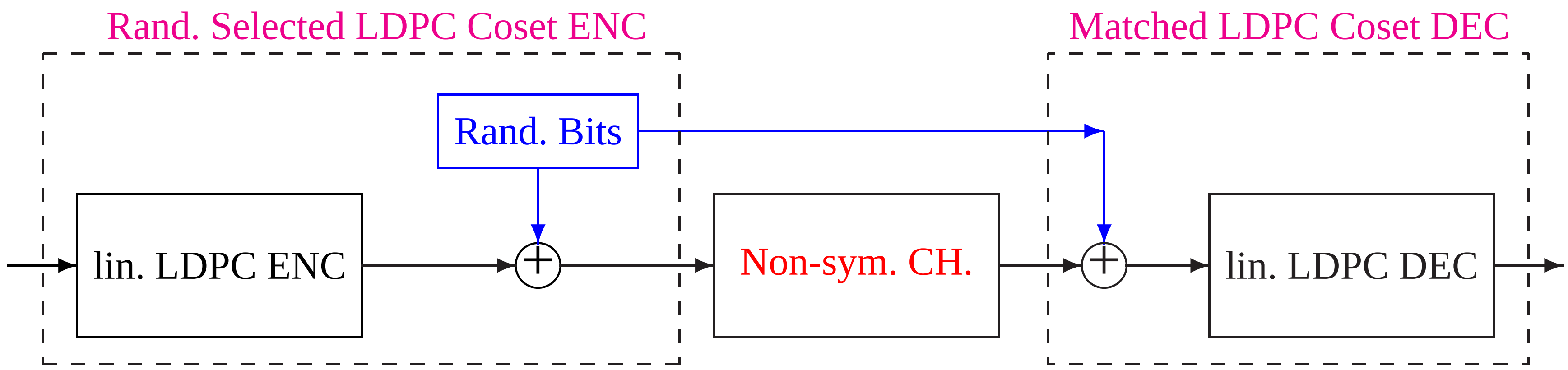
$$\mathbf{A}\tilde{\mathbf{x}} = \mathbf{A}(\mathbf{x} + \mathbf{b}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{b} = \mathbf{A}\mathbf{b} = \mathbf{s}$$

- Averaged performance of the LDPC coset code ensemble

The Existing Typicality Result

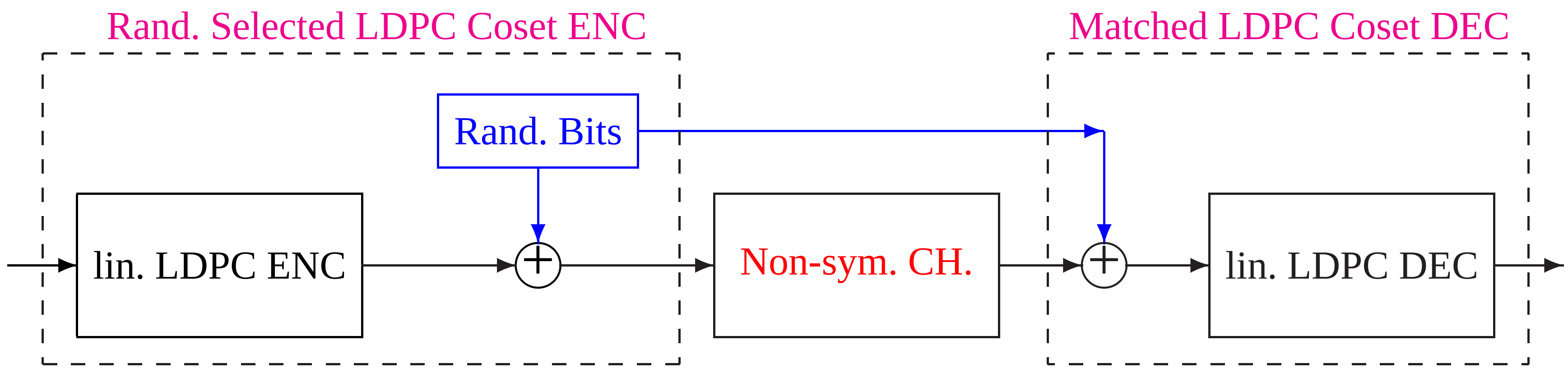


The Existing Typicality Result



- The difficulty: Maintaining the synchronization of two random strings.

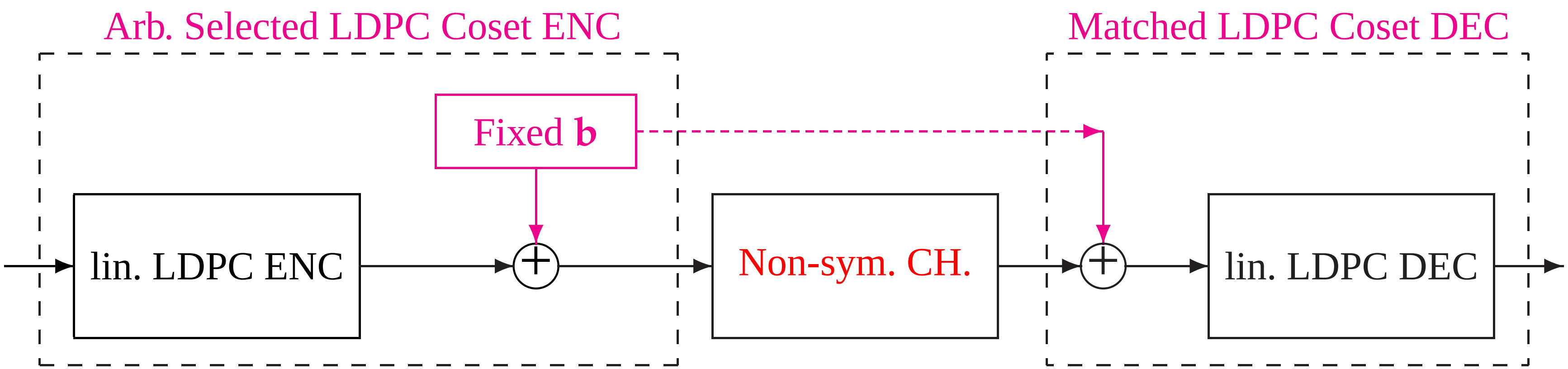
The Existing Typicality Result



- The difficulty: Maintaining the synchronization of two random strings.
- [Kavčić 03]: Asymptotically, almost all $\mathbf{s} \in \{0,1\}^{n(1-R)}$ are typical. Namely,

$$\lim_{n\to\infty} \mathsf{P}\left(\mathbf{s} : |p_e(\mathbf{s}) - \mathsf{E}_{\mathbf{s}}\{p_e(\mathbf{s})\}| < \epsilon\right) = 1, \ \ \forall \epsilon > 0.$$

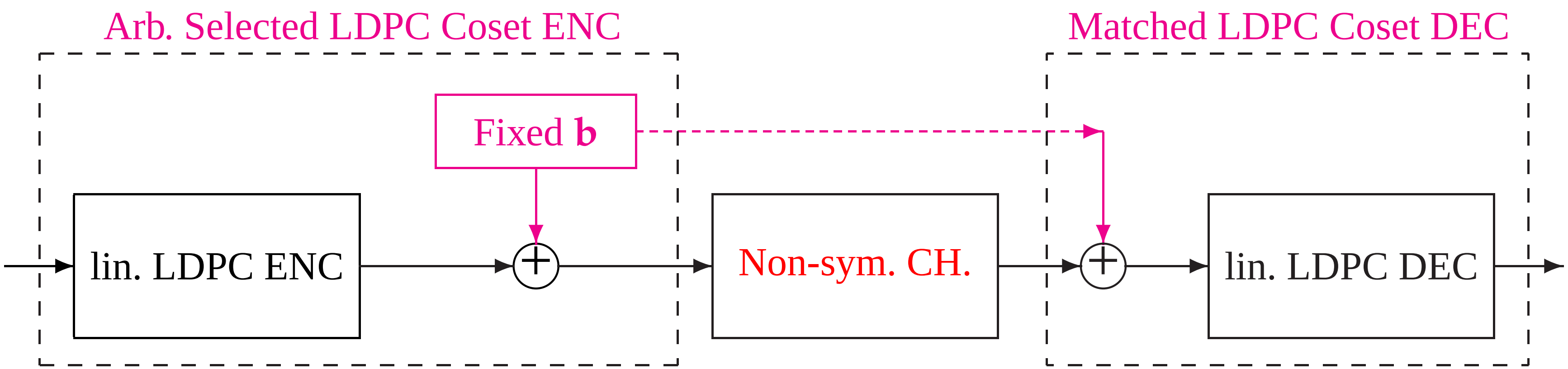
The Existing Typicality Result



- The difficulty: Maintaining the synchronization of two random strings.
- [Kavčić 03]: Asymptotically, almost all $\mathbf{s} \in \{0,1\}^{n(1-R)}$ are typical. Namely,

$$\lim_{n\to\infty} \mathsf{P}\left(\mathbf{s} : |p_e(\mathbf{s}) - \mathsf{E}_{\mathbf{s}}\{p_e(\mathbf{s})\}| < \epsilon\right) = 1, \quad \forall \epsilon > 0.$$

The Existing Typicality Result

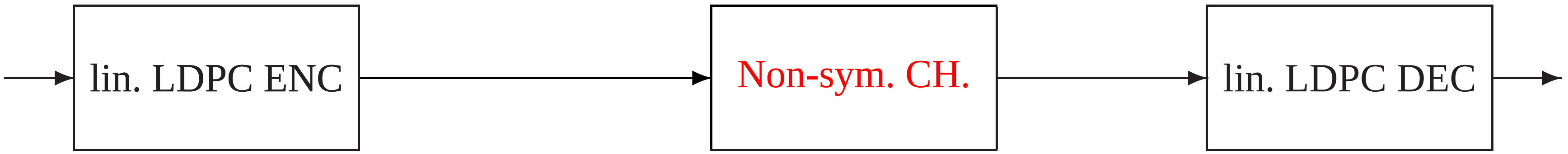


- The difficulty: Maintaining the synchronization of two random strings.
- [Kavčić 03]: Asymptotically, almost all $\mathbf{s} \in \{0,1\}^{n(1-R)}$ are typical. Namely,

$$\lim_{n\to\infty} \mathsf{P}\left(\mathbf{s} : |p_e(\mathbf{s}) - \mathsf{E}_{\mathbf{s}}\{p_e(\mathbf{s})\}| < \epsilon\right) = 1, \ \ \forall \epsilon > 0.$$

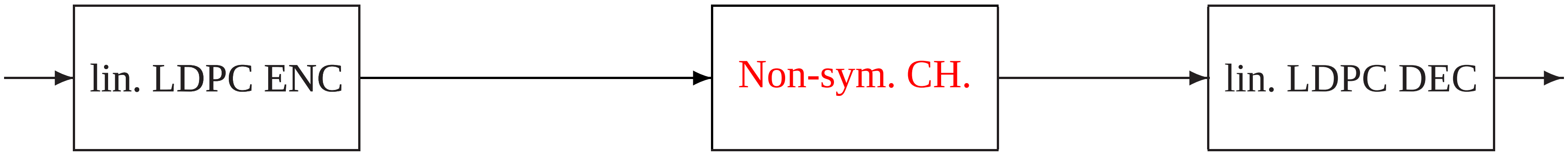
- Due to their hardware uniformity, linear codes are always the superior choice.

Second Approach: Linear Codes w. Codeword Averaging



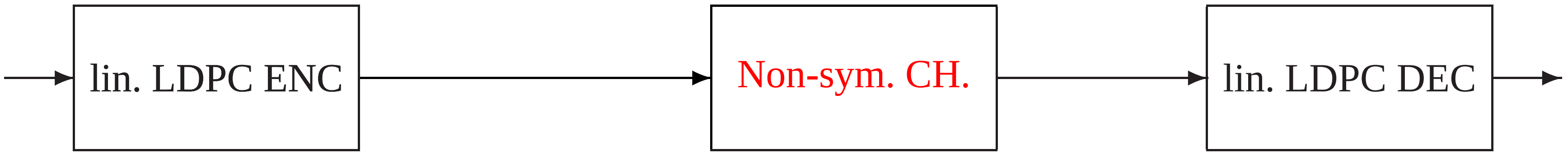
- Codeword-dependent performance

Second Approach: Linear Codes w. Codeword Averaging



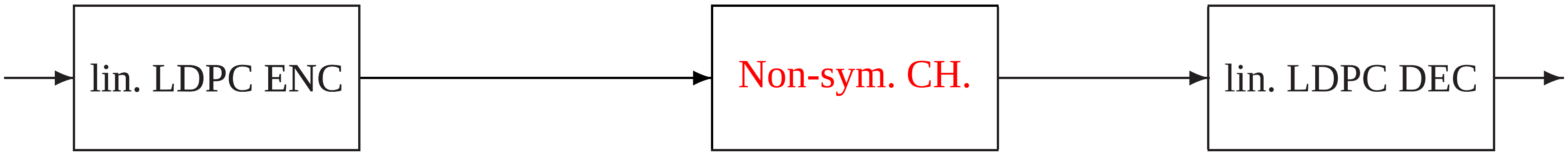
- Codeword-dependent performance **Codeword Averaging**

Second Approach: Linear Codes w. Codeword Averaging

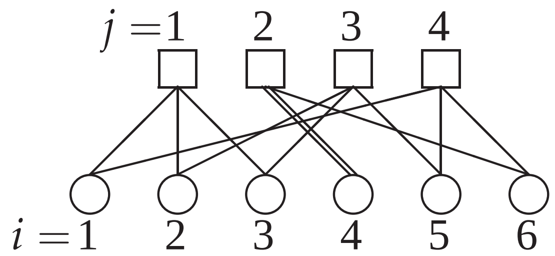


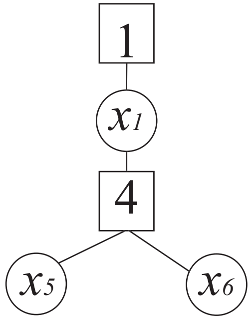
- Codeword-dependent performance **Codeword Averaging**
- A generalized density evolution.

Second Approach: Linear Codes w. Codeword Averaging

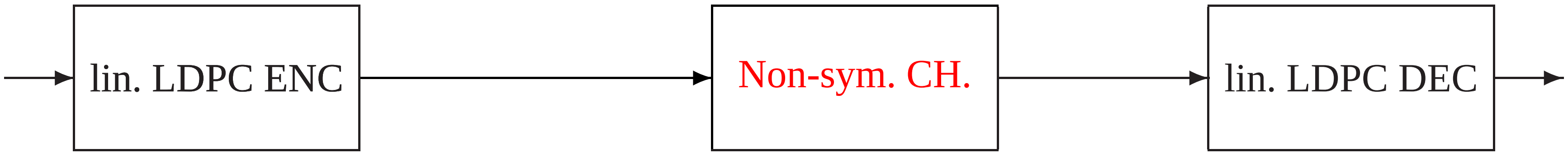


- Codeword-dependent performance **Codeword Averaging**
- A generalized density evolution.

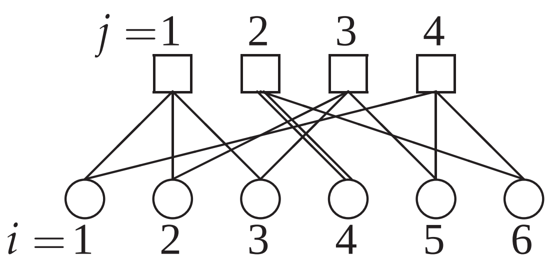




Second Approach: Linear Codes w. Codeword Averaging

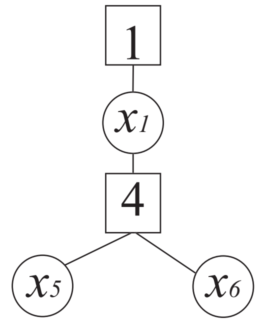


- Codeword-dependent performance **Codeword Averaging**
- A generalized density evolution.



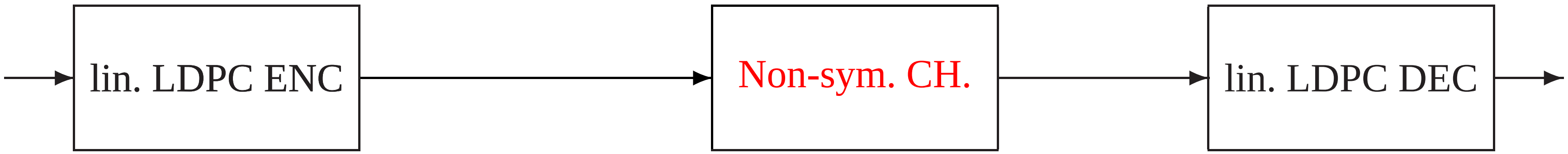
$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$\mathbf{Ax} = \mathbf{0} \iff x_6 = 0$

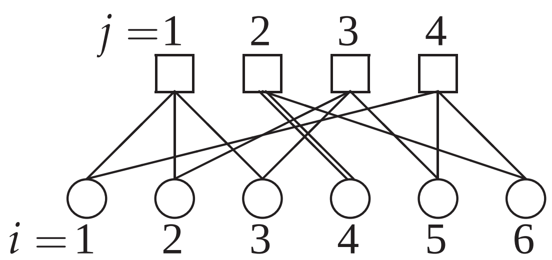


$\mathbf{X}^1_{(1,1)} := \{x_1x_5x_6 :$
$x_1x_5x_6 = 000, 011, 101, 110\}$

Second Approach: Linear Codes w. Codeword Averaging

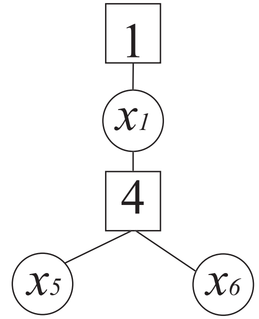


- Codeword-dependent performance **Codeword Averaging**
- A generalized density evolution.



$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\mathbf{A}\mathbf{x} = \mathbf{0} \Longleftrightarrow x_6 = 0$$



$$\mathbf{X}^1_{(1,1)} := \{x_1x_5x_6 : x_1x_5x_6 = 000, 011, 101, 110\}$$

> Averaging the trimmed tree code is not equivalent to averaging the original code.

Perfect Projection Condition

Definition 1 (Perfect Projection) *The supporting tree $\mathcal{N}^{2l}$ is perfectly projected, if for any codeword $\mathbf{x}_t$ of the tree code $\mathbf{X}_t$,*

$$\frac{\left|\{\mathbf{x}\in\mathbf{X} : \mathbf{x}|_{\text{tree}} = \mathbf{x}_t\}\right|}{|\mathbf{X}|} = \frac{1}{|\mathbf{X}_t|}.$$

Perfect Projection Condition

Definition 1 (Perfect Projection) *The supporting tree $\mathcal{N}^{2l}$ is perfectly projected, if for any codeword $\mathbf{x}_t$ of the tree code $\mathbf{X}_t$,*

$$\frac{\left|\{\mathbf{x}\in\mathbf{X}:\mathbf{x}|_{\text{tree}}=\mathbf{x}_t\}\right|}{|\mathbf{X}|}=\frac{1}{|\mathbf{X}_t|}.$$

In other words, averaging the trimmed tree code is equivalent to averaging over the original code.

Perfect Projection Condition

Definition 1 (Perfect Projection) *The supporting tree $\mathcal{N}^{2l}$ is perfectly projected, if for any codeword $\mathbf{x}_t$ of the tree code $\mathbf{X}_t$,*

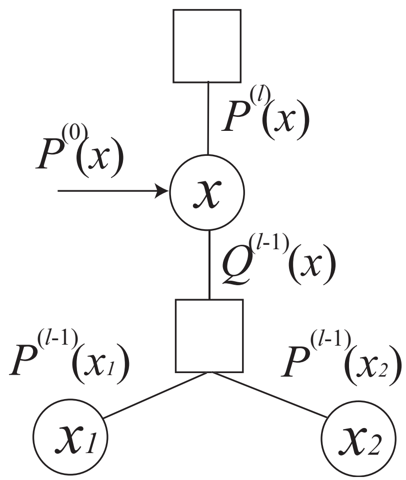
$$\frac{\left|\{\mathbf{x} \in \mathbf{X} : \mathbf{x}|_{\text{tree}} = \mathbf{x}_t\}\right|}{|\mathbf{X}|} = \frac{1}{|\mathbf{X}_t|}.$$

In other words, averaging the trimmed tree code is equivalent to averaging over the original code.

We then have

$$\begin{aligned} P^{(l)}(x) &:= \left\langle P^{(l)}(\mathbf{x}) \right\rangle_{\{\mathbf{x} \in \mathbf{X} : \mathbf{x}|_0 = x\}} = \left\langle P^{(l)}(\mathbf{x}_t) \right\rangle_{\{\mathbf{x}_t \in \mathbf{X}_t : \mathbf{x}_t|_0 = x\}}, \\ p_e^{(l)} &= \frac{1}{2} \left(\int_{m=-\infty}^{0} P^{(l)}(0)(dm) + \int_{m=-\infty}^{0} P^{(l)}(1)(dm) \right). \end{aligned}$$

New Iterative Formula for DE



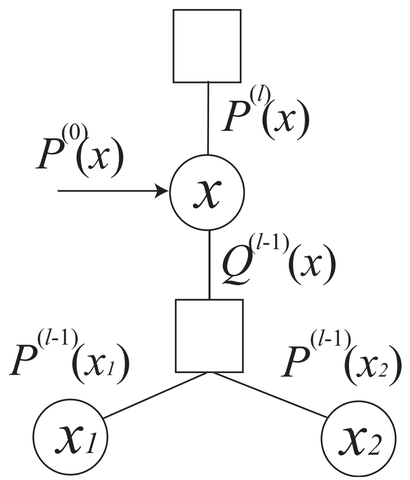
$x = 0, \;\; x_1 x_2 = 00, 11$

$x = 1, \;\; x_1 x_2 = 01, 10$

$$\forall x \in \{0,1\}, \;\; P^{(l)}(x) = P^{(0)}(x) \otimes \left(Q^{(l-1)}(x)\right)^{\otimes(d_v-1)}$$

$$Q^{(l-1)}(x) = \Gamma^{-1}\left(\frac{1}{2^{d_c-2}} \sum_{\mathbf{x}^1 \in \mathbf{X}^1(x)} \bigotimes_{v=1}^{d_c-1} \Gamma\left(P^{(l-1)}(x_v)\right)\right)$$

New Iterative Formula for DE



$$x = 0, \quad x_1x_2 = 00, 11$$

$$x = 1, \quad x_1x_2 = 01, 10$$

$$\forall x \in \{0,1\}, \quad P^{(l)}(x) = P^{(0)}(x) \otimes \left(Q^{(l-1)}(x)\right)^{\otimes(d_v-1)}$$

$$Q^{(l-1)}(x) = \Gamma^{-1}\left(\frac{1}{2^{d_c-2}} \sum_{\mathbf{x}^1 \in \mathbf{X}^1(x)} \bigotimes_{v=1}^{d_c-1} \Gamma\left(P^{(l-1)}(x_v)\right)\right)$$

$$= \Gamma^{-1}\left(\left(\Gamma\left(\frac{P^{(l-1)}(0) + P^{(l-1)}(1)}{2}\right)\right)^{\otimes(d_c-1)} + (-1)^x \left(\Gamma\left(\frac{P^{(l-1)}(0) - P^{(l-1)}(1)}{2}\right)\right)^{\otimes(d_c-1)}\right).$$

Comparisons

- Linear code ensemble:

$$P^{(l)}_{linear}(x) = P^{(0)}_{linear}(x) \otimes \left(Q^{(l-1)}_{linear}(x)\right)^{\otimes(d_v-1)}$$

$$Q^{(l-1)}_{linear}(x) = \Gamma^{-1}\left(\left(\Gamma\left(\frac{P^{(l-1)}_{linear}(0)+P^{(l-1)}_{linear}(1)}{2}\right)\right)^{\otimes(d_c-1)} + (-1)^x\left(\Gamma\left(\frac{P^{(l-1)}_{linear}(0)-P^{(l-1)}_{linear}(1)}{2}\right)\right)^{\otimes(d_c-1)}\right)$$

$$\left\langle P^{(l)}_{linear}\right\rangle = \frac{P^{(l)}_{linear}(0)+P^{(l)}_{linear}(1)}{2}.$$

Comparisons

- Linear code ensemble:

$$
\begin{aligned}
P^{(l)}_{linear}(x) &= P^{(0)}_{linear}(x) \otimes \left(Q^{(l-1)}_{linear}(x)\right)^{\otimes(d_v-1)} \\
Q^{(l-1)}_{linear}(x) &= \Gamma^{-1}\Bigg(\left(\Gamma\left(\frac{P^{(l-1)}_{linear}(0)+P^{(l-1)}_{linear}(1)}{2}\right)\right)^{\otimes(d_c-1)} \\
&\qquad +(-1)^x\left(\Gamma\left(\frac{P^{(l-1)}_{linear}(0)-P^{(l-1)}_{linear}(1)}{2}\right)\right)^{\otimes(d_c-1)}\Bigg) \\
\left\langle P^{(l)}_{linear}\right\rangle &= \frac{P^{(l)}_{linear}(0)+P^{(l)}_{linear}(1)}{2}.
\end{aligned}
$$

- Coset code ensemble:

$$
\begin{aligned}
P^{(0)}_{coset} &= \frac{P^{(0)}_{linear}(0)+P^{(0)}_{linear}(1)}{2} \\
P^{(l)}_{coset} &= P^{(0)}_{coset} \otimes \left(Q^{(l-1)}_{coset}\right)^{\otimes(d_v-1)} \\
Q^{(l-1)}_{coset} &= \Gamma^{-1}\left(\left(\Gamma\left(P^{(l-1)}_{coset}\right)\right)^{\otimes(d_c-1)}\right)
\end{aligned}
$$

Similarities

- **Symmetry**:

- **Monotonicity**:

- **Stability Conditions**:

Similarities

- **Symmetry**:
  - For all $l \in \mathbb{N}$, both $\left\langle P^{(l)}_{linear} \right\rangle$ and $P^{(l)}_{coset}$ are symmetric.
- **Monotonicity**:
- **Stability Conditions**:

Similarities

- **Symmetry**:
  - For all $l \in \mathbb{N}$, both $\left\langle P^{(l)}_{linear} \right\rangle$ and $P^{(l)}_{coset}$ are symmetric.
- **Monotonicity**: Let

$$p^{(l)}_{e,linear} = \int_{m=\infty}^{0^-} \left\langle P^{(l)}_{linear} \right\rangle (dm)$$

$$p^{(l)}_{e,coset} = \int_{m=\infty}^{0^-} P^{(l)}_{coset}(dm).$$

- **Stability Conditions**:

Similarities

- **Symmetry**:
  - For all $l \in \mathbb{N}$, both $\left\langle P^{(l)}_{linear} \right\rangle$ and $P^{(l)}_{coset}$ are symmetric.
- **Monotonicity**: Let

$$\begin{aligned} p^{(l)}_{e,linear} &= \int_{m=\infty}^{0^-} \left\langle P^{(l)}_{linear} \right\rangle (dm) \\ p^{(l)}_{e,coset} &= \int_{m=\infty}^{0^-} P^{(l)}_{coset}(dm). \end{aligned}$$

  - Both $p^{(l)}_{e,linear}$ and $p^{(l)}_{e,coset}$ are monotonically decreasing w.r.t. l.
- **Stability Conditions**:

Similarities

- **Symmetry**:
  - For all $l \in \mathbb{N}$, both $\left\langle P^{(l)}_{linear} \right\rangle$ and $P^{(l)}_{coset}$ are symmetric.
- **Monotonicity**: Let

$$p^{(l)}_{e,linear} = \int_{m=\infty}^{0^-} \left\langle P^{(l)}_{linear} \right\rangle (dm)$$

$$p^{(l)}_{e,coset} = \int_{m=\infty}^{0^-} P^{(l)}_{coset}(dm).$$

  - Both $p^{(l)}_{e,linear}$ and $p^{(l)}_{e,coset}$ are monotonically decreasing w.r.t. l.
- **Stability Conditions**: Let $\lambda(x) := \sum \lambda_k x^{k-1}$ and $\rho(x) := \sum \rho_k x^{k-1}$ denote the edge degree distribution poly., and

$$\left\langle CB^{(0)} \right\rangle = \int e^{-\frac{m}{2}} \left\langle P^{(0)}_{linear} \right\rangle (dm) = \int e^{\frac{m}{2}} P^{(0)}_{coset}(dm).$$

Similarities

- **Symmetry**:
  - For all $l \in \mathbb{N}$, both $\left\langle P^{(l)}_{linear} \right\rangle$ and $P^{(l)}_{coset}$ are symmetric.
- **Monotonicity**: Let

$$p^{(l)}_{e,linear} = \int_{m=\infty}^{0^-} \left\langle P^{(l)}_{linear} \right\rangle (dm)$$

$$p^{(l)}_{e,coset} = \int_{m=\infty}^{0^-} P^{(l)}_{coset}(dm).$$

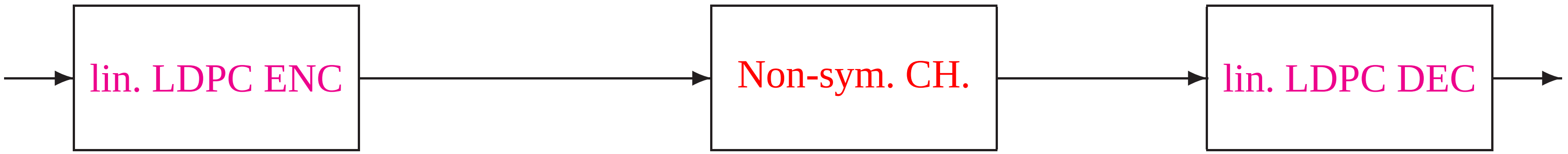
  - Both $p^{(l)}_{e,linear}$ and $p^{(l)}_{e,coset}$ are monotonically decreasing w.r.t. l.
- **Stability Conditions**: Let $\lambda(x) := \sum \lambda_k x^{k-1}$ and $\rho(x) := \sum \rho_k x^{k-1}$ denote the edge degree distribution poly., and

$$\left\langle CB^{(0)} \right\rangle = \int e^{-\frac{m}{2}} \left\langle P^{(0)}_{linear} \right\rangle (dm) = \int e^{\frac{m}{2}} P^{(0)}_{coset}(dm).$$

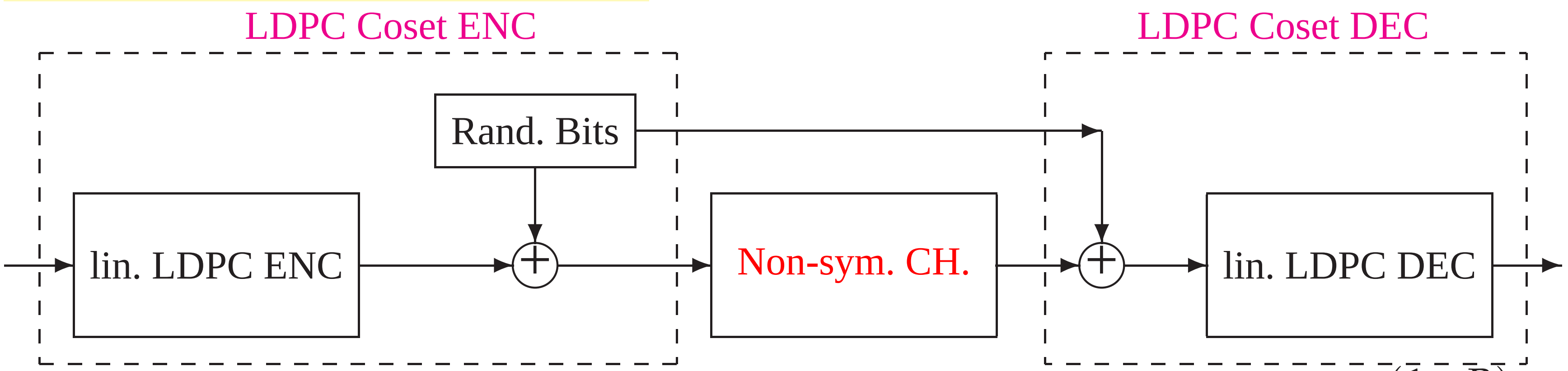
  - Having the same stability conditions: $\left\langle CB^{(0)} \right\rangle \overset{?}{<} \frac{1}{\lambda_2 \rho'(1)}$.

Comparisons

Linear codes:



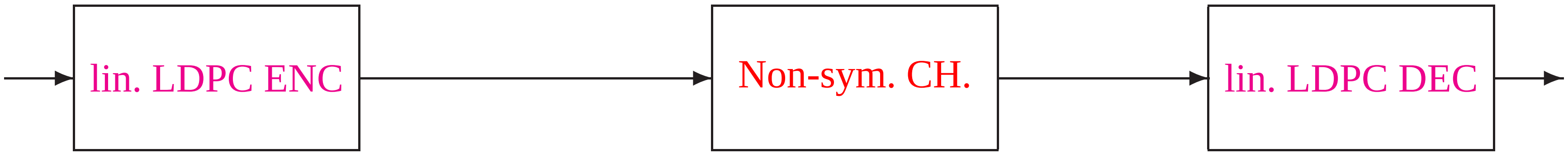
The coset code ensemble:



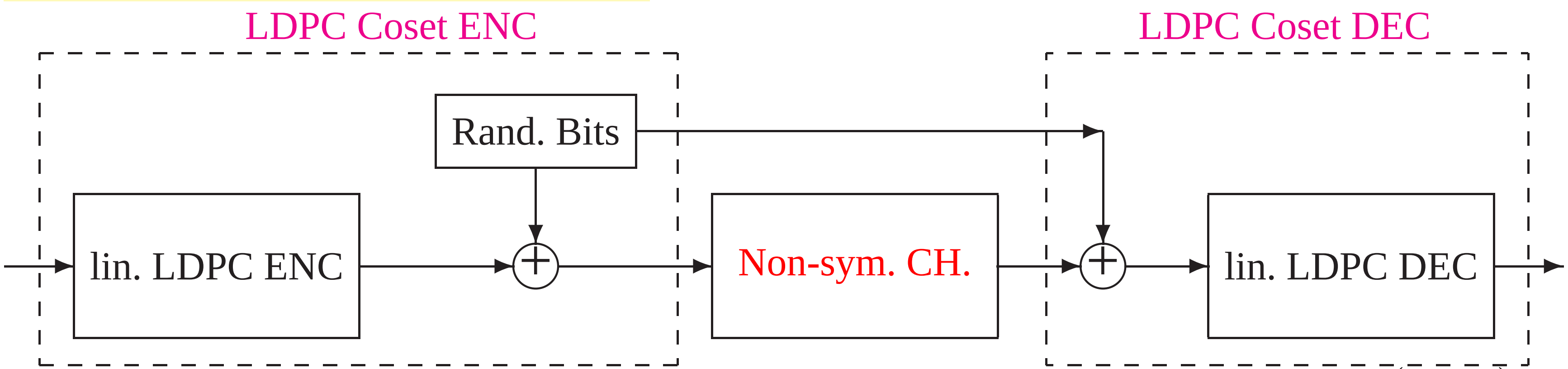
LDPC coset code ensemble: $\mathbf{A}\mathbf{x} = \mathbf{s}$ and $\mathbf{s} \in_{\text{rand.}} \{0,1\}^{n(1-R)}$.

Comparisons

Linear codes:



The coset code ensemble:

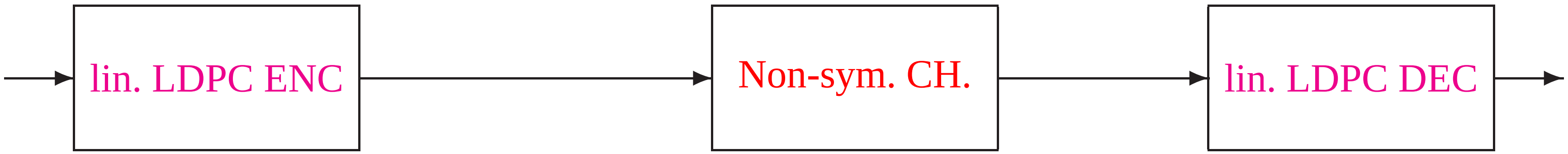


LDPC coset code ensemble: $\mathbf{Ax} = \mathbf{s}$ and $\mathbf{s} \in_{\text{rand.}} \{0,1\}^{n(1-R)}$.

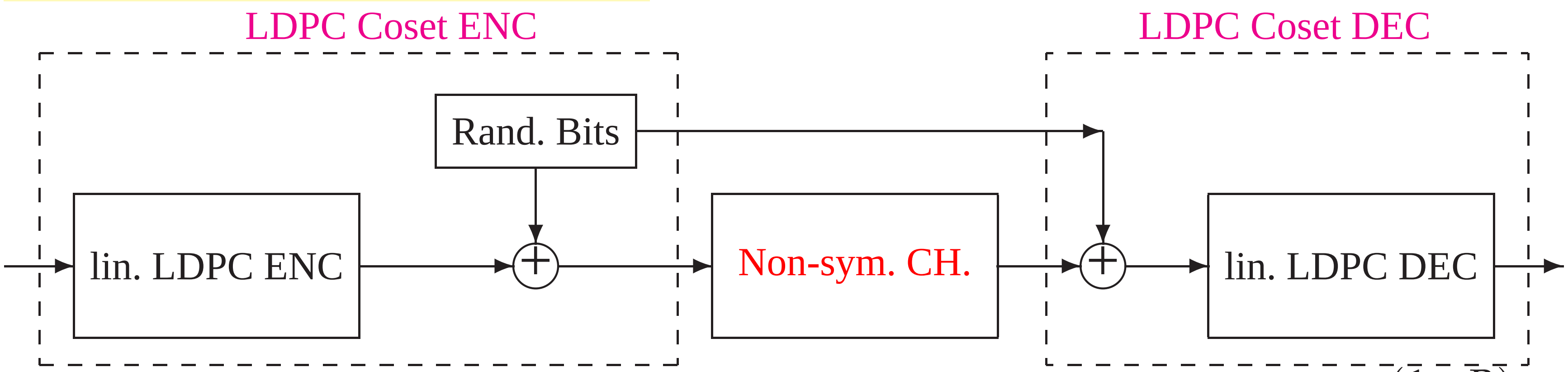
- Their stability conditions and many other properties coincide.

Comparisons

Linear codes:



The coset code ensemble:

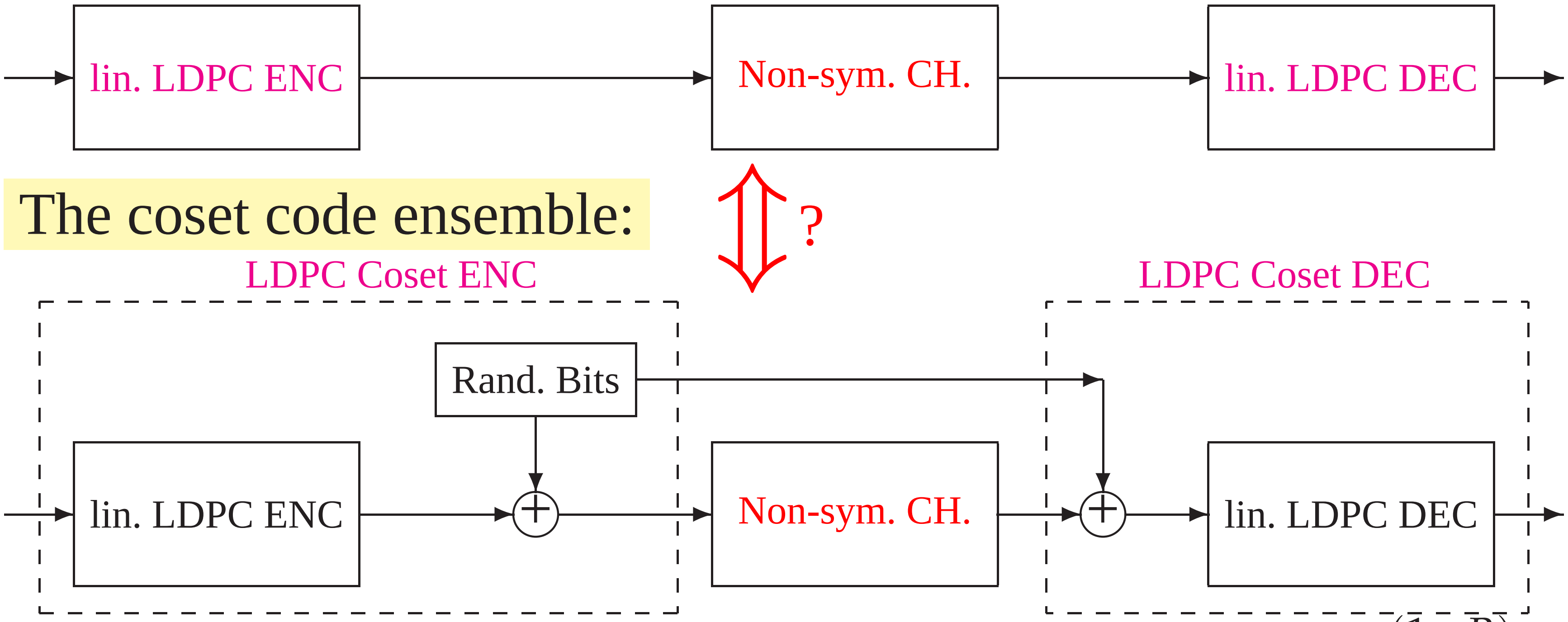


LDPC coset code ensemble: $\mathbf{Ax} = \mathbf{s}$ and $\mathbf{s} \in_{\text{rand.}} \{0,1\}^{n(1-R)}$.

- Their stability conditions and many other properties coincide.
- Nearly identical performance under Monte-Carlo simulations.

Comparisons

Linear codes:



LDPC coset code ensemble: $\mathbf{Ax} = \mathbf{s}$ and $\mathbf{s} \in_{\text{rand.}} \{0,1\}^{n(1-R)}$.

- Their stability conditions and many other properties coincide.
- Nearly identical performance under Monte-Carlo simulations.

A Short Answer

Not equivalent:

$$P^{(l)}_{linear}(x) = P^{(0)}_{linear}(x) \otimes \left(Q^{(l-1)}_{linear}(x)\right)^{\otimes(d_v-1)}$$

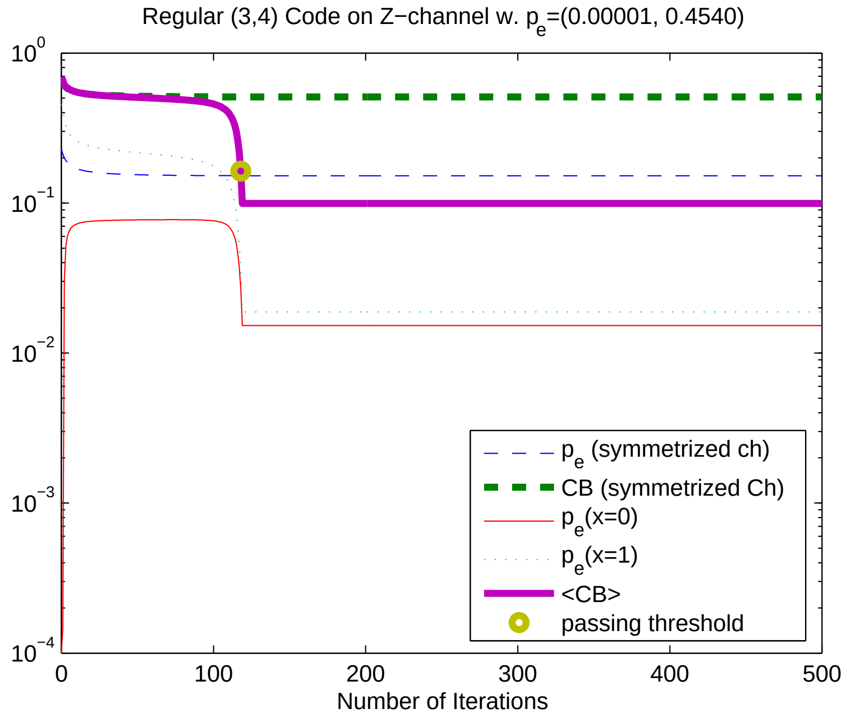
$$Q^{(l-1)}_{linear}(x) = \Gamma^{-1}\left(\left(\Gamma\left(\frac{P^{(l-1)}_{linear}(0)+P^{(l-1)}_{linear}(1)}{2}\right)\right)^{\otimes(d_c-1)} + (-1)^x\left(\Gamma\left(\frac{P^{(l-1)}_{linear}(0)-P^{(l-1)}_{linear}(1)}{2}\right)\right)^{\otimes(d_c-1)}\right)$$

A Short Answer

Not equivalent:

$$P^{(l)}_{linear}(x) = P^{(0)}_{linear}(x) \otimes \left(Q^{(l-1)}_{linear}(x)\right)^{\otimes(d_v-1)}$$

$$Q^{(l-1)}_{linear}(x) = \Gamma^{-1}\left(\left(\Gamma\left(\frac{P^{(l-1)}_{linear}(0)+P^{(l-1)}_{linear}(1)}{2}\right)\right)^{\otimes(d_c-1)} + (-1)^x\left(\Gamma\left(\frac{P^{(l-1)}_{linear}(0)-P^{(l-1)}_{linear}(1)}{2}\right)\right)^{\otimes(d_c-1)}\right)$$



Typicality of Linear LDPC Codes

(λ, ρ)	(x^2, x^3)	(x^2, x^5)	$(x^2, 0.5x^2 + 0.5x^3)$	$(x^2, 0.5x^4 + 0.5x^5)$
Linear	0.4540	0.2305	0.5888	0.2689
Coset	0.4527	0.2304	0.5908	0.2690

Typicality of Linear LDPC Codes

(λ,ρ)	(x^2,x^3)	(x^2,x^5)	$(x^2,0.5x^2+0.5x^3)$	$(x^2,0.5x^4+0.5x^5)$
Linear	0.4540	0.2305	0.5888	0.2689
	$\updownarrow$ 0.29%	$\updownarrow$ 0.043%	$\updownarrow$ 0.17%	$\updownarrow$ 0.037%
Coset	0.4527	0.2304	0.5908	0.2690

Typicality of Linear LDPC Codes

(λ, ρ)	(x^2, x^3)	(x^2, x^5)	$(x^2, 0.5x^2 + 0.5x^3)$	$(x^2, 0.5x^4 + 0.5x^5)$
Linear	0.4540	0.2305	0.5888	0.2689
	$\updownarrow$ 0.29%	$\updownarrow$ 0.043%	$\updownarrow$ 0.17%	$\updownarrow$ 0.037%
Coset	0.4527	0.2304	0.5908	0.2690

Theorem 1 (Main Theorem) *Consider non-symmetric channels and a fixed pair of degree polynomials* λ *and* ρ*. The shifted check node polynomial is denoted by* $\rho_\Delta = x^\Delta \cdot \rho$*. Let* $P^{(l)}_{linear}$ *and* $P^{(l)}_{coset}$ *denote the evolved densities of the linear and coset code ensemble with degrees* (λ, ρ_Δ)*. Then,* $\forall l_0 \in \mathbb{N}, \lim_{\Delta \to \infty} \left\langle P^{(l_0)}_{linear} \right\rangle \stackrel{\mathcal{D}}{=} P^{(l_0)}_{coset}$ *in distribution, with the convergence rate being* $\mathcal{O}\left(\text{const}^\Delta\right)$ *for some* $\text{const} < 1$.

Intuition

- Consider the 1st check node iteration. Suppose for all possible $\left\{P^{(0)}_{linear}(x)\right\}$, $Q^{(0)}_{linear}(0) = Q^{(0)}_{linear}(1) = Q^{(0)}_{coset}$. Then

$$\begin{aligned} P^{(1)}_{linear}(x) &= P^{(0)}_{linear}(x) \otimes \left(Q^{(0)}_{coset}\right)^{\otimes(d_v-1)} \\ \left\langle P^{(1)}_{linear}\right\rangle &= \frac{P^{(0)}_{linear}(0) + P^{(0)}_{linear}(1)}{2} \otimes \left(Q^{(0)}_{coset}\right)^{\otimes(d_v-1)} = P^{(1)}_{coset}. \end{aligned}$$

Intuition

- Consider the 1st check node iteration. Suppose for all possible $\left\{P^{(0)}_{linear}(x)\right\}$, $Q^{(0)}_{linear}(0) = Q^{(0)}_{linear}(1) = Q^{(0)}_{coset}$. Then

$$\begin{aligned} P^{(1)}_{linear}(x) &= P^{(0)}_{linear}(x) \otimes \left(Q^{(0)}_{coset}\right)^{\otimes(d_v-1)} \\ \left\langle P^{(1)}_{linear}\right\rangle &= \frac{P^{(0)}_{linear}(0) + P^{(0)}_{linear}(1)}{2} \otimes \left(Q^{(0)}_{coset}\right)^{\otimes(d_v-1)} = P^{(1)}_{coset}. \end{aligned}$$

- Furthermore, $Q^{(1)}_{linear}(0) = Q^{(1)}_{linear}(1) = Q^{(1)}_{coset}$.

Intuition

- Consider the 1st check node iteration. Suppose for all possible $\left\{P^{(0)}_{linear}(x)\right\}$, $Q^{(0)}_{linear}(0) = Q^{(0)}_{linear}(1) = Q^{(0)}_{coset}$. Then

$$\begin{aligned} P^{(1)}_{linear}(x) &= P^{(0)}_{linear}(x) \otimes \left(Q^{(0)}_{coset}\right)^{\otimes(d_v-1)} \\ \left\langle P^{(1)}_{linear}\right\rangle &= \frac{P^{(0)}_{linear}(0) + P^{(0)}_{linear}(1)}{2} \otimes \left(Q^{(0)}_{coset}\right)^{\otimes(d_v-1)} = P^{(1)}_{coset}. \end{aligned}$$

- Furthermore, $Q^{(1)}_{linear}(0) = Q^{(1)}_{linear}(1) = Q^{(1)}_{coset}$.
- Since the iterative equations of DE are continuous, we need only to prove that for all possible $\left\{P^{(0)}_{linear}(x)\right\}$,

$$\lim_{\Delta\to\infty} Q^{(0)}_{linear}(0) \stackrel{\mathcal{D}}{=} \lim_{\Delta\to\infty} Q^{(0)}_{linear}(1) \stackrel{\mathcal{D}}{=} Q^{(0)}_{coset}$$

Proof of the Main Theorem

- $m = \log \frac{\mathsf{P}(X=0|Y)}{\mathsf{P}(X=1|Y)}$.

Proof of the Main Theorem

- $m = \log \frac{\mathsf{P}(X=0|Y)}{\mathsf{P}(X=1|Y)}$.
- $\gamma(m) = (1_{\{m<0\}}, \log \cot \left|\frac{m}{2}\right|)$

Proof of the Main Theorem

- $m = \log \frac{\mathsf{P}(X=0|Y)}{\mathsf{P}(X=1|Y)}$.
- $\gamma(m) = (1_{\{m<0\}}, \log\cot\left|\frac{m}{2}\right|)$
- $P^{(0)}_{linear}(x)(dm) \mapsto P'_x(d\gamma)$ and $Q^{(0)}_{linear}(x)(dm) \mapsto Q'_x(d\gamma)$.

Proof of the Main Theorem

- $m = \log \frac{\mathsf{P}(X=0|Y)}{\mathsf{P}(X=1|Y)}$.
- $\gamma(m) = (1_{\{m<0\}}, \log \cot \left|\frac{m}{2}\right|)$
- $P^{(0)}_{linear}(x)(dm) \mapsto P'_x(d\gamma)$ and $Q^{(0)}_{linear}(x)(dm) \mapsto Q'_x(d\gamma)$.
- The characteristic function: $\Phi_P(\lambda_1, \lambda_2) = \mathsf{E}(-1)^{\lambda_1\gamma_1} e^{i\lambda_2\gamma_2}$.

Proof of the Main Theorem

- $m = \log \frac{\mathsf{P}(X=0|Y)}{\mathsf{P}(X=1|Y)}$.
- $\gamma(m) = (1_{\{m<0\}}, \log\cot\left|\frac{m}{2}\right|)$
- $P^{(0)}_{linear}(x)(dm) \mapsto P'_x(d\gamma)$ and $Q^{(0)}_{linear}(x)(dm) \mapsto Q'_x(d\gamma)$.
- The characteristic function: $\Phi_P(\lambda_1, \lambda_2) = \mathsf{E}(-1)^{\lambda_1\gamma_1} e^{i\lambda_2\gamma_2}$.
- By induction,

$$\Phi_{Q'_0}(\lambda_1,\lambda_2) - \Phi_{Q'_1}(\lambda_1,\lambda_2) = 2\left(\frac{\Phi_{P'_0}(\lambda_1,\lambda_2) - \Phi_{P'_1}(\lambda_1,\lambda_2)}{2}\right)^{\Delta}.$$

Proof of the Main Theorem

- $m = \log \frac{\mathsf{P}(X=0|Y)}{\mathsf{P}(X=1|Y)}$.
- $\gamma(m) = (1_{\{m<0\}}, \log\cot\left|\frac{m}{2}\right|)$
- $P^{(0)}_{linear}(x)(dm) \mapsto P'_x(d\gamma)$ and $Q^{(0)}_{linear}(x)(dm) \mapsto Q'_x(d\gamma)$.
- The characteristic function: $\Phi_P(\lambda_1, \lambda_2) = \mathsf{E}(-1)^{\lambda_1\gamma_1}e^{i\lambda_2\gamma_2}$.
- By induction,

$$\Phi_{Q'_0}(\lambda_1, \frac{\lambda_2}{\Delta}) - \Phi_{Q'_1}(\lambda_1, \frac{\lambda_2}{\Delta}) = 2\left(\frac{\Phi_{P'_0}(\lambda_1, \frac{\lambda_2}{\Delta}) - \Phi_{P'_1}(\lambda_1, \frac{\lambda_2}{\Delta})}{2}\right)^{\Delta}.$$

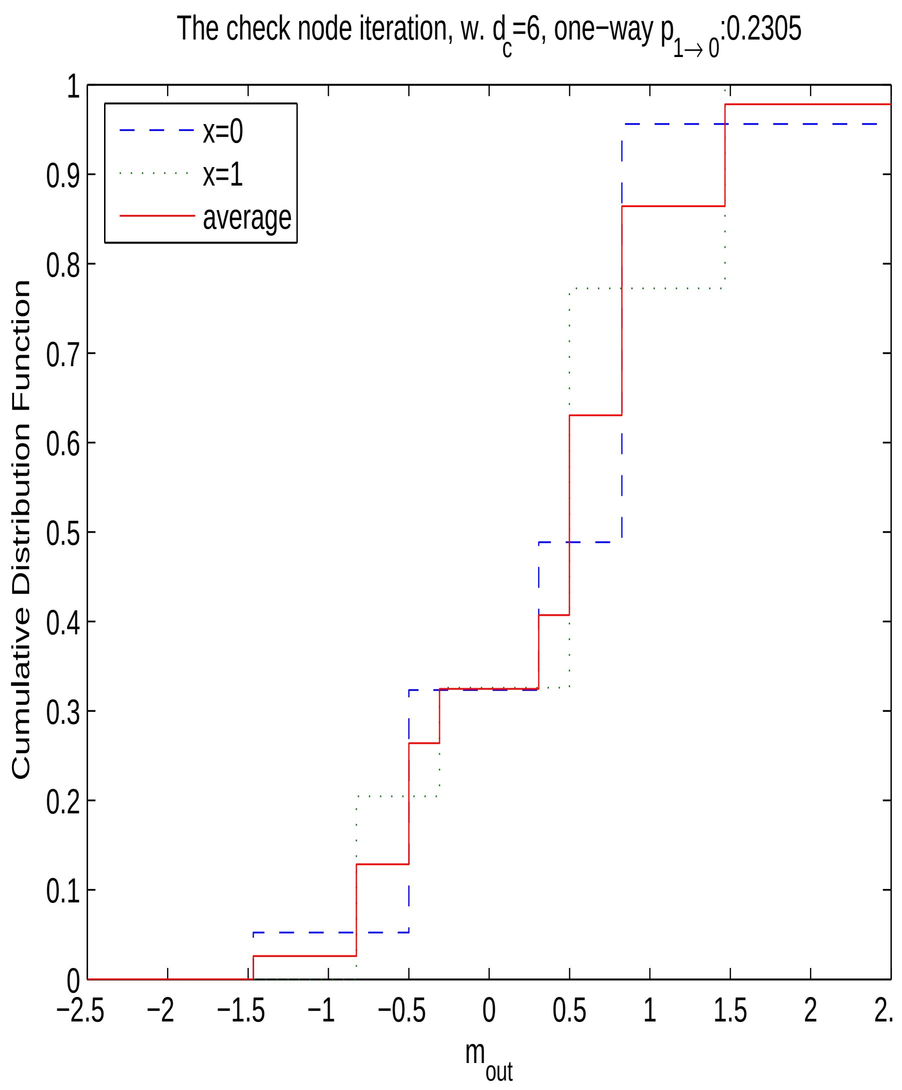
Proof of the Main Theorem

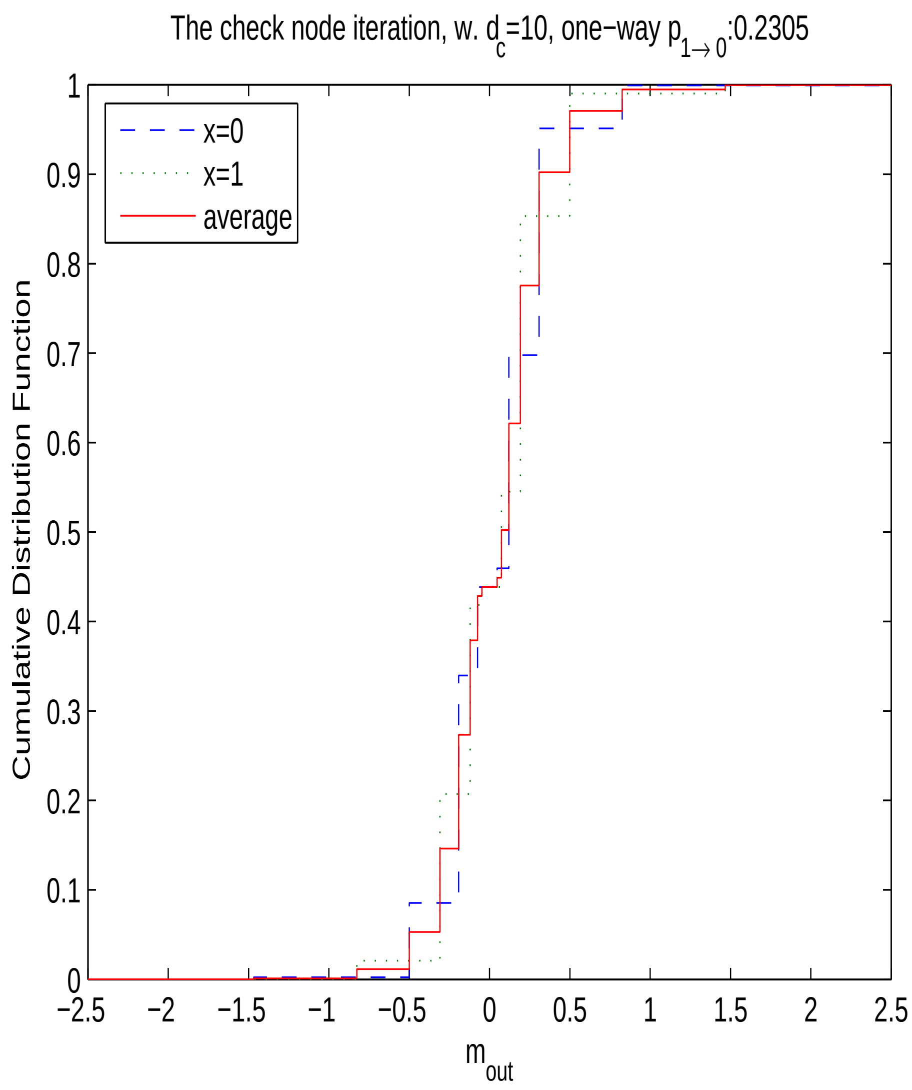
- $m = \log \frac{\mathsf{P}(X=0|Y)}{\mathsf{P}(X=1|Y)}$.
- $\gamma(m) = (1_{\{m<0\}}, \log \cot \left|\frac{m}{2}\right|)$
- $P^{(0)}_{linear}(x)(dm) \mapsto P'_x(d\gamma)$ and $Q^{(0)}_{linear}(x)(dm) \mapsto Q'_x(d\gamma)$.
- The characteristic function: $\Phi_P(\lambda_1, \lambda_2) = \mathsf{E}(-1)^{\lambda_1\gamma_1}e^{i\lambda_2\gamma_2}$.
- By induction,

$$\Phi_{Q'_0}(\lambda_1, \frac{\lambda_2}{\Delta}) - \Phi_{Q'_1}(\lambda_1, \frac{\lambda_2}{\Delta}) = 2\left(\frac{\Phi_{P'_0}(\lambda_1, \frac{\lambda_2}{\Delta}) - \Phi_{P'_1}(\lambda_1, \frac{\lambda_2}{\Delta})}{2}\right)^{\Delta}.$$

- By Taylor's expansion, the RHS converges to zero for all λ_1, λ_2 and the convergence rate is exponential.

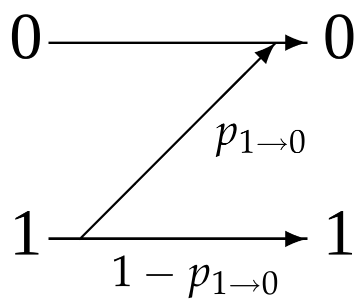
The Weak Convergence





Typicality in Terms of the Decodable Threshold

Z-Channels:

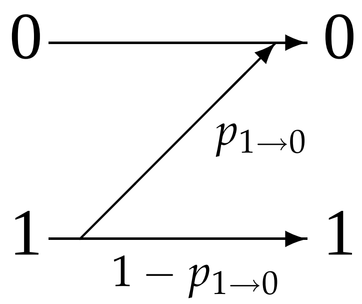


$$p^*_{1\to 0,linear} := \sup\left\{p_{1\to 0} > 0 : \lim_{l\to\infty} p^{(l)}_{e,linear} = 0\right\}$$

$$\text{and } p^*_{1\to 0,coset} := \sup\left\{p_{1\to 0} > 0 : \lim_{l\to\infty} p^{(l)}_{e,coset} = 0\right\}.$$

Typicality in Terms of the Decodable Threshold

Z-Channels:



$$p^*_{1\to 0,linear} := \sup \left\{ p_{1\to 0} > 0 : \lim_{l\to\infty} p^{(l)}_{e,linear} = 0 \right\}$$

$$\text{and } p^*_{1\to 0,coset} := \sup \left\{ p_{1\to 0} > 0 : \lim_{l\to\infty} p^{(l)}_{e,coset} = 0 \right\}.$$

Corollary 1 (Typicality Results for Z-Channels) *For any* $\epsilon > 0$, *there exists a* $\Delta \in \mathbb{N}$ *such that*

$$\left| p^*_{1\to 0,linear} - p^*_{1\to 0,coset} \right| < \epsilon.$$

Proof of Corollary 1

$$p^*_{1\to 0,linear} > p^*_{1\to 0,coset} - \epsilon := p_{temp}$$

Proof of Corollary 1

$$p^*_{1\to 0,linear} > p^*_{1\to 0,coset} - \epsilon := p_{temp}$$

- Def: $\exists l_0, P^{(l_0)}_{coset}$ is in its stability region $\Leftrightarrow \lim_{l\to\infty} p^{(l)}_{e,coset} = 0$.

Proof of Corollary 1

$$p^*_{1\to 0,linear} > p^*_{1\to 0,coset} - \epsilon := p_{temp}$$

- Def: $\exists l_0, P^{(l_0)}_{coset}$ is in its stability region $\Leftrightarrow \lim_{l\to\infty} p^{(l)}_{e,coset} = 0$.
- By definition, when $p_{1\to 0} = p_{temp}$, $\exists l_0$ such that after l_0 iterations, $P^{(l_0)}_{coset}$ is in its stability region, which is an open set.

Proof of Corollary 1

$$p^*_{1\to 0,linear} > p^*_{1\to 0,coset} - \epsilon := p_{temp}$$

- Def: $\exists l_0, P^{(l_0)}_{coset}$ is in its stability region $\Leftrightarrow \lim_{l\to\infty} p^{(l)}_{e,coset} = 0$.
- By definition, when $p_{1\to 0} = p_{temp}$, $\exists l_0$ such that after l_0 iterations, $P^{(l_0)}_{coset}$ is in its stability region, which is an open set.
- Note: The stability region is continuous w.r.t. weak convergence. By Theorem 1, there exists a $\Delta \in \mathbb{N}$ such that $\left\langle P^{(l_0)}_{linear} \right\rangle$ is also in the stability region of the coset code ensemble.

Proof of Corollary 1

$$p^*_{1\to 0,linear} > p^*_{1\to 0,coset} - \epsilon := p_{temp}$$

- Def: $\exists l_0, P^{(l_0)}_{coset}$ is in its stability region $\Leftrightarrow \lim_{l\to\infty} p^{(l)}_{e,coset} = 0$.
- By definition, when $p_{1\to 0} = p_{temp}$, $\exists l_0$ such that after l_0 iterations, $P^{(l_0)}_{coset}$ is in its stability region, which is an open set.
- Note: The stability region is continuous w.r.t. weak convergence. By Theorem 1, there exists a $\Delta \in \mathbb{N}$ such that $\left\langle P^{(l_0)}_{linear} \right\rangle$ is also in the stability region of the coset code ensemble.
- Note: The stability regions of the coset code ensembles and the linear codes are identical.

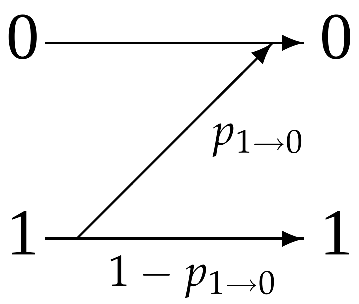
Proof of Corollary 1

$$p^*_{1\to 0,linear} > p^*_{1\to 0,coset} - \epsilon := p_{temp}$$

- Def: $\exists l_0, P^{(l_0)}_{coset}$ is in its stability region $\Leftrightarrow \lim_{l\to\infty} p^{(l)}_{e,coset} = 0$.
- By definition, when $p_{1\to 0} = p_{temp}$, $\exists l_0$ such that after l_0 iterations, $P^{(l_0)}_{coset}$ is in its stability region, which is an open set.
- Note: The stability region is continuous w.r.t. weak convergence. By Theorem 1, there exists a $\Delta \in \mathbb{N}$ such that $\left\langle P^{(l_0)}_{linear} \right\rangle$ is also in the stability region of the coset code ensemble.
- Note: The stability regions of the coset code ensembles and the linear codes are identical.
- By the definition of the stability condition, $\lim_{l\to\infty} p^{(l)}_{e,linear} = 0$.

Typicality in Terms of the Decodable Threshold

Z-Channels:



$$p^*_{1\to 0,linear} := \sup\left\{p_{1\to 0} > 0 : \lim_{l\to\infty} p^{(l)}_{e,linear} = 0\right\}$$

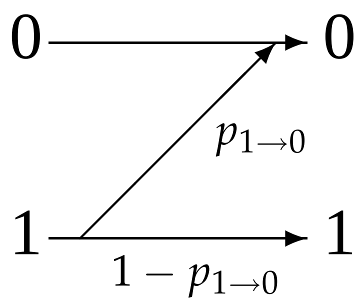
$$\text{and } p^*_{1\to 0,coset} := \sup\left\{p_{1\to 0} > 0 : \lim_{l\to\infty} p^{(l)}_{e,coset} = 0\right\}.$$

Corollary 1 (Typicality Results for Z-Channels) *For any $\epsilon > 0$, there exists a $\Delta \in \mathbb{N}$ such that*

$$\left|p^*_{1\to 0,linear} - p^*_{1\to 0,coset}\right| < \epsilon.$$

Typicality in Terms of the Decodable Threshold

Z-Channels:



$$p^*_{1\to 0,linear} := \sup\left\{p_{1\to 0}>0 : \lim_{l\to\infty} p^{(l)}_{e,linear}=0\right\}$$

$$\text{and } p^*_{1\to 0,coset} := \sup\left\{p_{1\to 0}>0 : \lim_{l\to\infty} p^{(l)}_{e,coset}=0\right\}.$$

Corollary 1 (Typicality Results for Z-Channels) *For any $\epsilon > 0$, there exists a $\Delta \in \mathbb{N}$ such that*

$$\left|p^*_{1\to 0,linear} - p^*_{1\to 0,coset}\right| < \epsilon.$$

Similar corollaries can be easily derived for other types of channel models.

The Convergence Rates

- Exponential convergence rate $\mathcal{O}\left(\mathsf{const}^{\Delta}\right)$: The thresholds are nearly identical (the discrepancy is $< 0.05\%$) when $d_c \geq 6$.

The Convergence Rates

- Exponential convergence rate $\mathcal{O}\left(\mathsf{const}^{\Delta}\right)$: The thresholds are nearly identical (the discrepancy is $< 0.05\%$) when $d_c \geq 6$.
- Revisit the check node iteration:

The Convergence Rates

- Exponential convergence rate $\mathcal{O}\left(\mathsf{const}^{\Delta}\right)$: The thresholds are nearly identical (the discrepancy is $< 0.05\%$) when $d_c \geq 6$.
- Revisit the check node iteration:
  - Consider a check node of degree 10000.

The Convergence Rates

- Exponential convergence rate $\mathcal{O}\left(\mathsf{const}^{\Delta}\right)$: The thresholds are nearly identical (the discrepancy is $< 0.05\%$) when $d_c \geq 6$.
- Revisit the check node iteration:
  - Consider a check node of degree 10000.
  - All information will be erased after the check node, and it is equivalent to an erasure channel.

$$\lim_{\Delta \to \infty} Q^{(1)}_{linear}(0) \stackrel{\mathcal{D}}{=} \lim_{\Delta \to \infty} Q^{(1)}_{linear}(1) \stackrel{\mathcal{D}}{=} \lim_{\Delta \to \infty} Q^{(1)}_{coset} = \delta_0$$

The Convergence Rates

- Exponential convergence rate $\mathcal{O}\left(\text{const}^{\Delta}\right)$: The thresholds are nearly identical (the discrepancy is $< 0.05\%$) when $d_c \geq 6$.
- Revisit the check node iteration:
  - Consider a check node of degree 10000.
  - All information will be erased after the check node, and it is equivalent to an erasure channel.

$$\lim_{\Delta\to\infty} Q^{(1)}_{linear}(0) \stackrel{\mathcal{D}}{=} \lim_{\Delta\to\infty} Q^{(1)}_{linear}(1) \stackrel{\mathcal{D}}{=} \lim_{\Delta\to\infty} Q^{(1)}_{coset} = \delta_0$$

  - No error correcting capability. $p^*_{1\to 0,linear} = p^*_{1\to,coset} = 0.$

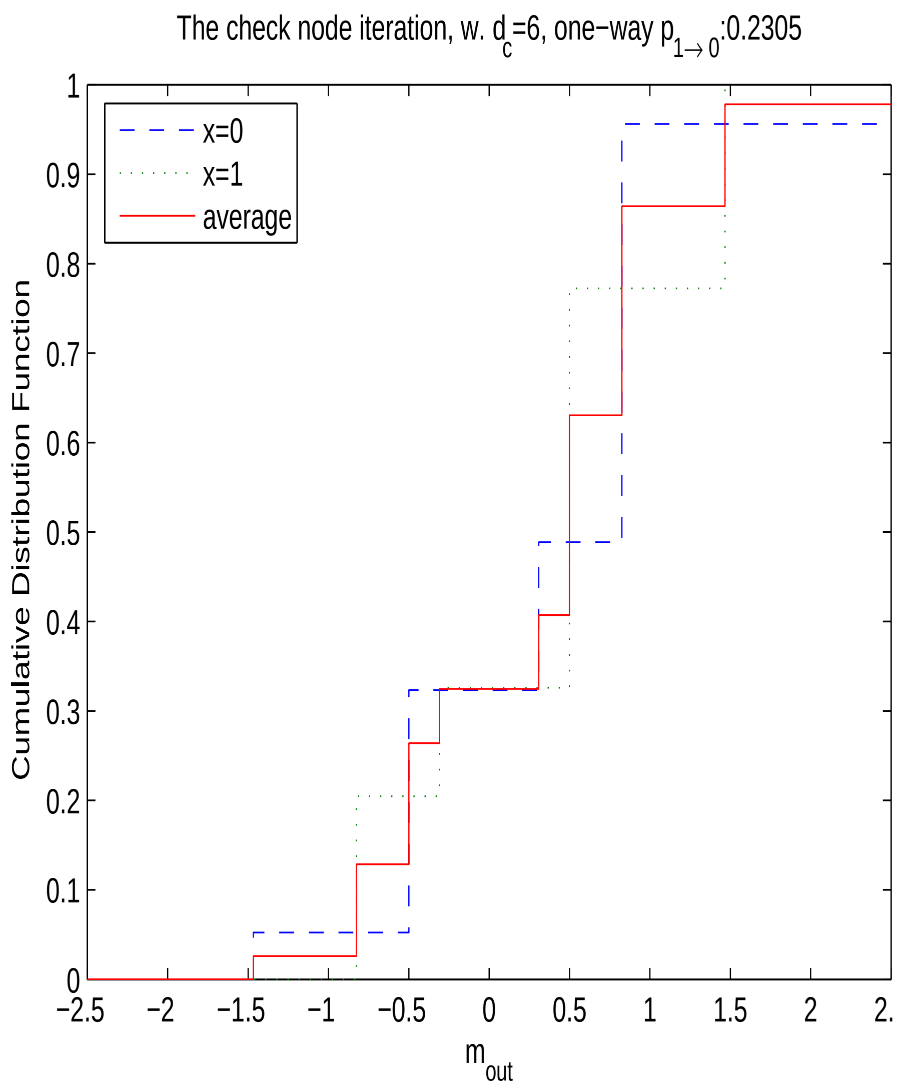
The Convergence Rates

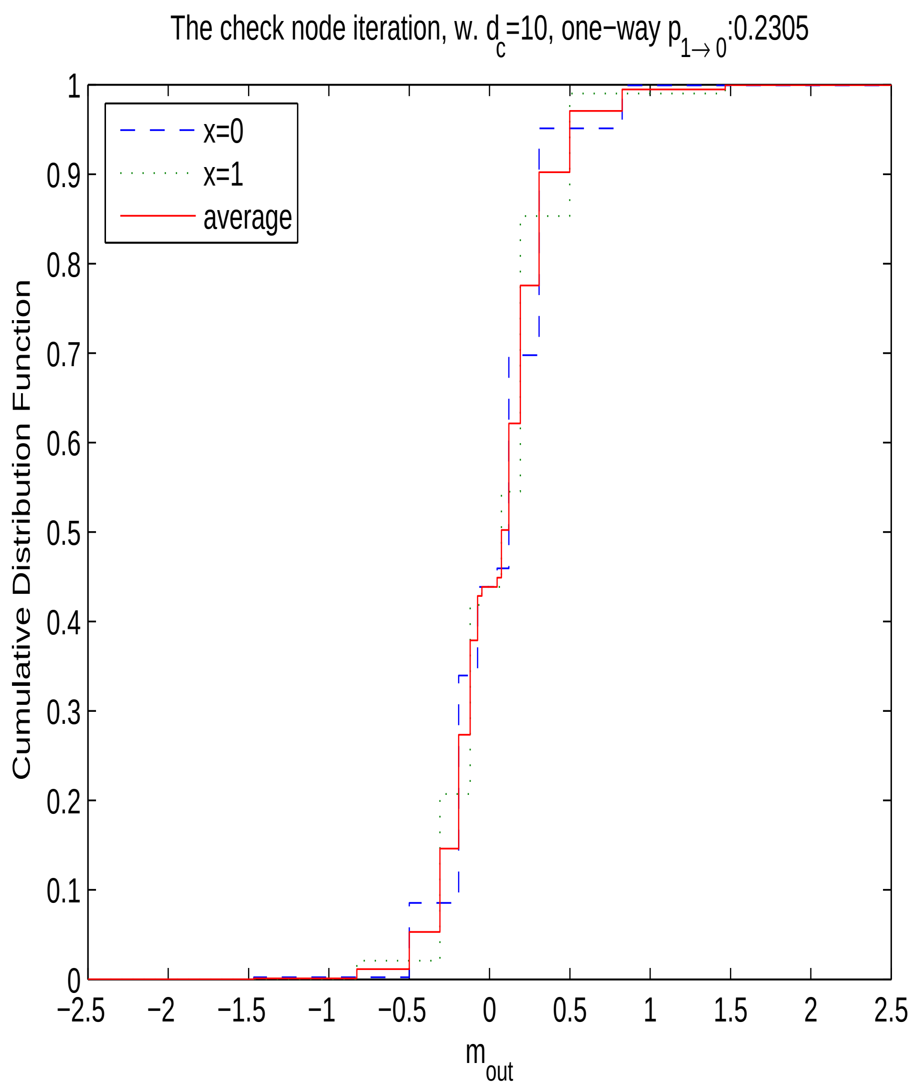
- Exponential convergence rate $\mathcal{O}\left(\text{const}^{\Delta}\right)$: The thresholds are nearly identical (the discrepancy is $< 0.05\%$) when $d_c \geq 6$.
- Revisit the check node iteration:
  - Consider a check node of degree 10000.
  - All information will be erased after the check node, and it is equivalent to an erasure channel.

$$\lim_{\Delta\to\infty} Q^{(1)}_{linear}(0) \stackrel{\mathcal{D}}{=} \lim_{\Delta\to\infty} Q^{(1)}_{linear}(1) \stackrel{\mathcal{D}}{=} \lim_{\Delta\to\infty} Q^{(1)}_{coset} = \delta_0$$

  - No error correcting capability. $p^*_{1\to 0,linear} = p^*_{1\to,coset} = 0.$
  - We have proved that the convergence rate is $\mathcal{O}\left(\frac{1}{\Delta}\right)$.

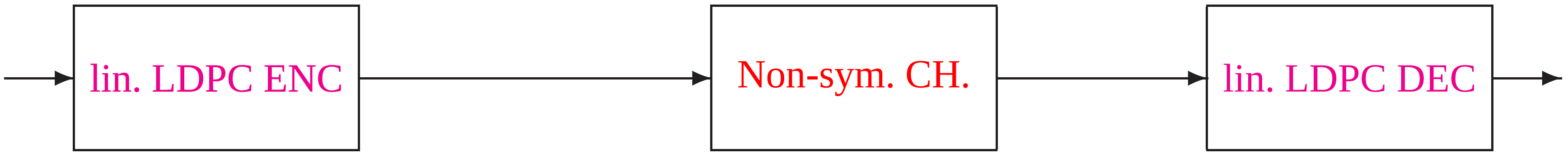
The Weak Convergence





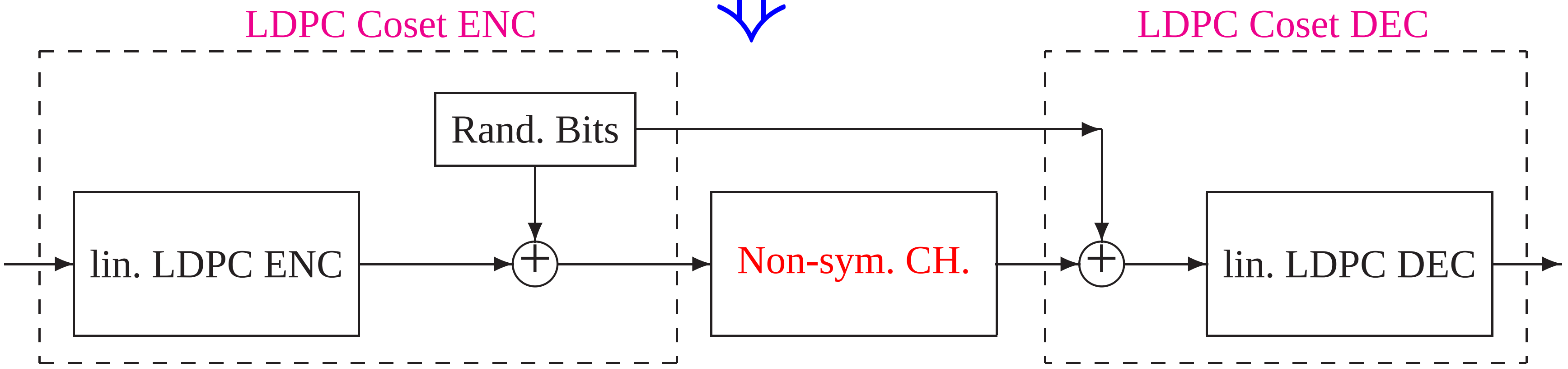
Comparisons

Linear codes:



$\Updownarrow$ almost equivalent for moderate d_c

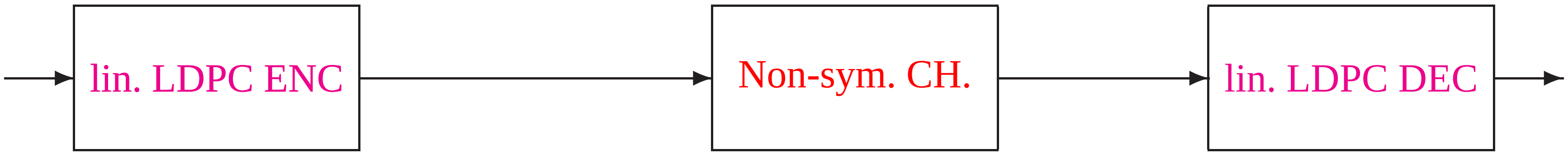
The LDPC code ensemble:



- [Kavčić 03]: Asymptotically, almost all $\mathbf{s} \in \{0,1\}^{n(1-R)}$ are typical.

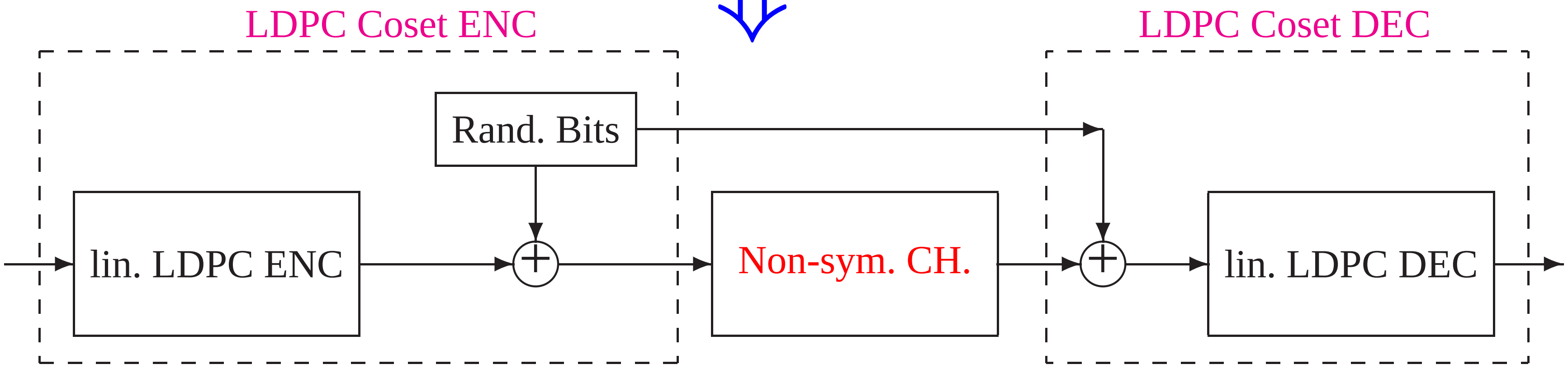
Comparisons

Linear codes:



The LDPC code ensemble:

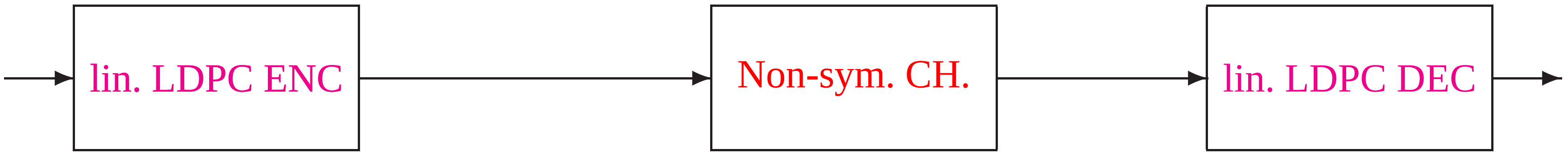
$\Updownarrow$ almost equivalent for moderate d_c



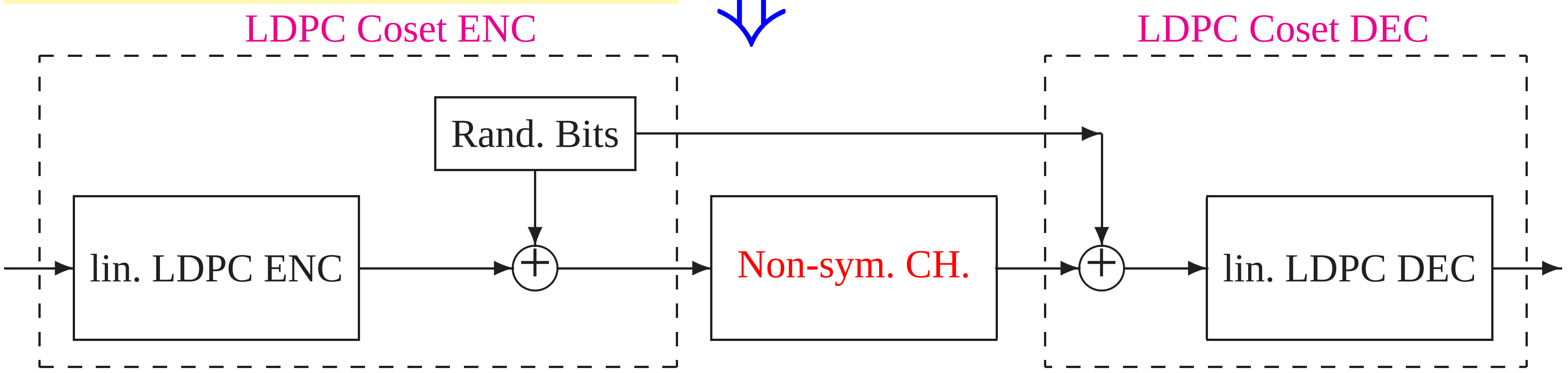
- [Kavčić 03]: Asymptotically, almost all $\mathbf{s} \in \{0,1\}^{n(1-R)}$ are typical.
- Our results can be viewed as a complementing theorem explicitly describing the typicality of individual $\mathbf{s}$ (including linear codes, $\mathbf{s} = \mathbf{0}$, as a special case).

Comparisons

Linear codes:



The LDPC code ensemble: $\Updownarrow$ almost equivalent for moderate d_c



- [Kavčić 03]: Asymptotically, almost all $\mathbf{s} \in \{0,1\}^{n(1-R)}$ are typical.
- Our results can be viewed as a complementing theorem explicitly describing the typicality of individual $\mathbf{s}$ (including linear codes, $\mathbf{s} = \mathbf{0}$, as a special case).
- Not much improvement left for choosing the optimal $\mathbf{s}$.

Applications

On a solid basis to interchangeably use the linear codes and the coset code ensemble when d_c is of moderate size.

Applications

On a solid basis to interchangeably use the linear codes and the coset code ensemble when d_c is of moderate size.

- Analysis: Use the fast density evolution (for coset codes) or the EXIT chart to optimize the degree distribution of linear codes for non-symmetric channels.

Applications

On a solid basis to interchangeably use the linear codes and the coset code ensemble when d_c is of moderate size.

- Analysis: Use the fast density evolution (for coset codes) or the EXIT chart to optimize the degree distribution of linear codes for non-symmetric channels.
- Finite code simulation: The codeword-averaged performance of linear codes vs. the all-zero codeword performance of coset codes.

Applications

On a solid basis to interchangeably use the linear codes and the coset code ensemble when d_c is of moderate size.

- Analysis: Use the fast density evolution (for coset codes) or the EXIT chart to optimize the degree distribution of linear codes for non-symmetric channels.
- Finite code simulation: The codeword-averaged performance of linear codes vs. the all-zero codeword performance of coset codes.
- Implementation: Use solely linear codes.

Summary

- Non-symmetric channels

Summary

- Non-symmetric channels
- Linear codes vs. the coset code ensemble

Summary

- Non-symmetric channels
- Linear codes vs. the coset code ensemble
- Convergence in distribution with respect to the check node degree d_c

Summary

- Non-symmetric channels
- Linear codes vs. the coset code ensemble
- Convergence in distribution with respect to the check node degree d_c
- Typicality results on Z channels: $< 0.05\%$ when $d_c \geq 6$

Summary

- Non-symmetric channels
- Linear codes vs. the coset code ensemble
- Convergence in distribution with respect to the check node degree d_c
- Typicality results on Z channels: $< 0.05\%$ when $d_c \geq 6$
- Analysis, simulations, and implementations