Low-Density Parity-Check Codes for Symmetric and Non-Symmetric Channels

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Department of Electrical Engineering, Princeton University

Content

- Part I — Low-density parity-check (LDPC) codes on symmetric & non-symmetric memoryless channels:

- Part II — EXtrinsic InformaTion (EXIT) chart & Finite-dim. bounds for LDPC codes.

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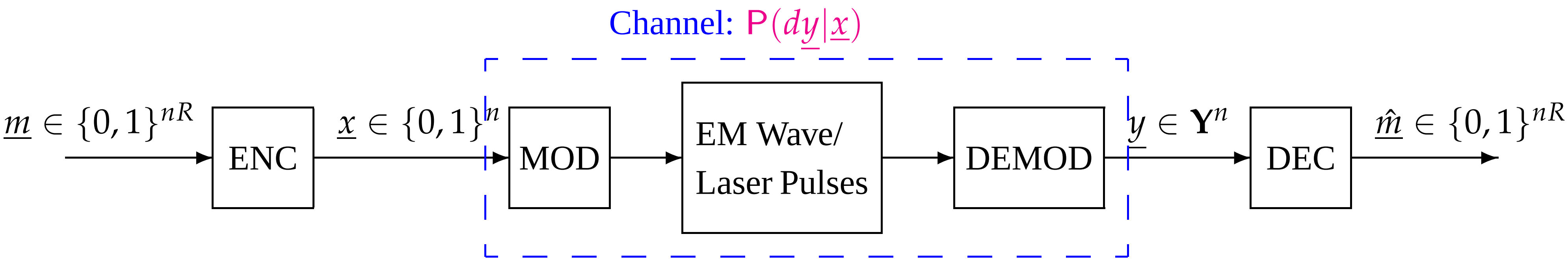
Error Correcting Codes



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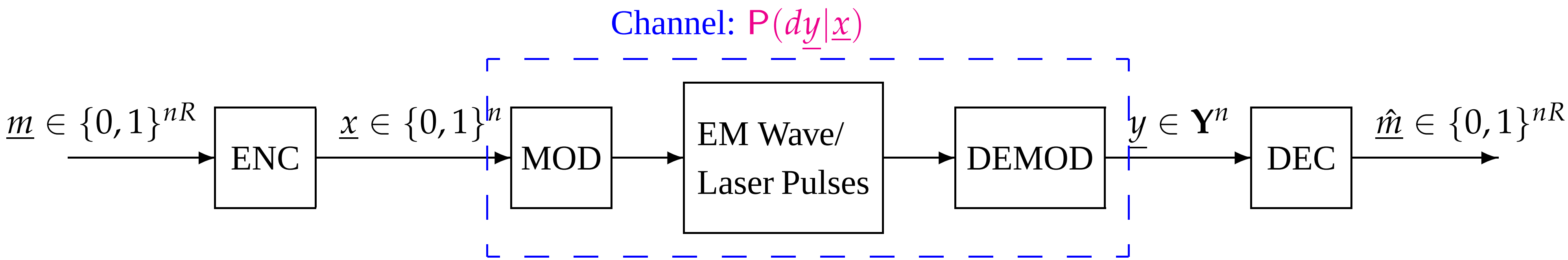

- Memoryless channels: $\mathsf{P}(d\underline{y}|\underline{x}) = \prod_{i=1}^{n} \mathsf{P}(dy_i|x_i)$.
- Symmetric channels: $\mathsf{P}(y|x=0) = \mathsf{P}(-y|x=1)$.

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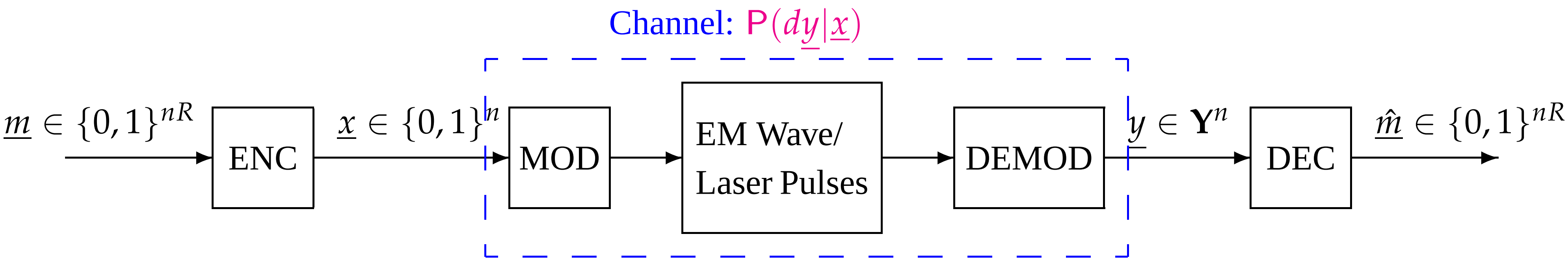

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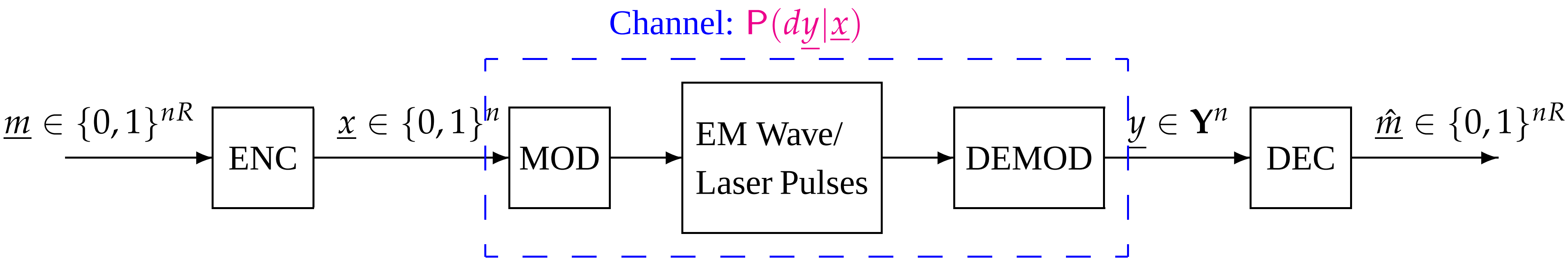

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- Shannon's channel coding theorem: $R < C := \max_{P_X} I(X;Y)$.
- Memoryless symmetric channels: Capacity-approaching error correcting codes have been constructed, including turbo codes, low-density parity-check (LDPC) codes, irregular RA codes, LT codes, concatenated tree codes, etc.
- Well established analysis tools.
- Performance: 0.1∼1.5dB away from capacity.

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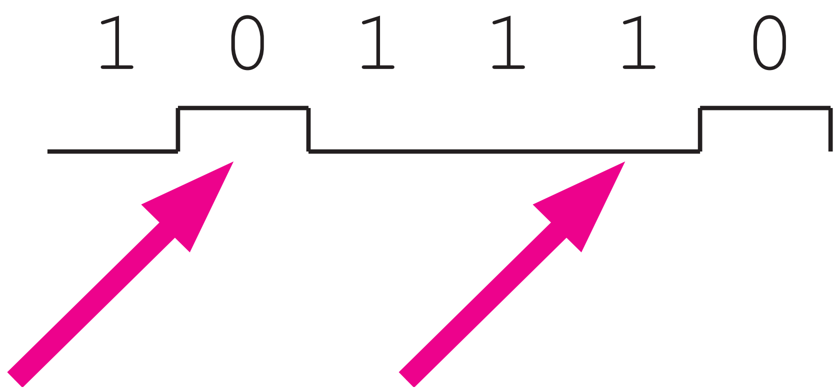

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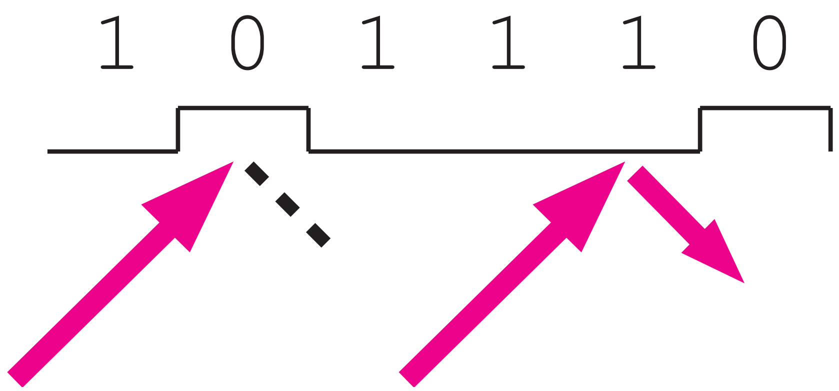

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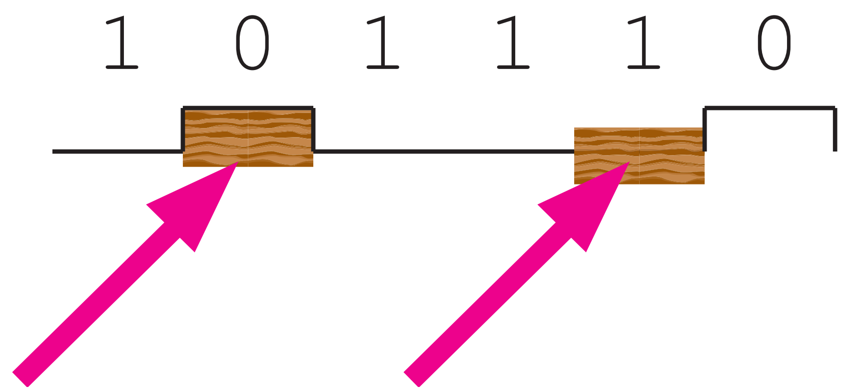

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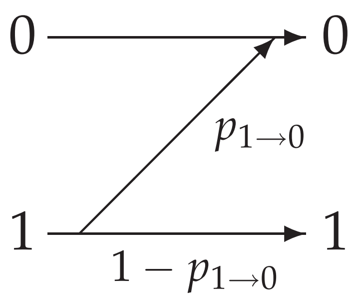

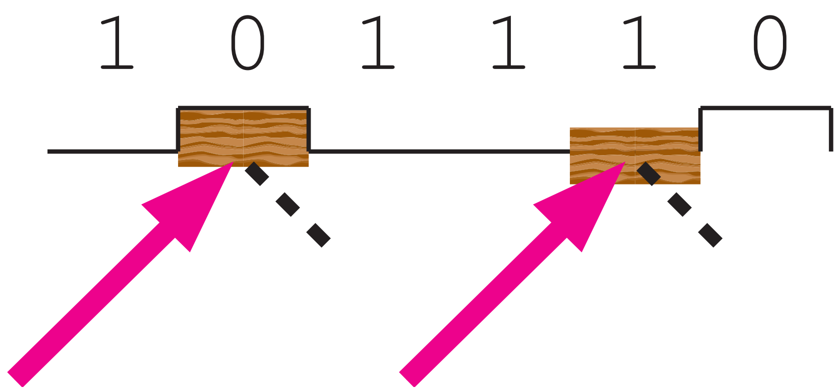

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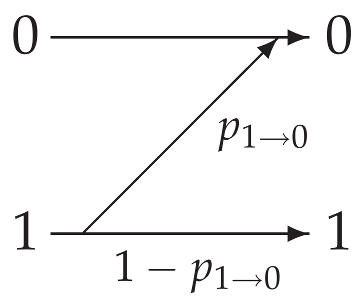
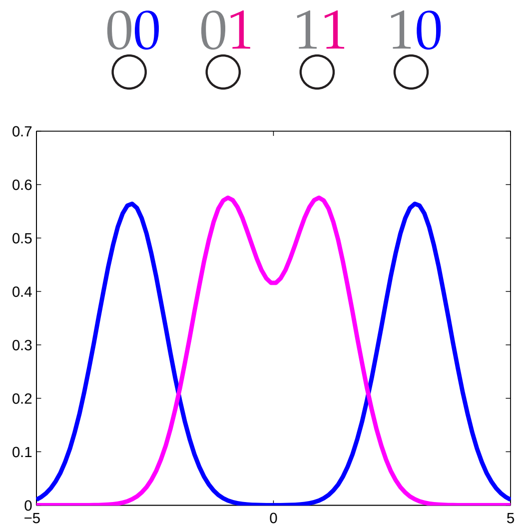


BICM



LDPC Codes & BP Decoders

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

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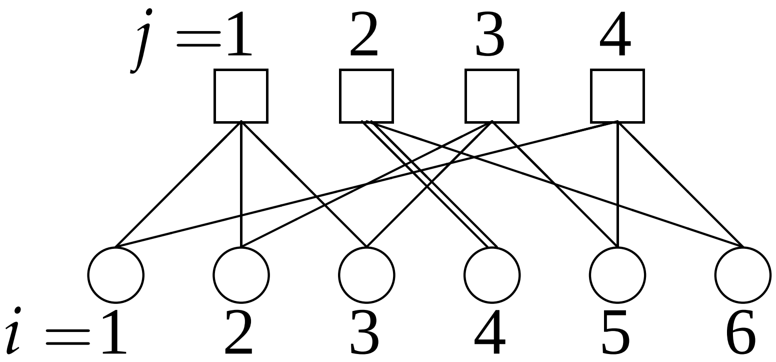

$\mathcal{C}^6(d_v, d_c), d_v = 2, d_c = 3$

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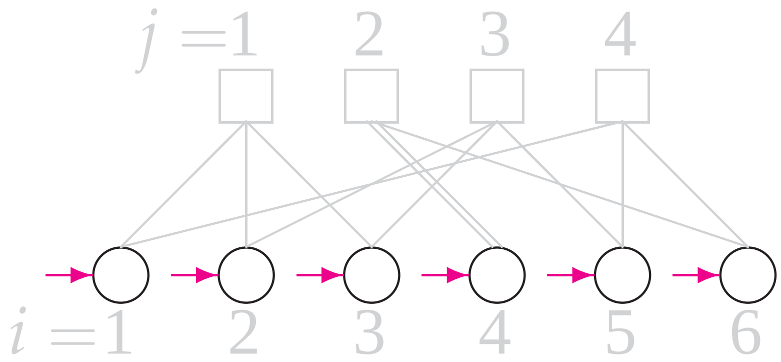

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Message Passing Algorithms

(i) m_0, (ii) $\Psi_v(m_0, m_1, \cdots, m_{d_v-1})$, (iii) $\Psi_c(m_1, \cdots, m_{d_c-1})$

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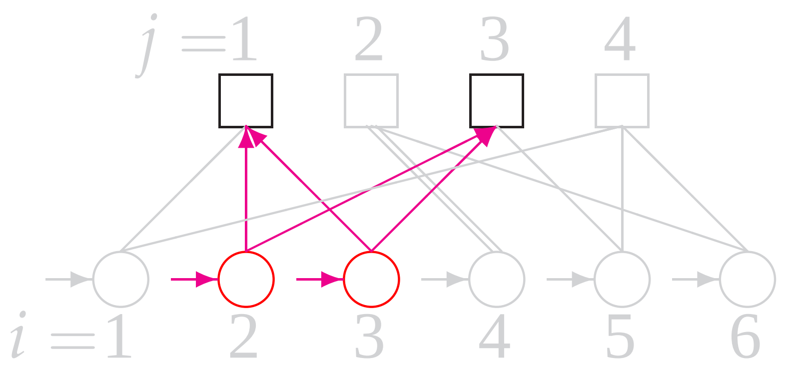

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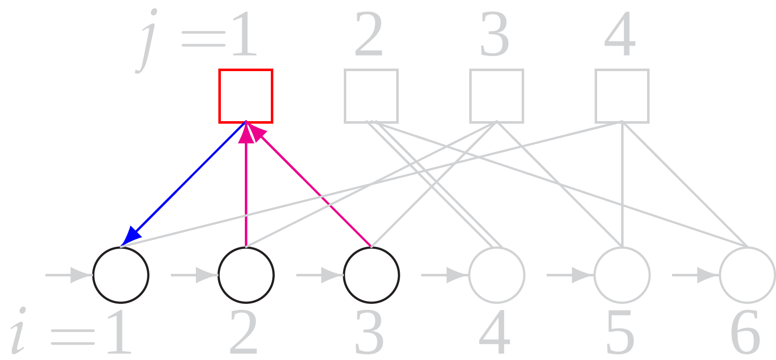

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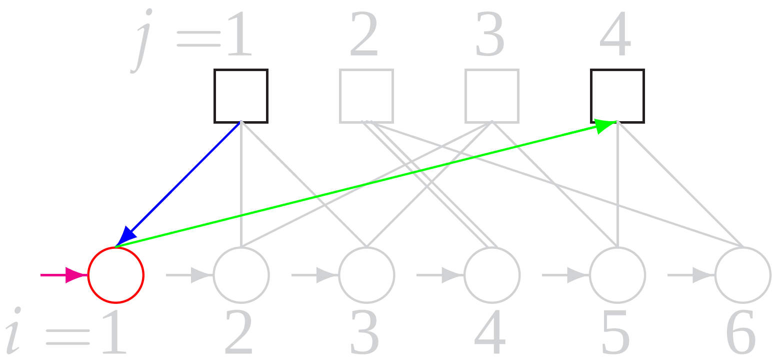

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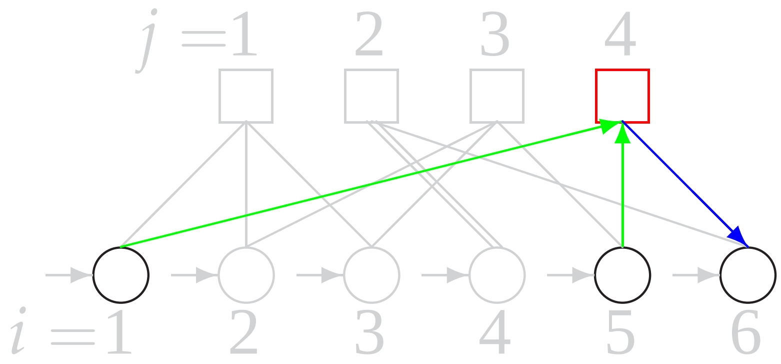

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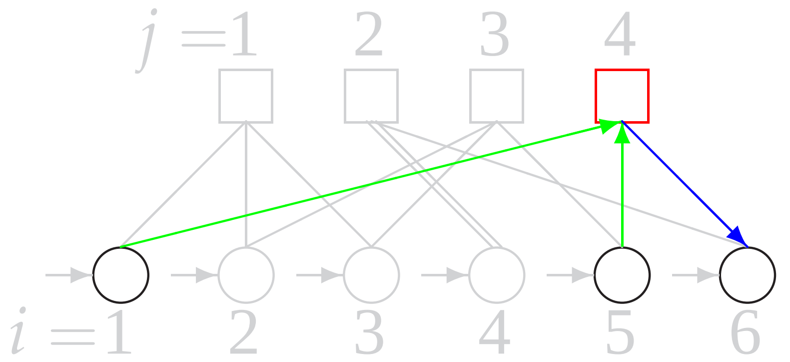

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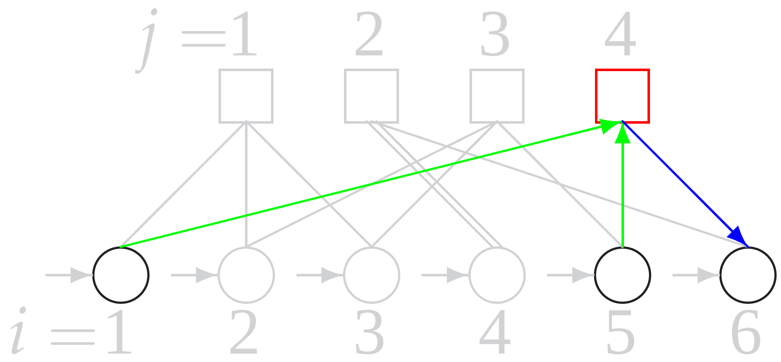

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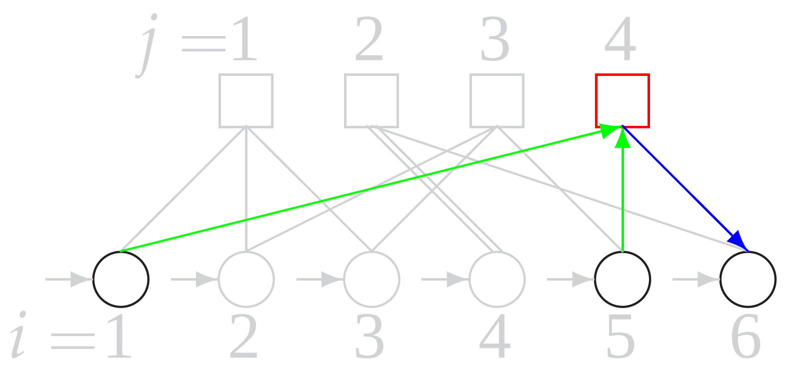

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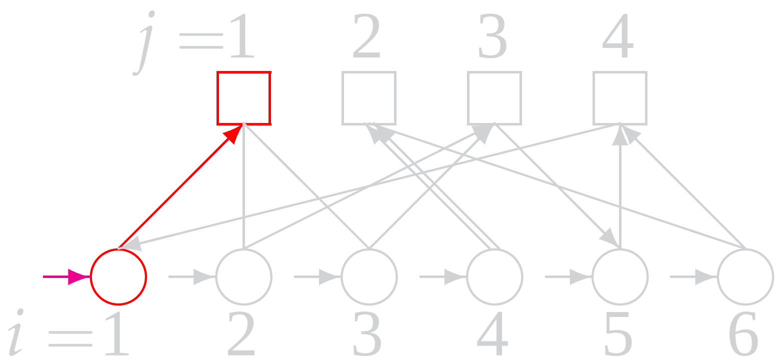
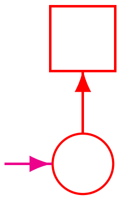
BP Decoder	Optimality	Analysis Tools	Simulation
Cycle-free + Non-Sym. Chs.	✓	✓	optimal
LDPC Codes + Sym. Chs.	?	✓	outstanding
LDPC Codes + Non-Sym. Chs.	?	?	very good

The Density Evolution





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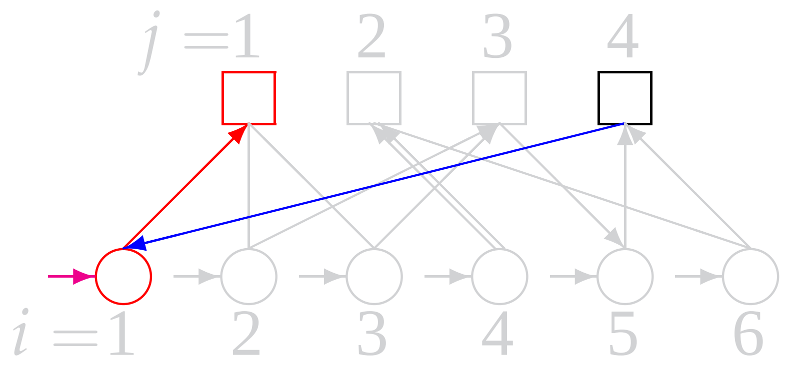

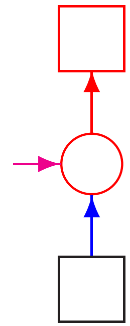

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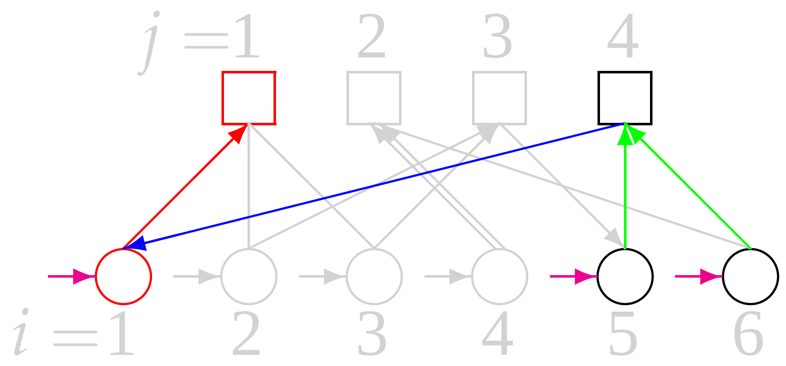

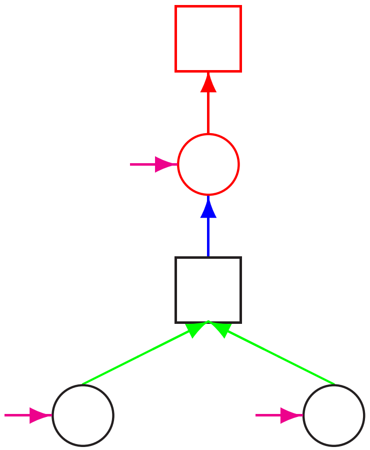

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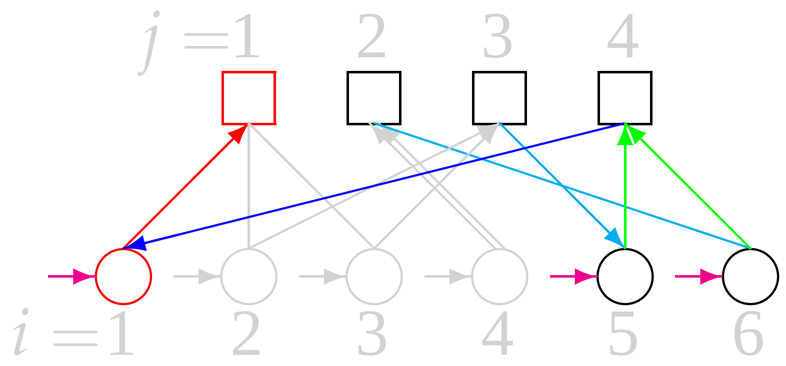

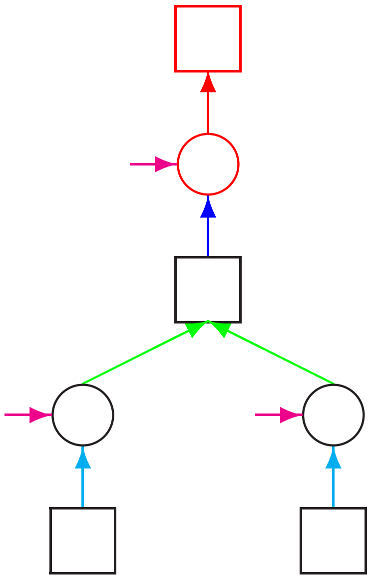

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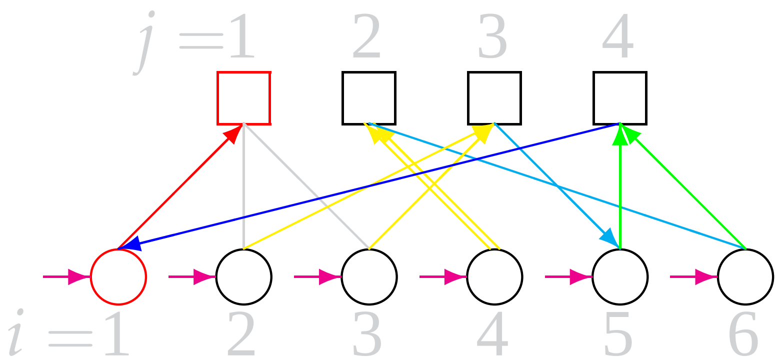

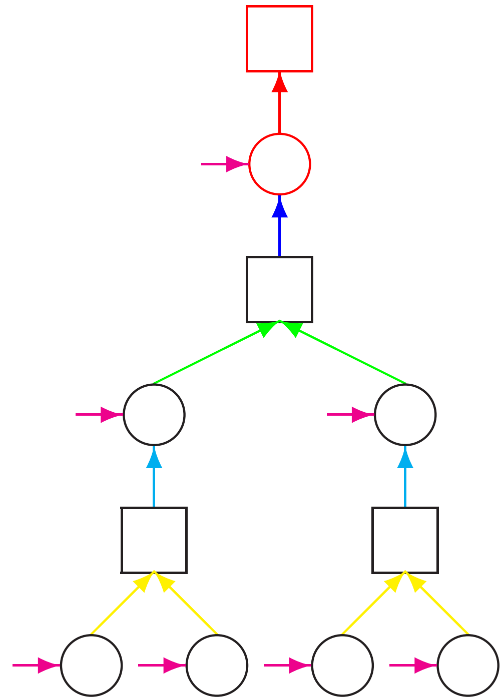

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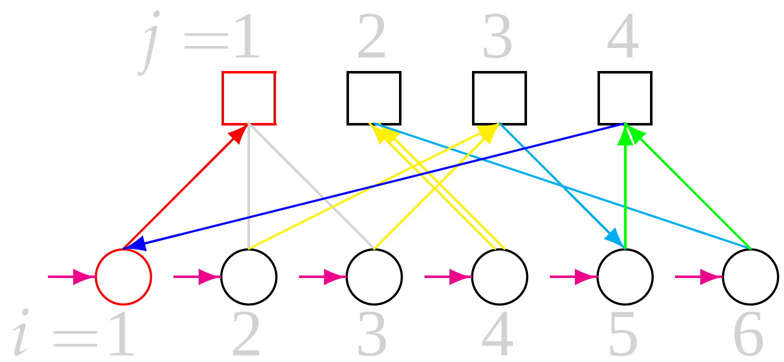
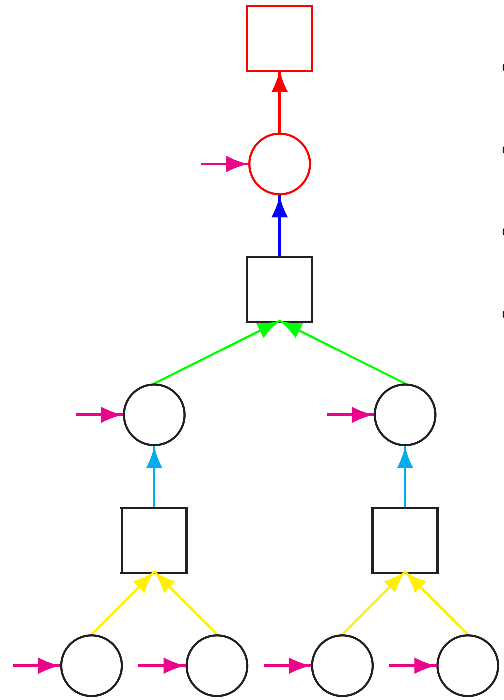
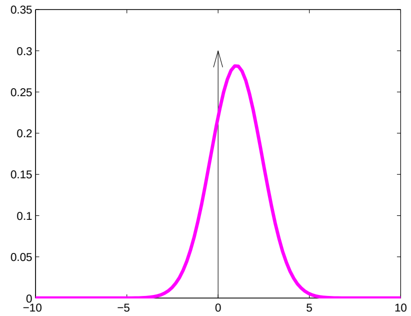
Sym. Chs: Assuming $\mathbf{x} = \mathbf{0}$.







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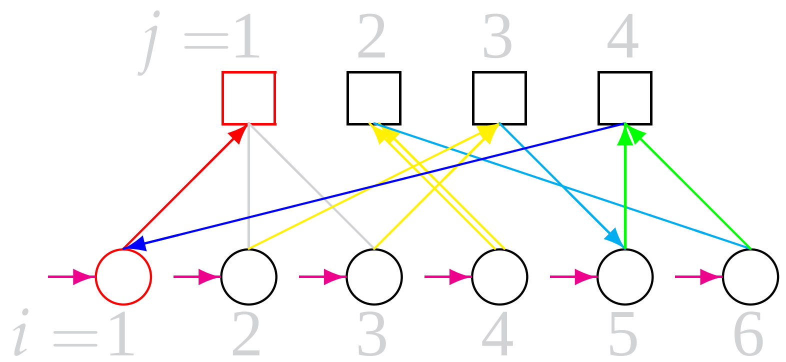

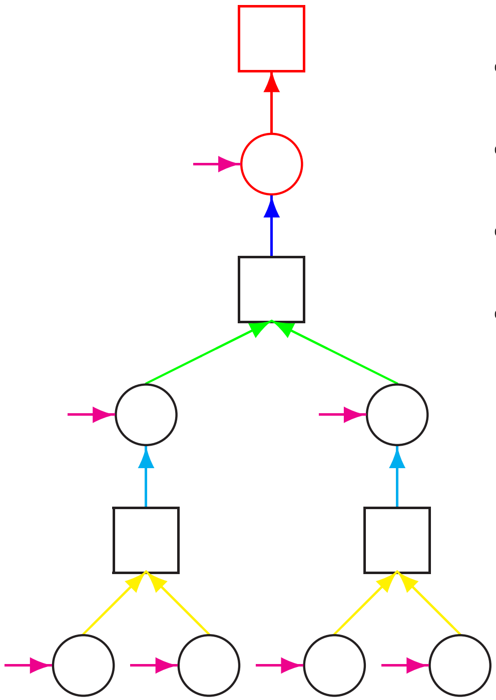

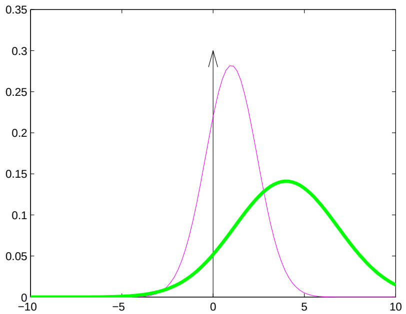

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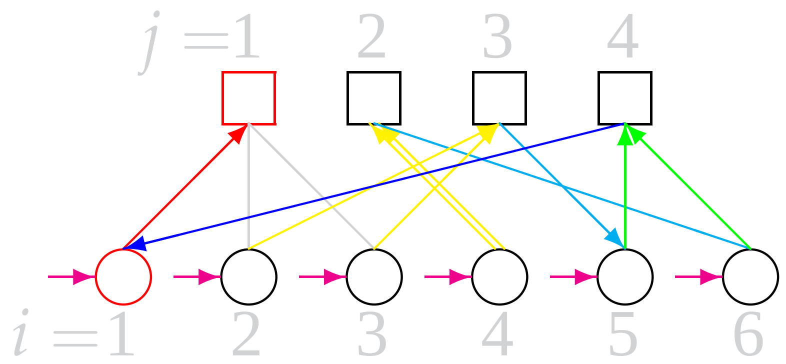

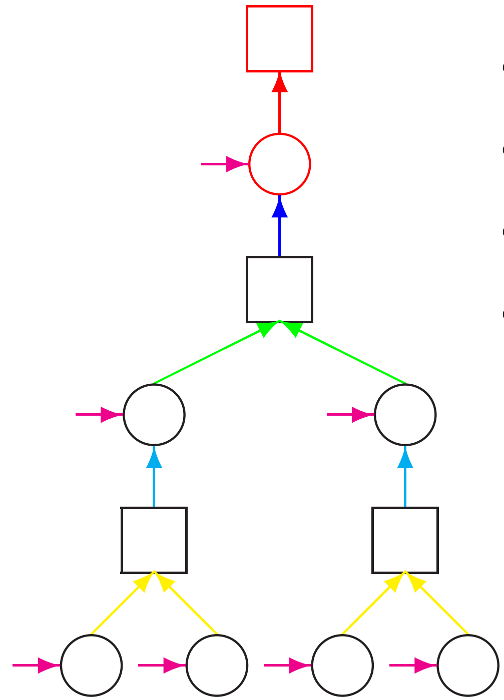

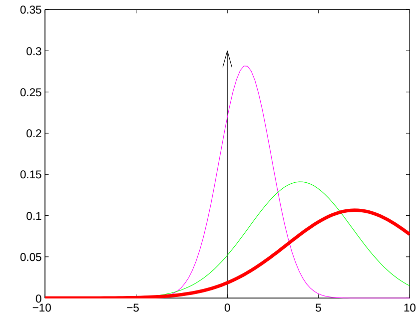

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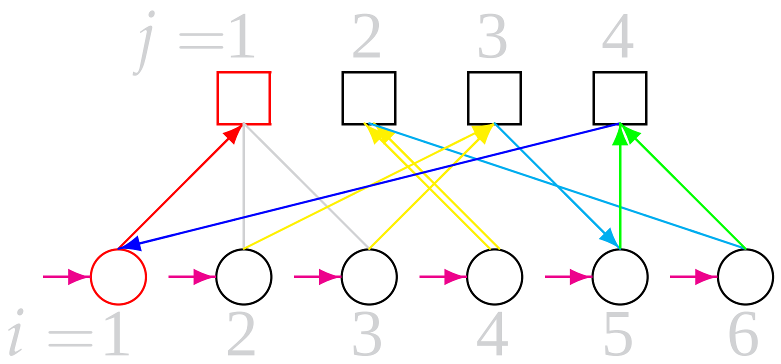

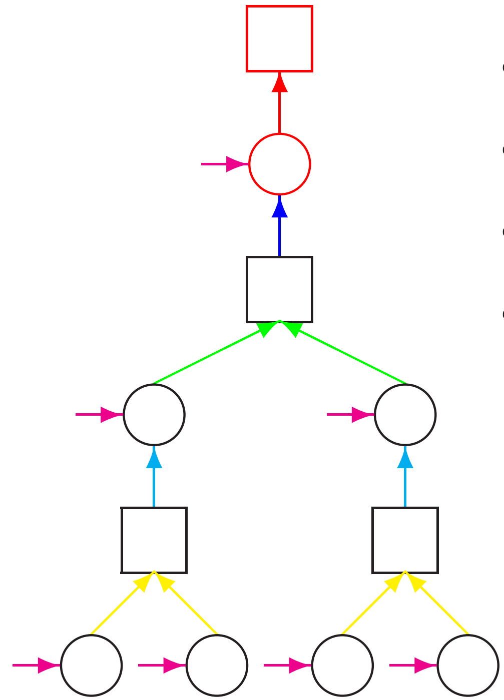

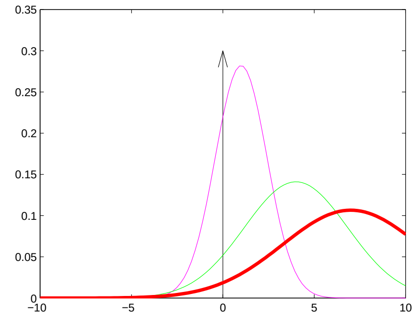

$$
\begin{aligned}
P^{(l)} &= P^{(0)} \otimes \left(Q^{(l-1)}\right)^{\otimes(d_v-1)} \\
Q^{(l-1)} &= \Gamma^{-1}\left(\left(\Gamma\left(P^{(l-1)}\right)\right)^{\otimes(d_c-1)}\right)
\end{aligned}
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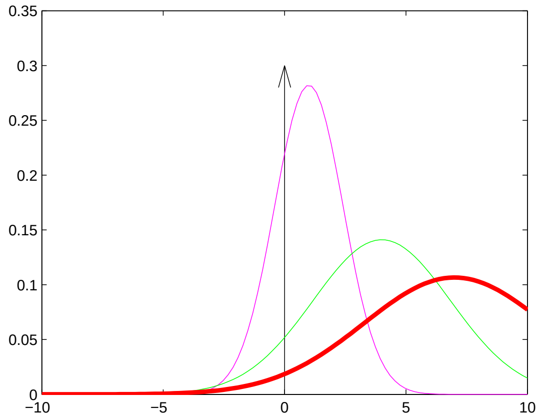

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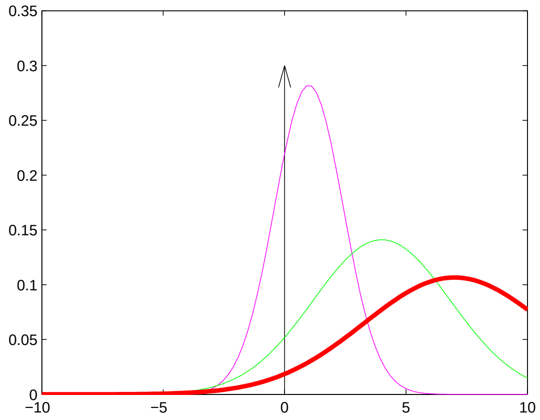
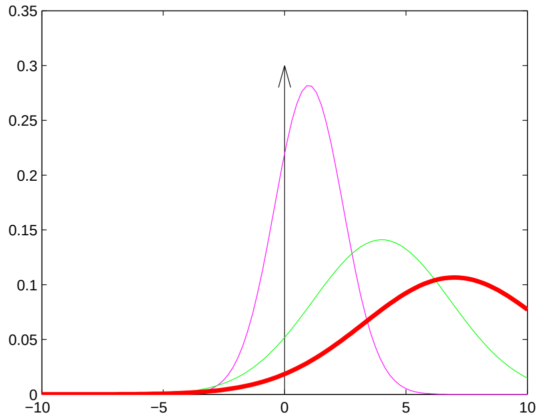
- In practice, the performance for $n = 10^4 \sim 10^6$ bits is well-predicted.

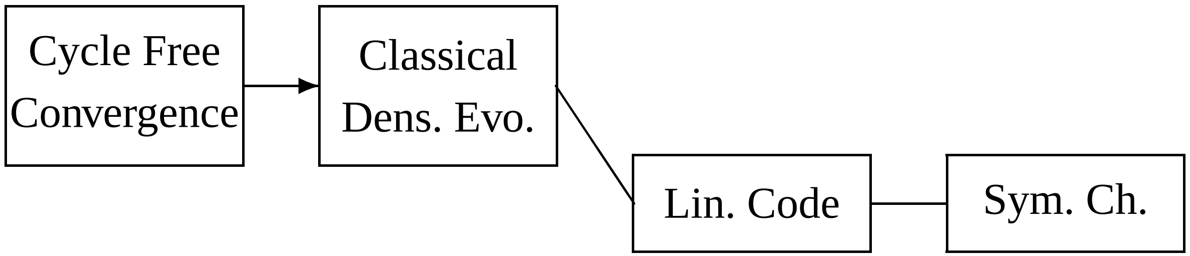


The Density Evolution (Cont'd)

- [Richardson et al. 01] The DE perfectly describes the decoding behavior for sufficiently large codeword length n.
- The tail error probability goes to zero $\Longleftrightarrow$ the channel of interest is decodable.



- In practice, the performance for $n = 10^4 \sim 10^6$ bits is well-predicted.
- Advantages:
  - Accurate performance prediction
  - Deterministic, one-time computation instead of random Monte Carlo simulations
  - Can be used as the ultimate performance metric for comparisions/code optimization



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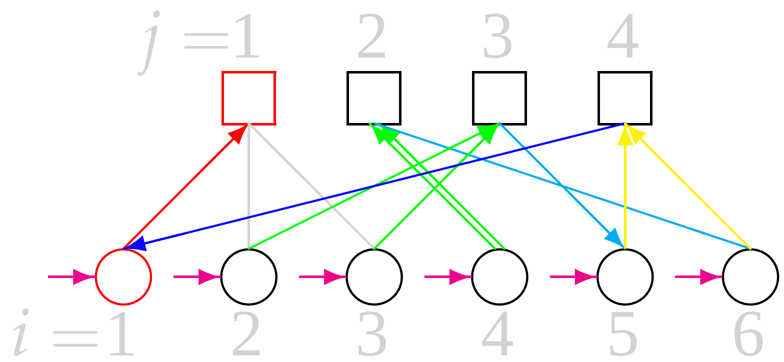
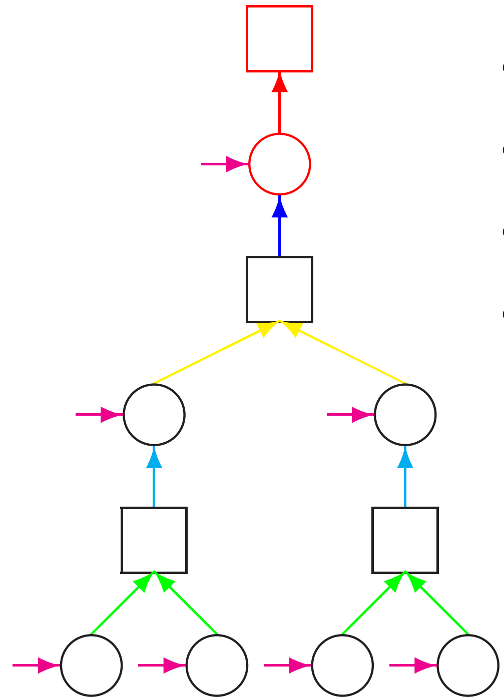
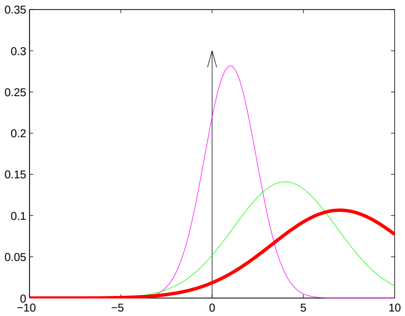
Codeword Dependence of DE

Sym. Chs: Assuming $\mathbf{x} = \mathbf{0}$.







$$P^{(l)} = P^{(0)} \otimes \left(Q^{(l-1)}\right)^{\otimes(d_v-1)}$$

$$Q^{(l-1)} = \Gamma^{-1}\left(\left(\Gamma\left(P^{(l-1)}\right)\right)^{\otimes(d_c-1)}\right)$$

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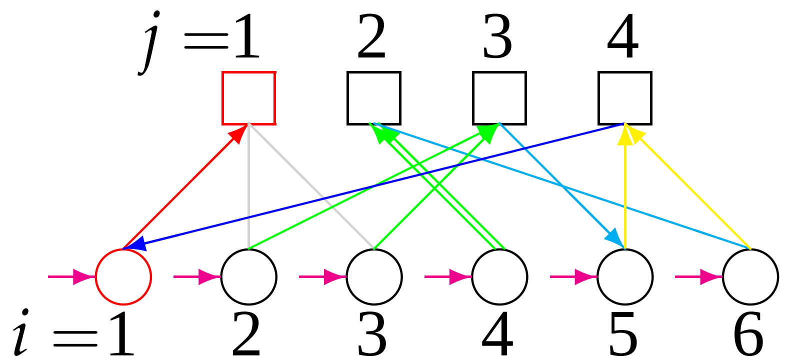
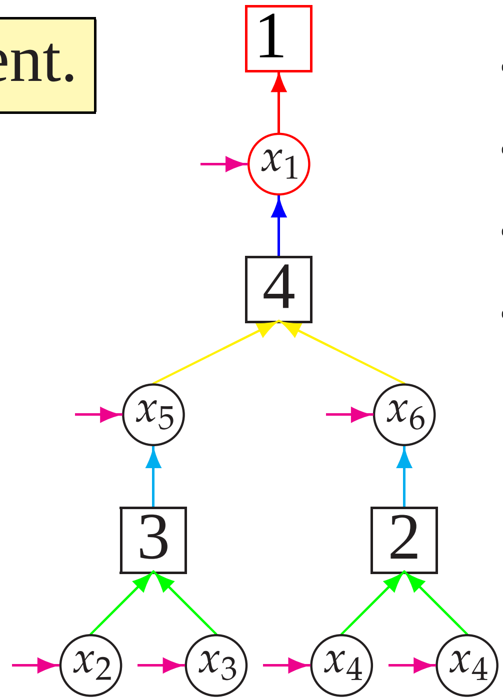
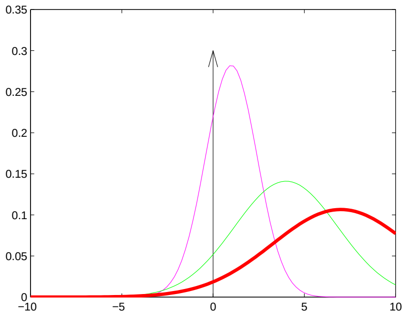
Non-sym. Chs: Codeword-dependent.







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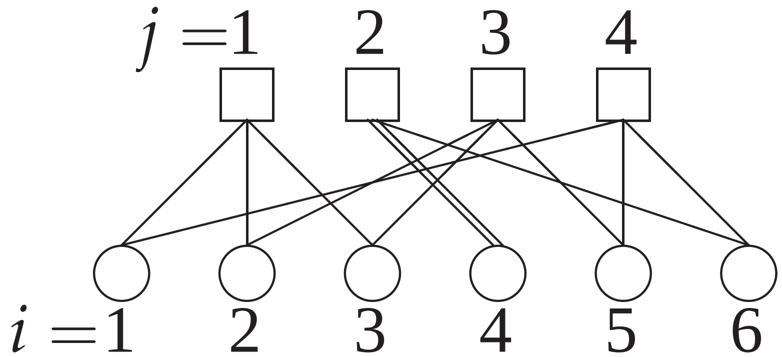

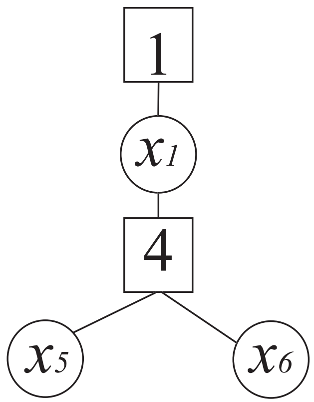

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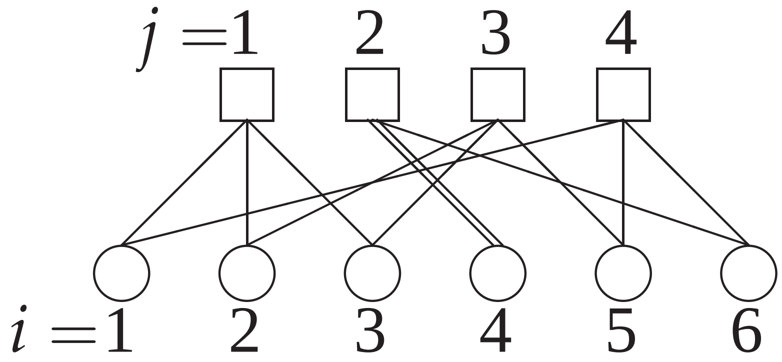
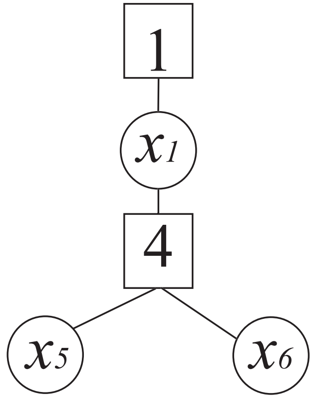


$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\mathbf{Hx} = \mathbf{0} \Longleftrightarrow x_6 = 0$$



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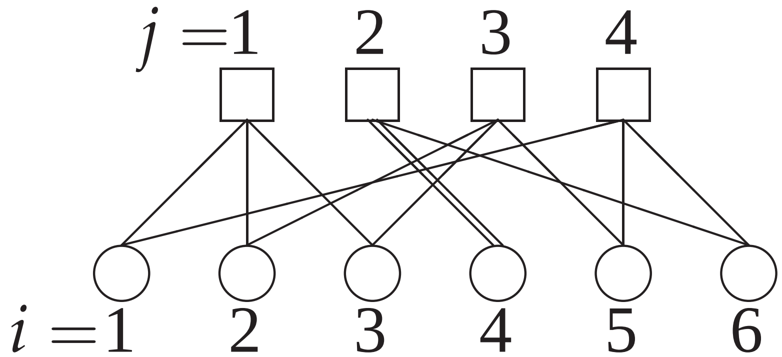

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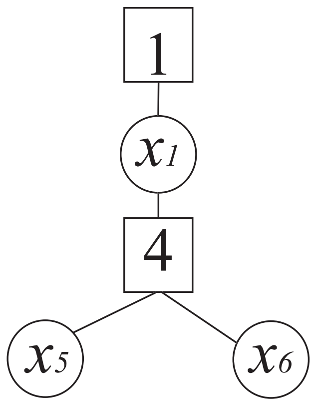

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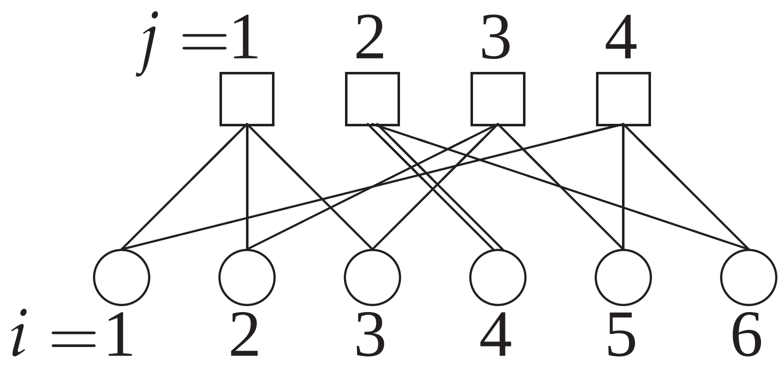

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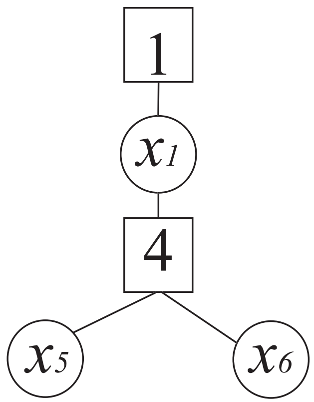

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No matter how large the tree is, averaging the trimmed tree code may not be equivalent to averaging the original code.

Perfect Projection Condition

Definition (Perfect Projection) The supporting tree $\mathcal{N}^{2l}$ is perfectly projected, if

$$\frac{\left|\{\mathbf{x}\in\mathbf{X}:\mathbf{x}|_{\text{tree}}=\mathbf{x}_t\}\right|}{|\mathbf{X}|}=\frac{1}{|\mathbf{X}_t|}.$$

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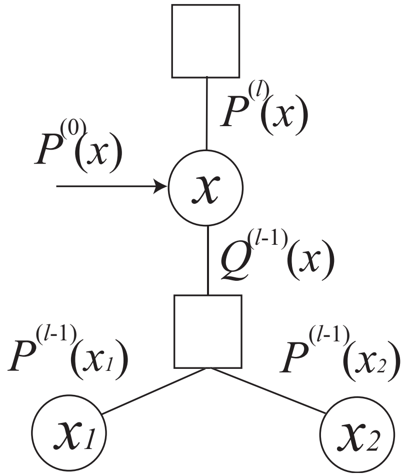
$$\begin{aligned} P^{(l)}(x) &:= \left\langle P^{(l)}(\mathbf{x})\right\rangle_{\{\mathbf{x}\in\mathbf{X}:\mathbf{x}|_0=x\}} = \left\langle P^{(l)}(\mathbf{x}_t)\right\rangle_{\{\mathbf{x}_t\in\mathbf{X}_t:\mathbf{x}_t|_0=x\}} \\ p_e^{(l)} &= \frac{1}{2}\left(\int_{m=-\infty}^{0} P^{(l)}(0)(dm) + \int_{m=-\infty}^{0} P^{(l)}(1)(dm)\right) \end{aligned}$$

New Iterative Formula for DE



$\because$ Parity-check equation,

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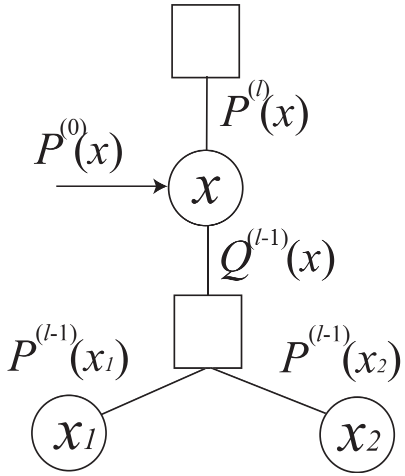

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A simplified expression

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- Existence of the cycle-free support tree.$\sim$ [Richardson *et al.* 01]

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For non-symmetric channels, the codeword-averaged performance is predicted by our generalized DE when the codeword length n is sufficiently large.

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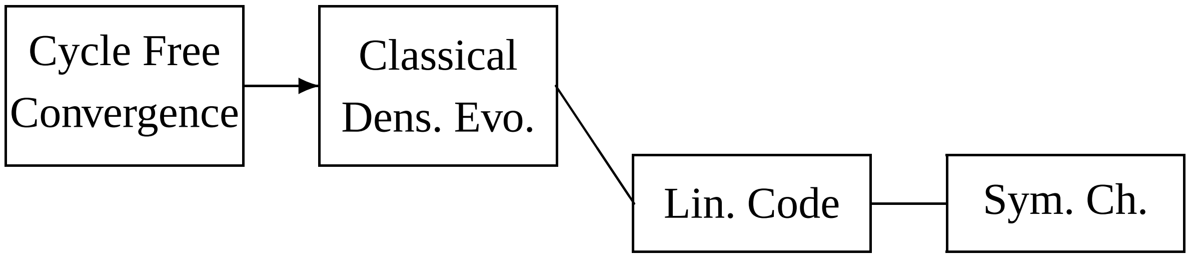
- The same computational complexity as the classical DE.



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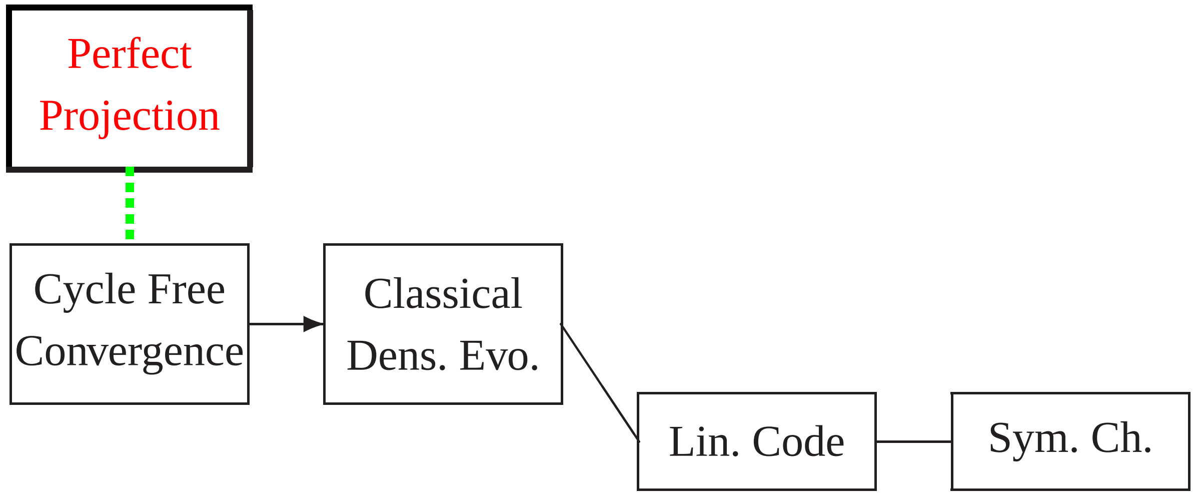

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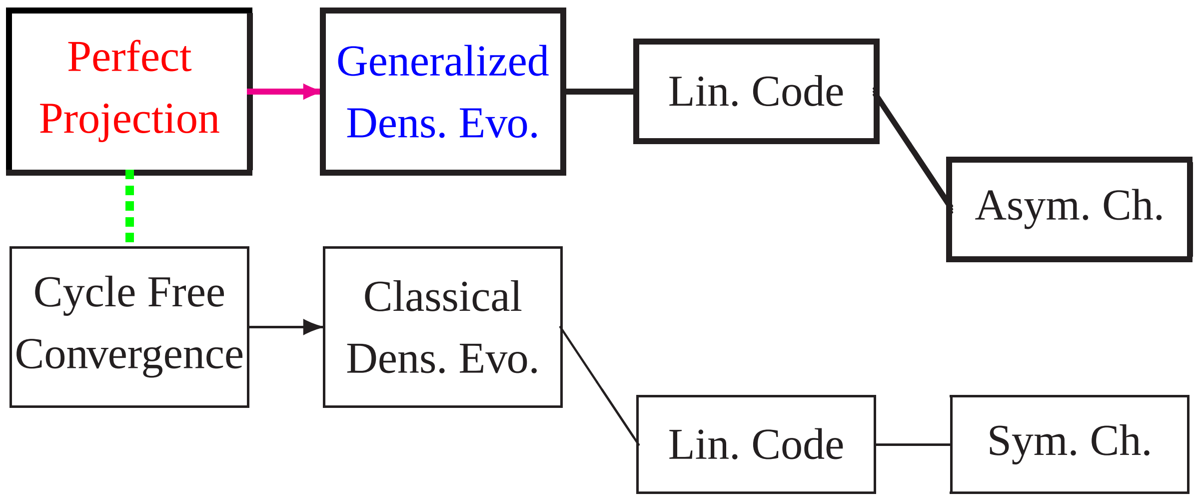

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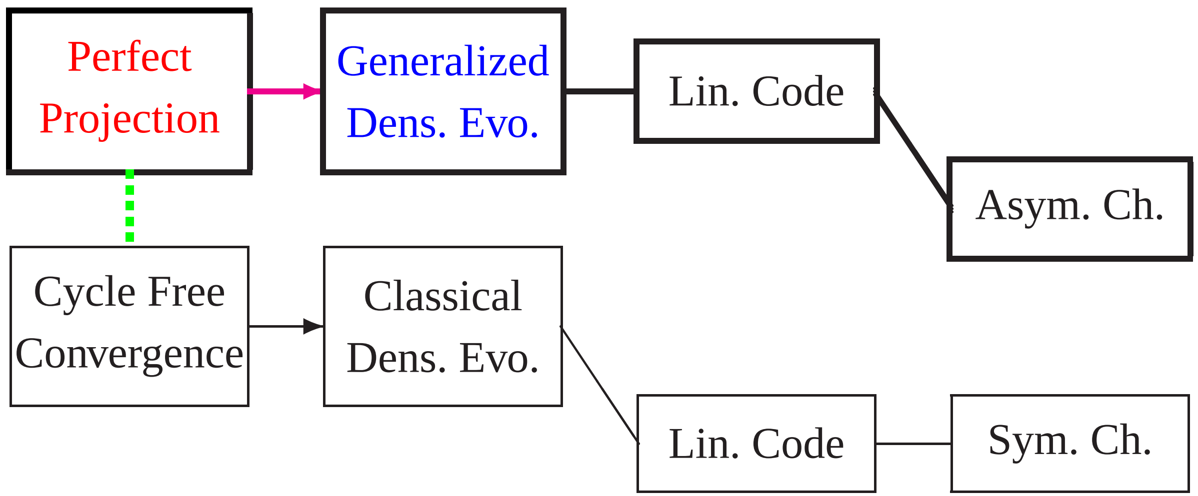

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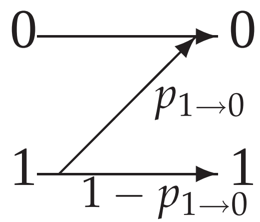
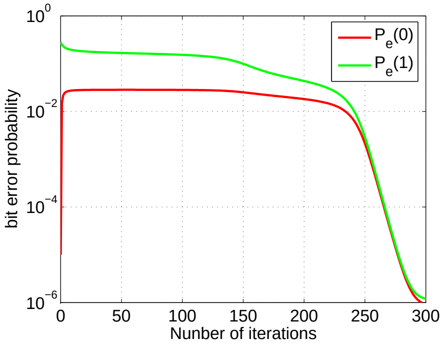
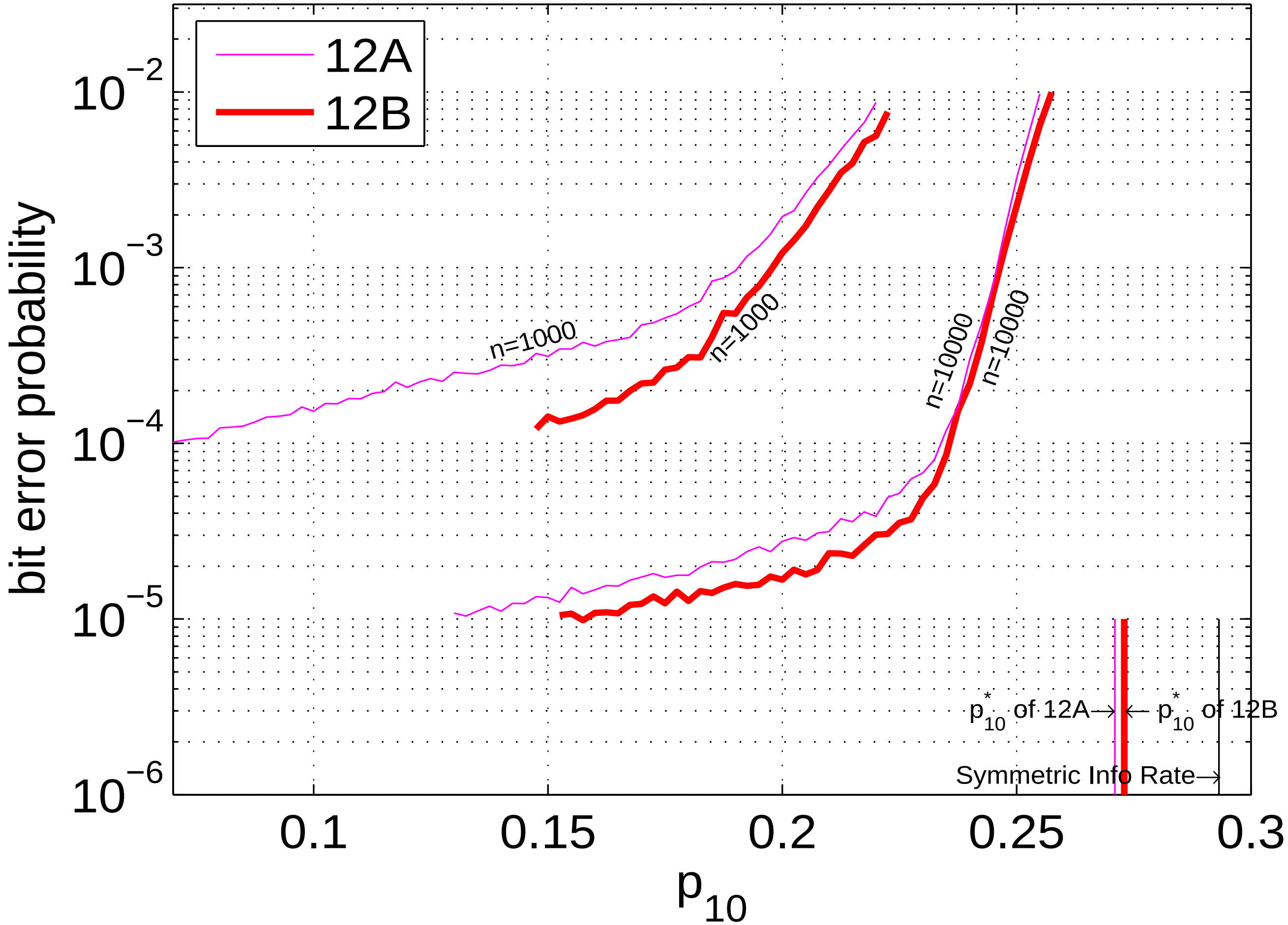
Code Optimization

Z-channels







12A: [Richardson01], optimized for BiAWGNC, maxDv=12.

12B: Optimized for z-channels, maxDv=12.

Optimized Rate 1/2 Codes

- Code complexity: $\max d_v = 20$ and $\max d_c = 10$.
  - Allowing 100 iterations: $p^*_{1\to 0} = 0.2741$.
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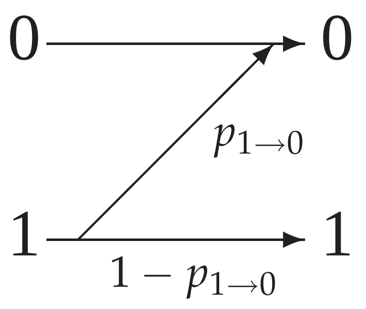

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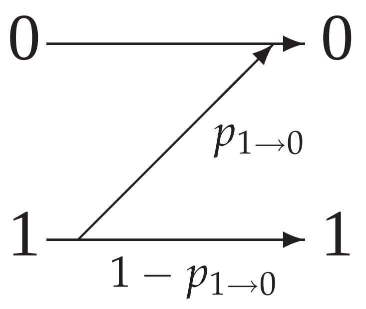

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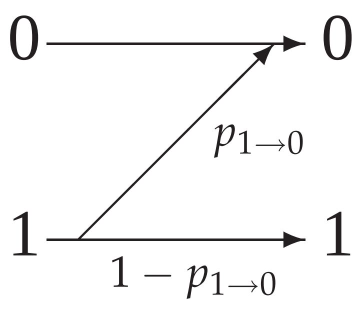

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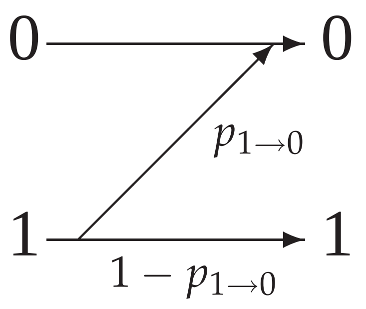

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- Not far
  - [Majani & Rumsey 91] showed that the ratio between the symmetric mutual information rate and the capacity is lower bounded by $\frac{e \ln 2}{2} \approx 94\%$.
  - [Shulman & Feder 04] further proved that the absolute difference is upper bounded by 0.011 bit/sym.

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- Describing the bipartite graph: Degree distribution polynomials,
  Variable nodes: $\lambda(x) := \sum \lambda_k x^{k-1}$
  Check nodes: $\rho(x) := \sum \rho_k x^{k-1}$.

A Stopping Criterion

if $\text{BNP}^{(0)} \geq \frac{1}{\lambda_2 \rho'(1)}$ **then**

    $\lim_{l \to \infty} p_e^{(l)} > 0 \Longleftrightarrow$ Undecodable.

else

    repeat

        Perform DE iteration

    until $\text{BNP}^{(l)} < \epsilon^*$

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8: **end if**

Example: Code 12B

$$\frac{1}{\lambda_2 \rho'(1)} = 0.6430$$

$$\text{BNP}^{(0)} = 0.5253$$

$$\epsilon^* = 0.0675$$

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Example: Code 12B

$\frac{1}{\lambda_2 \rho'(1)} = 0.6430$

$\text{BNP}^{(0)} = 0.5253$

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A Stopping Criterion

```
if $\mathrm{BNP}^{(0)} \geq \frac{1}{\lambda_2 \rho'(1)}$ then
    $\lim_{l\to\infty} p_e^{(l)} > 0 \iff$ Undecodable.
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LDPC Coset Code Ensemble

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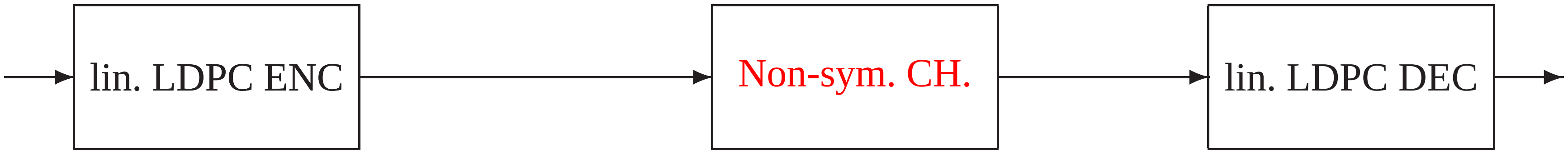

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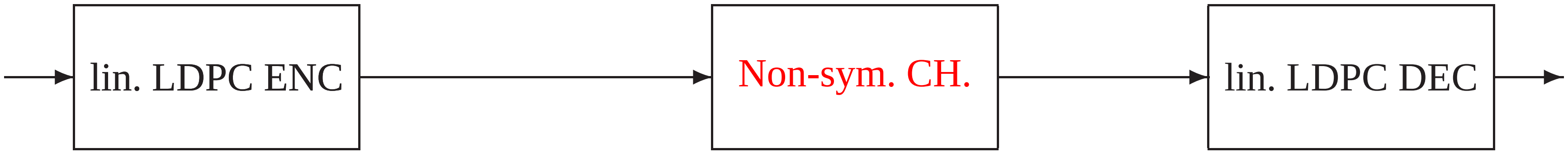

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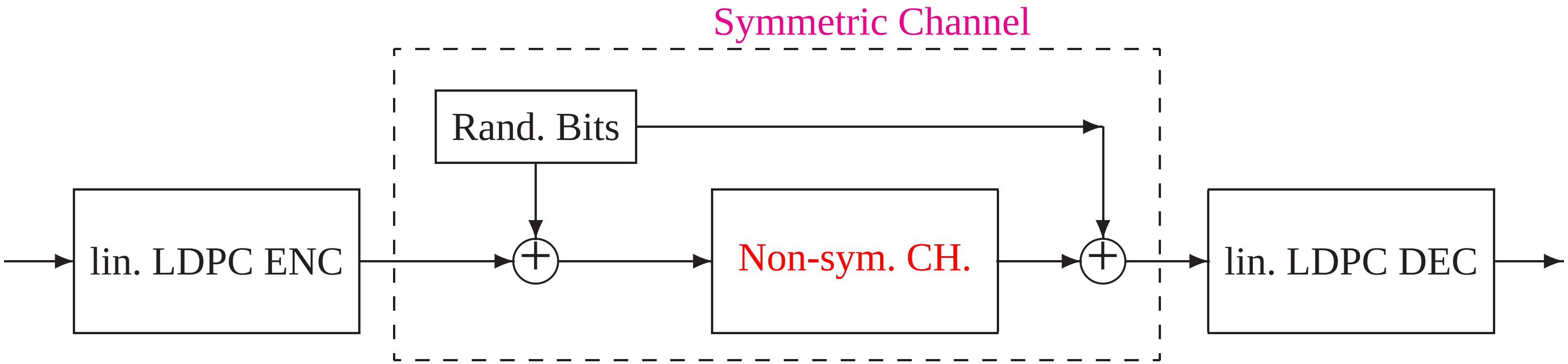

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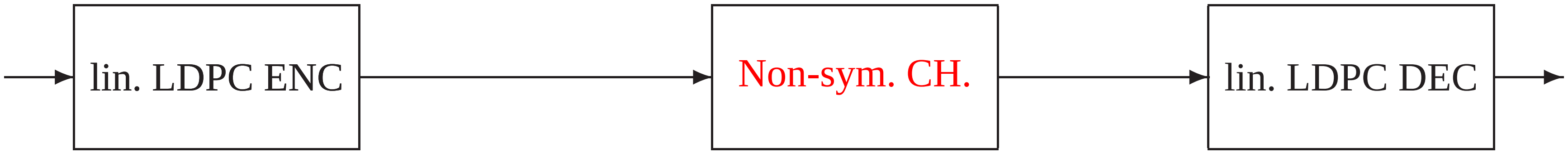

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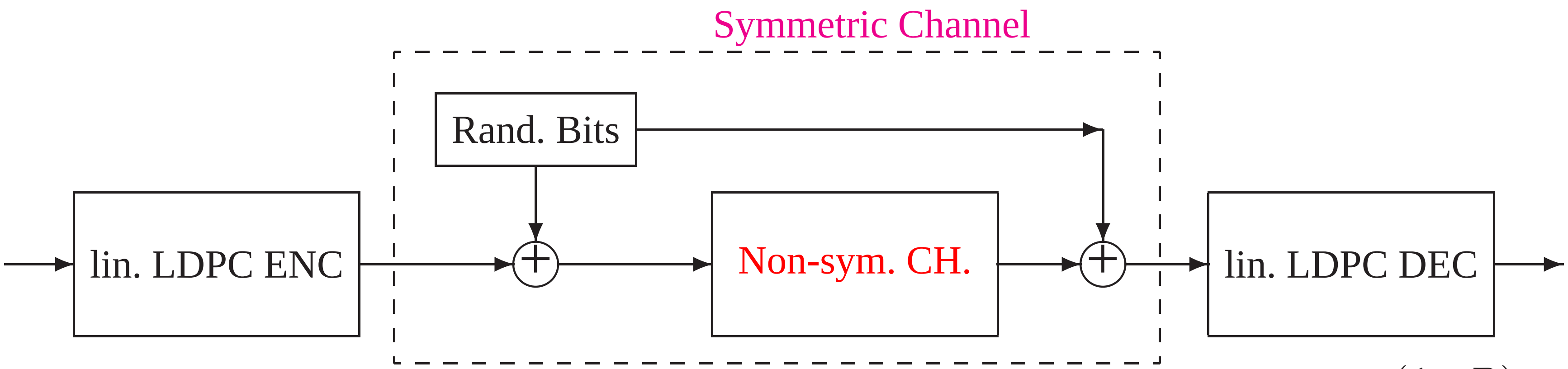

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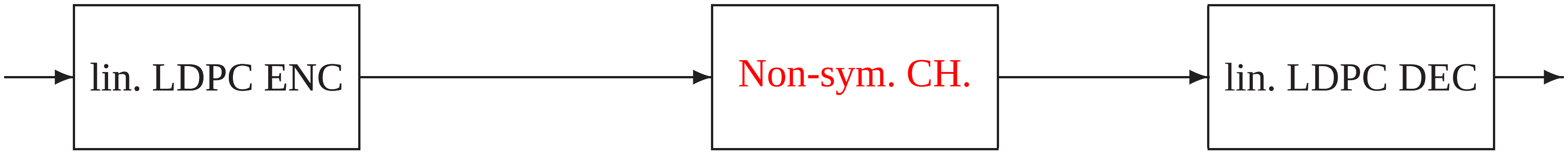

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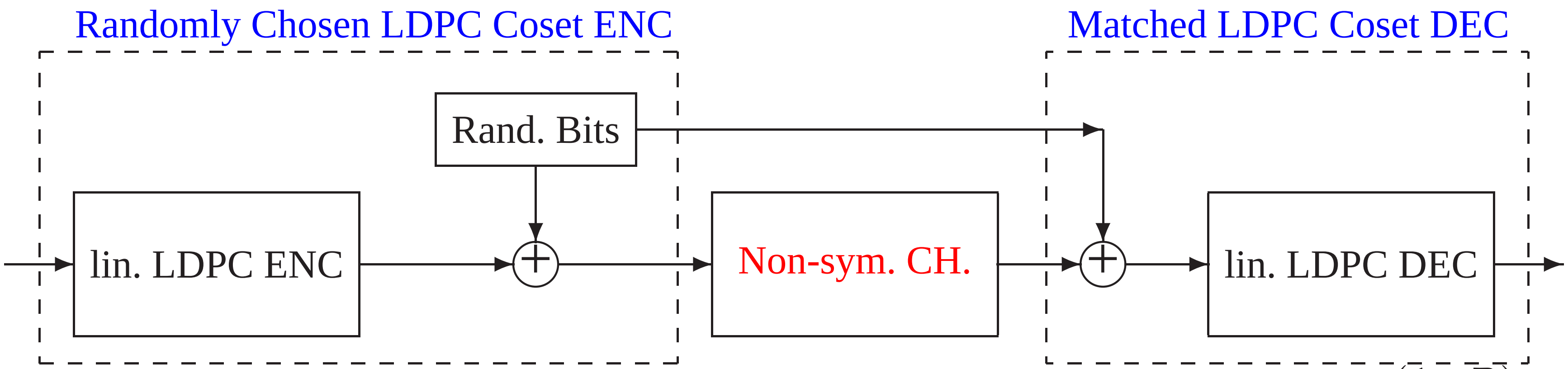

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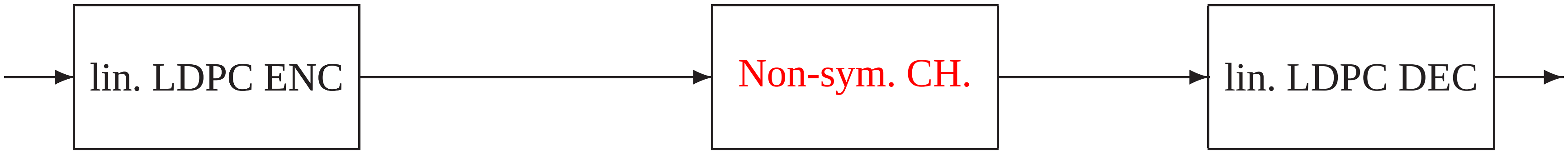

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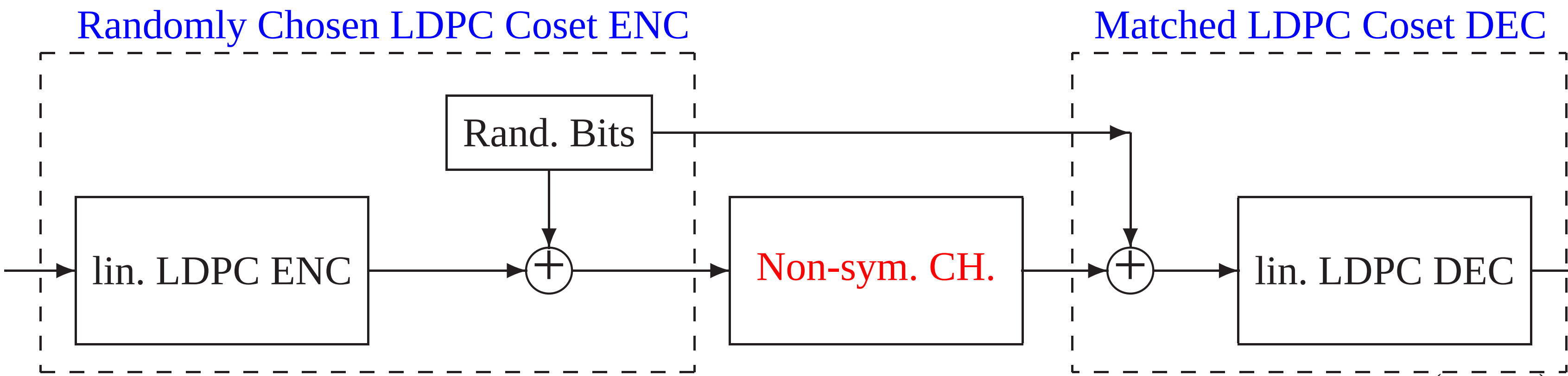

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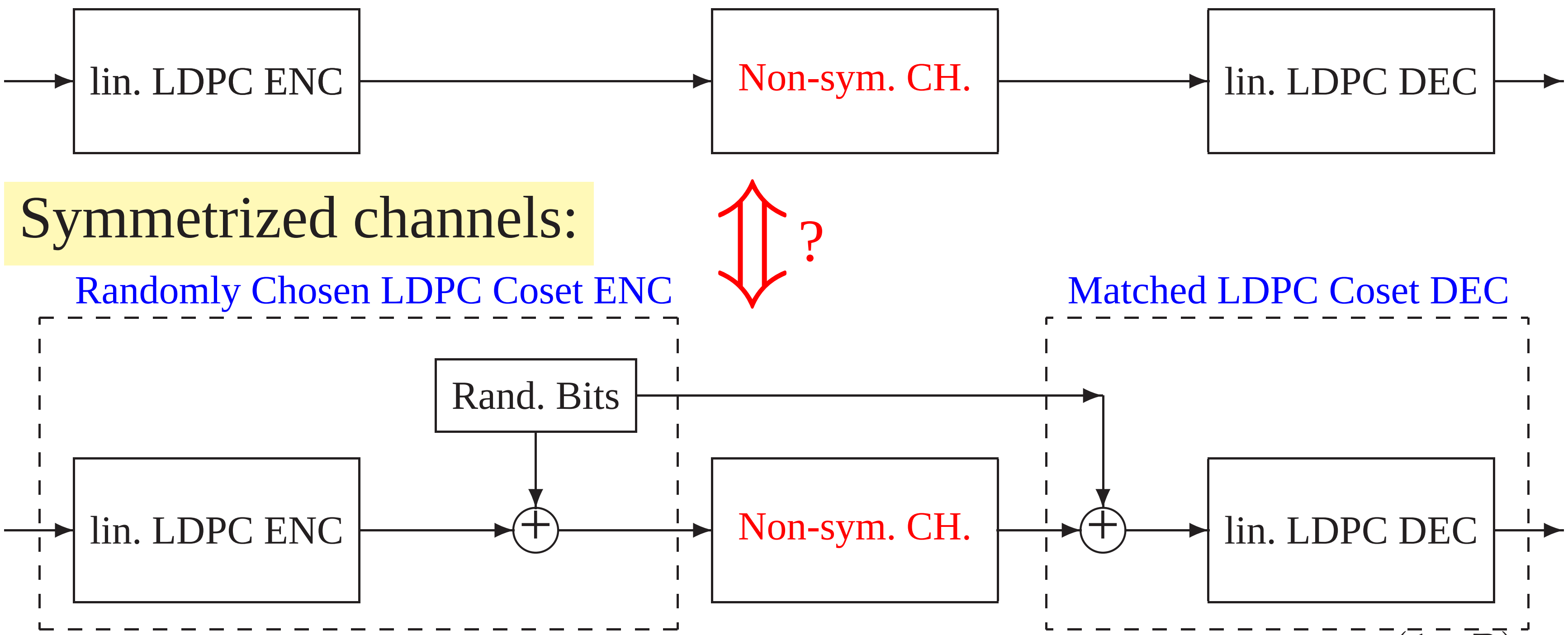

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Applications

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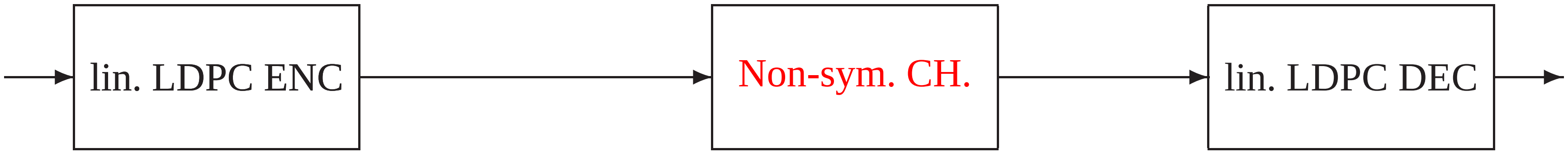

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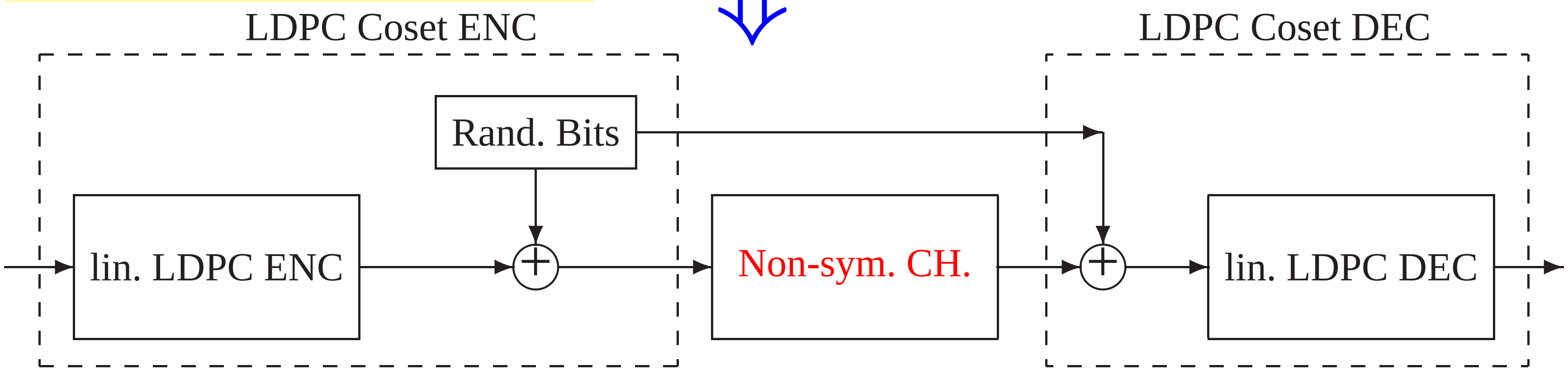

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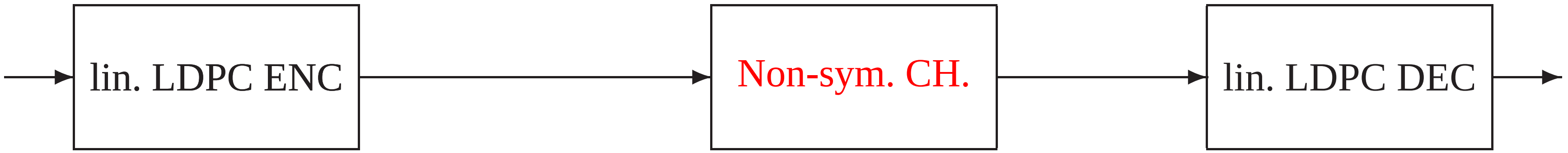

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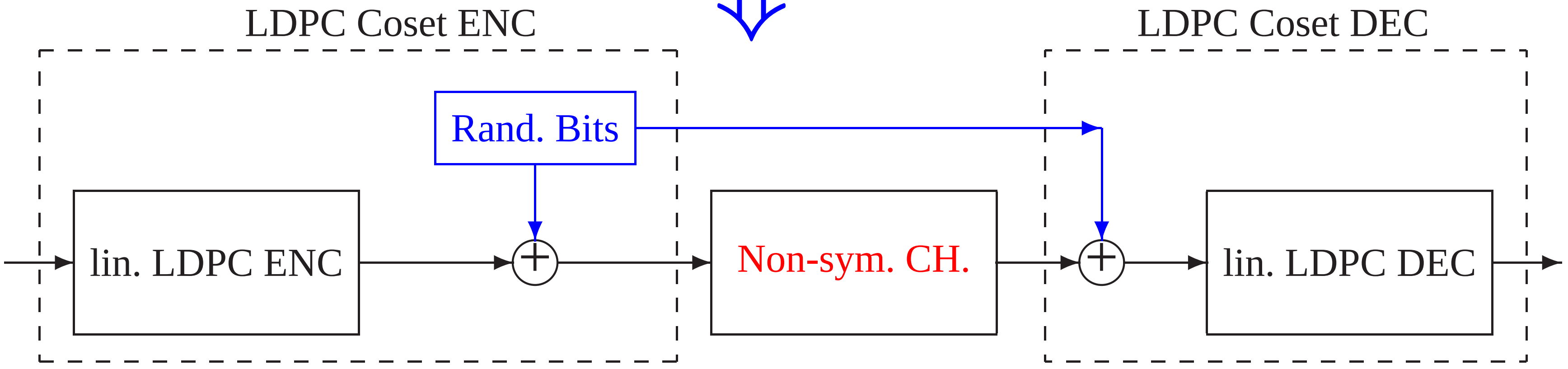

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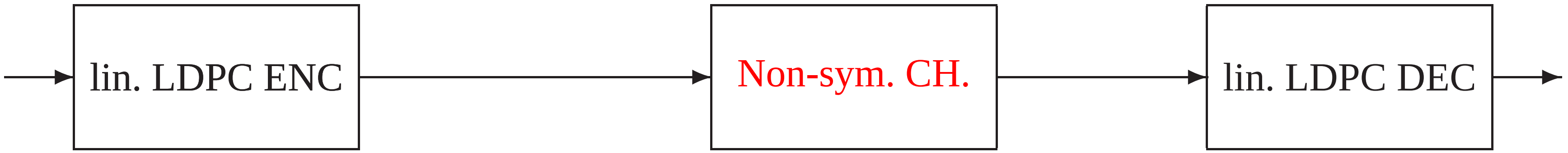

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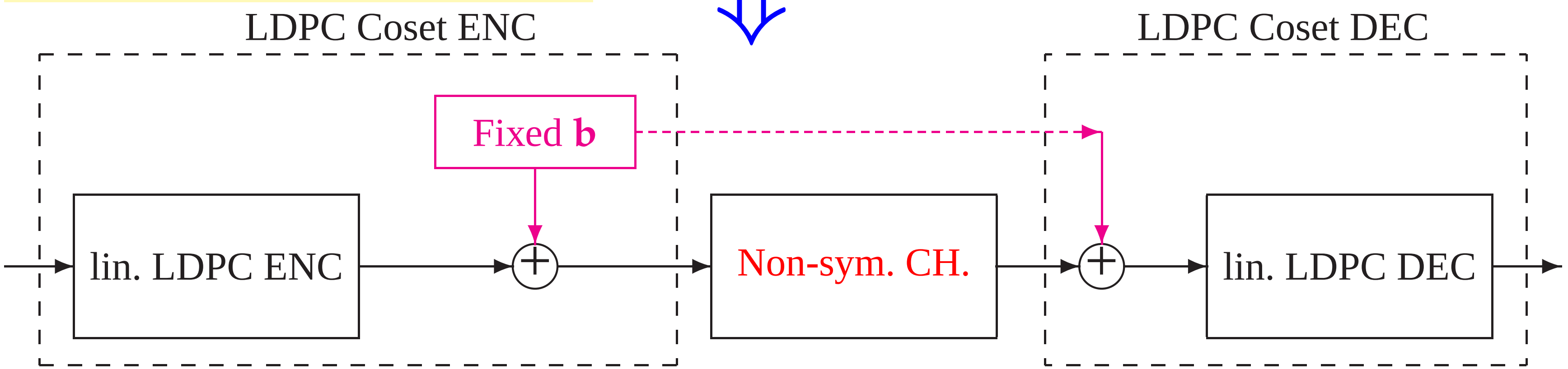

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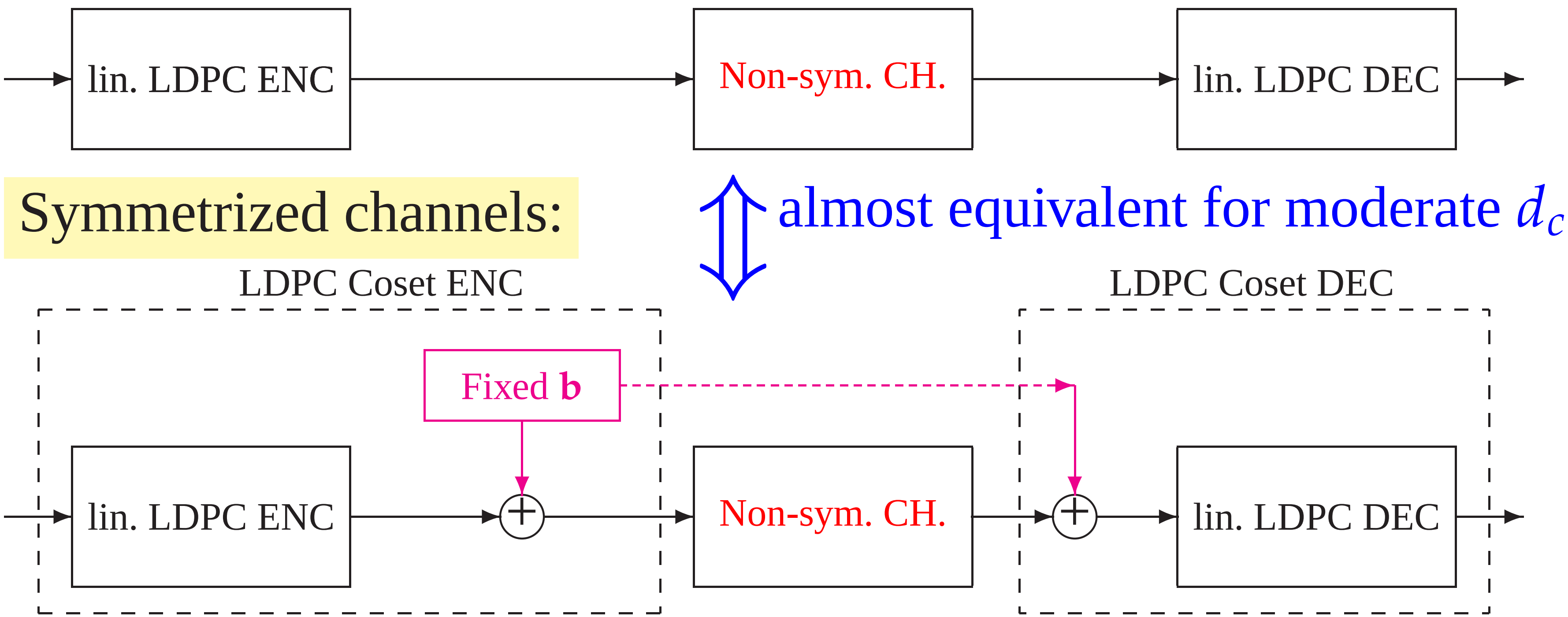

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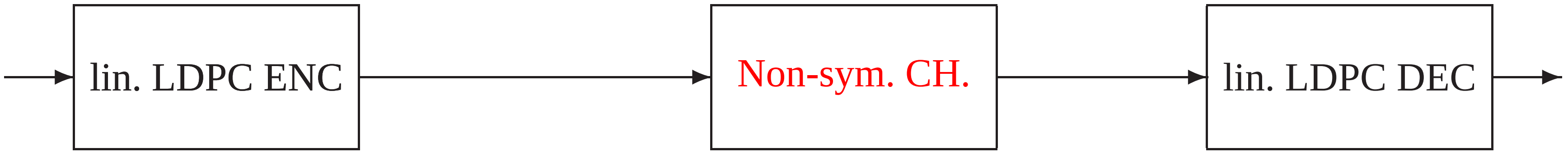

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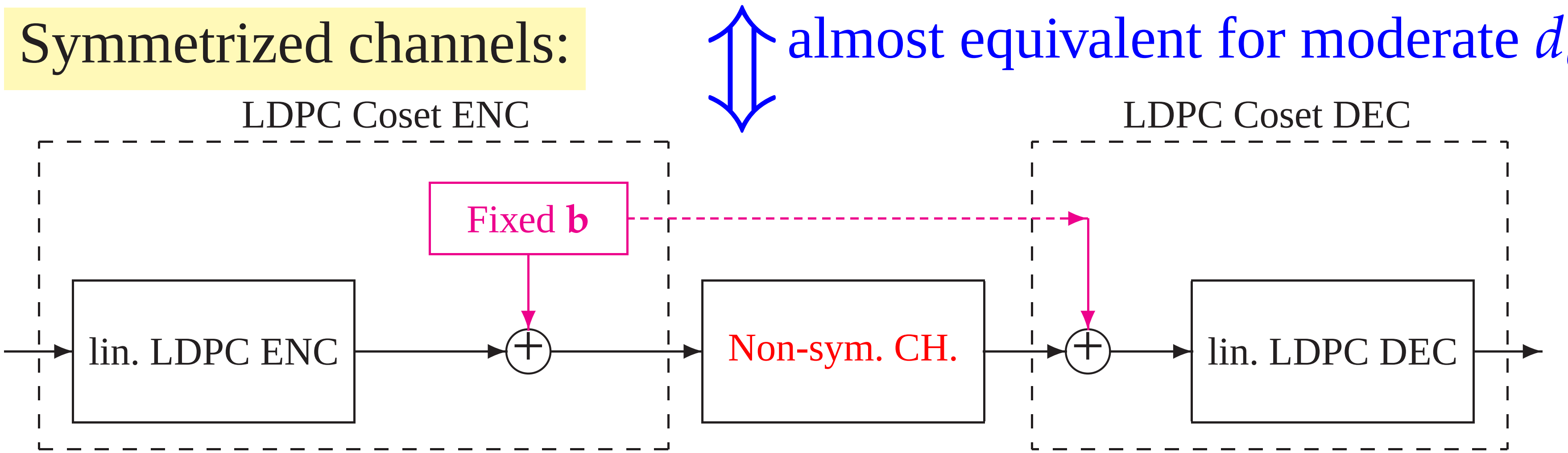

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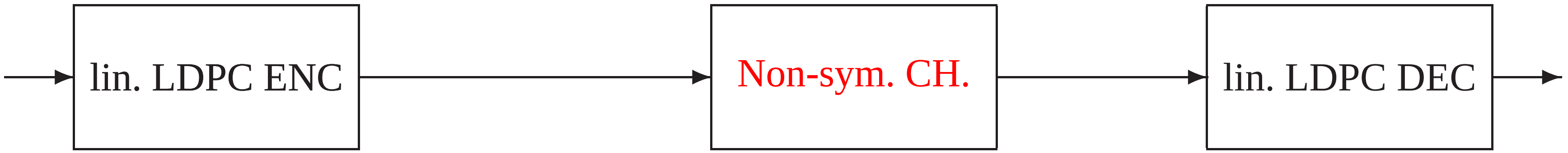

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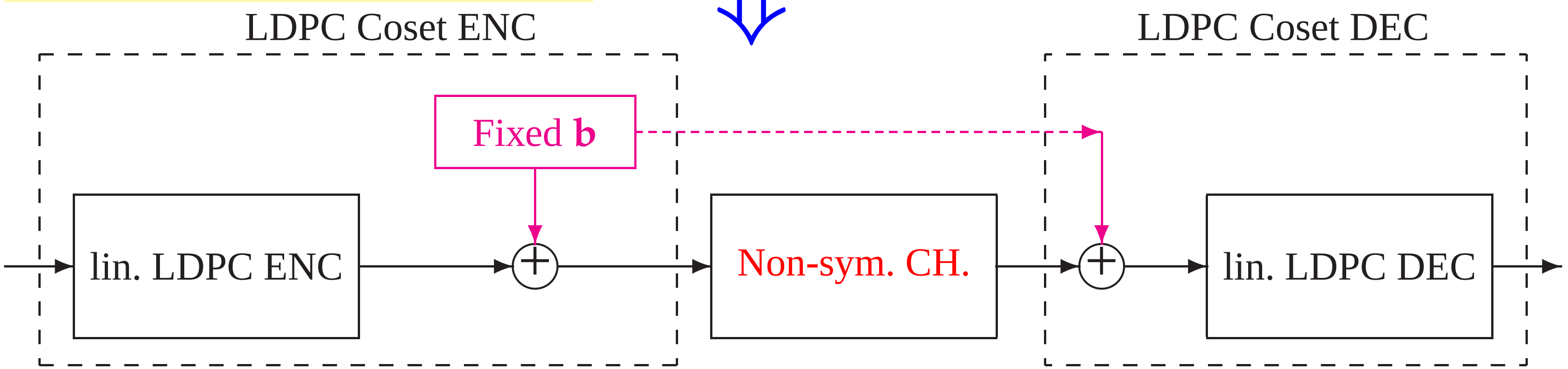

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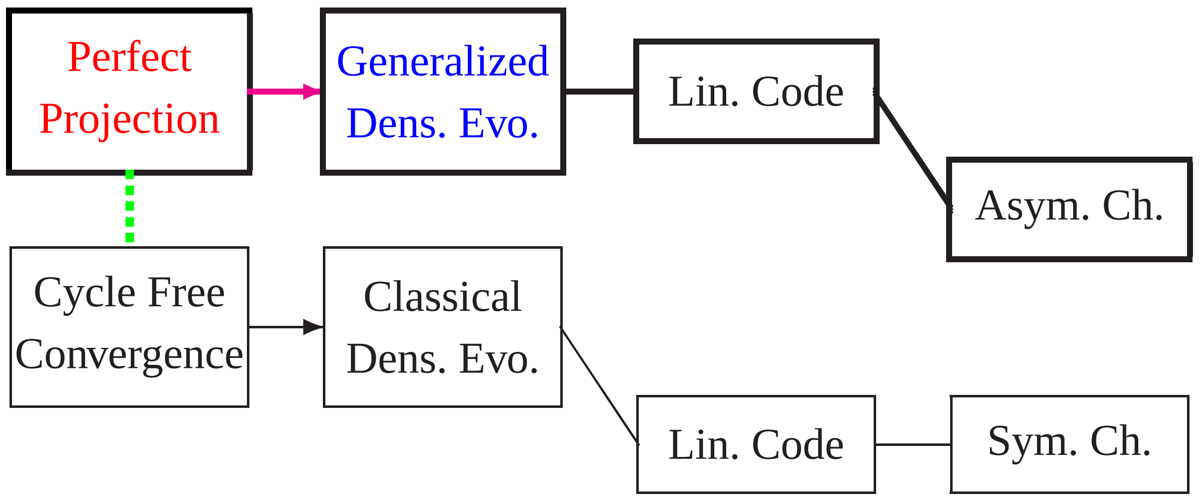
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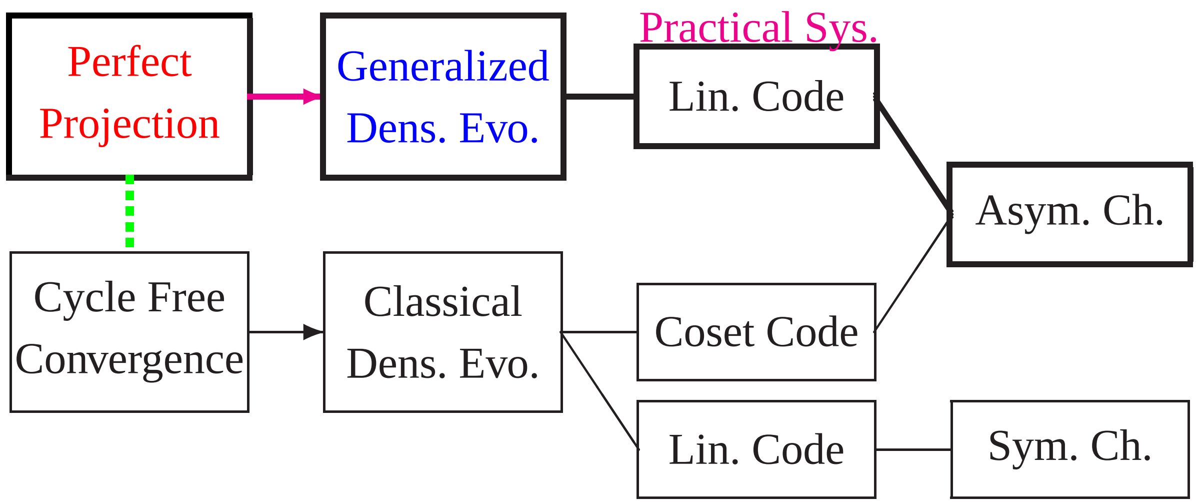
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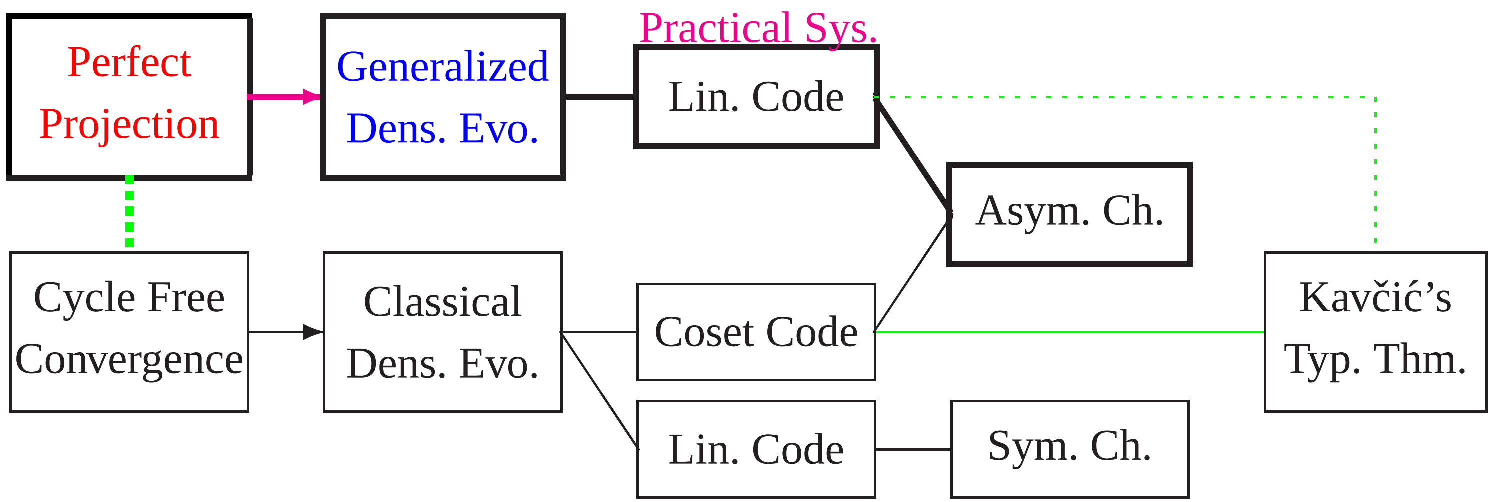
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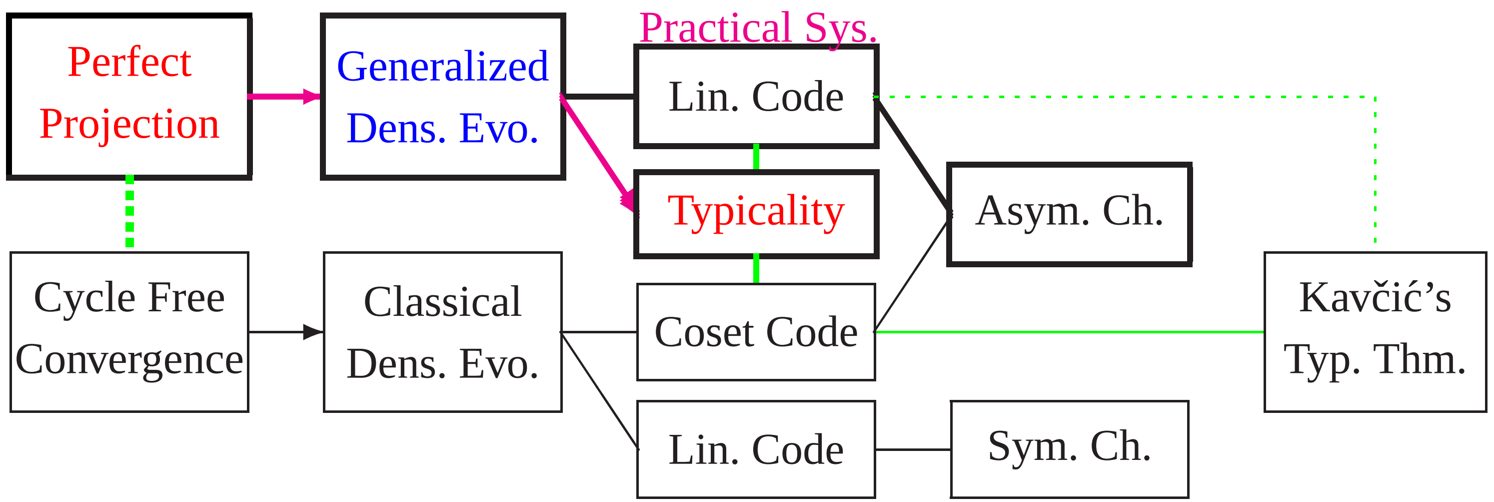
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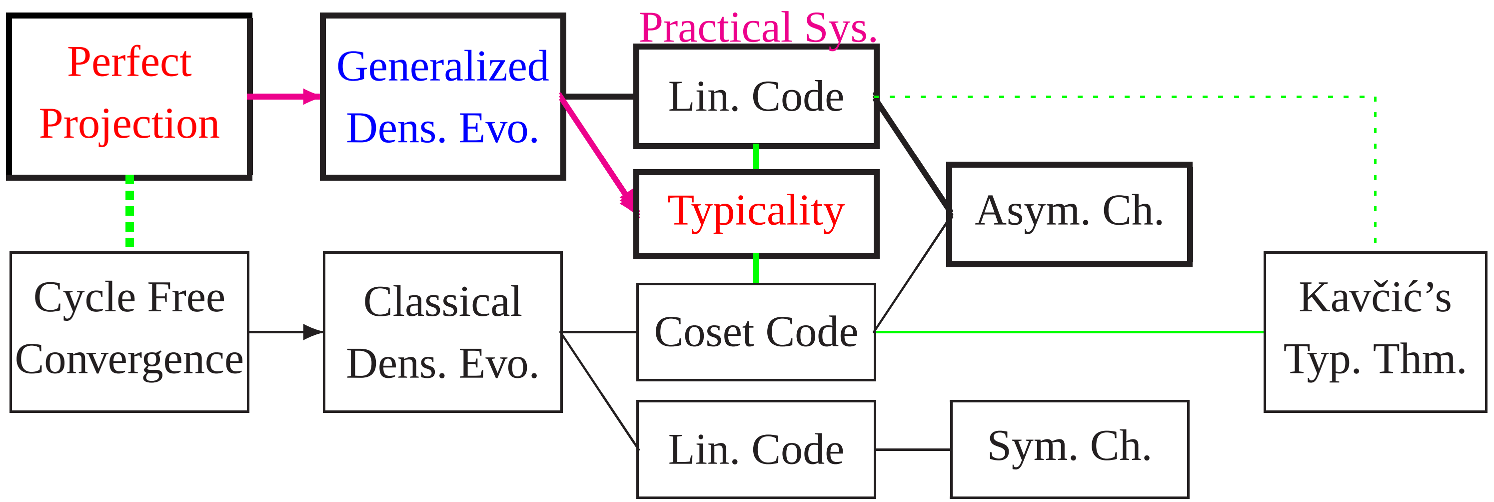

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Is BP locally optimal? Namely, given local observations on $\mathcal{N}^{(2l)}$ and complete knowledge of the codebook, can a maximum *a posteriori* probability (MAP) decoder do better?

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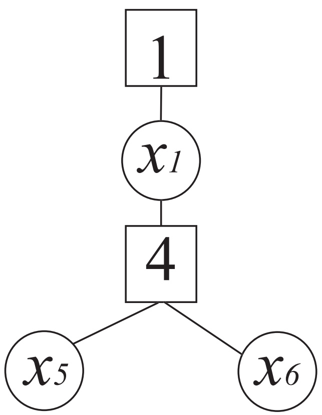

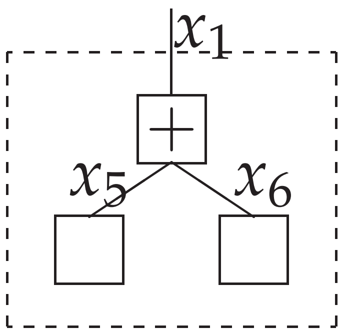

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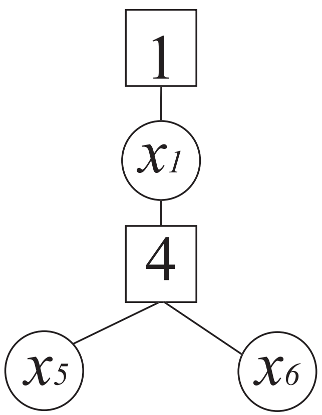

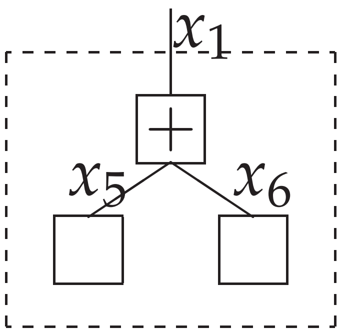

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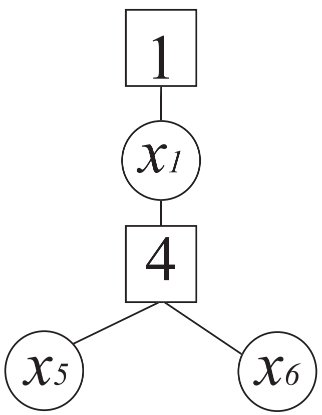

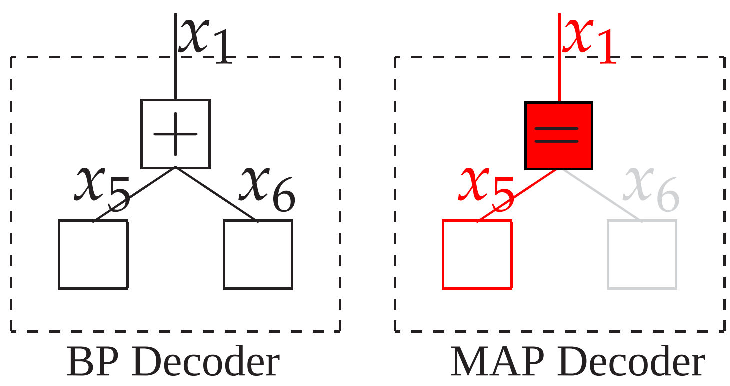

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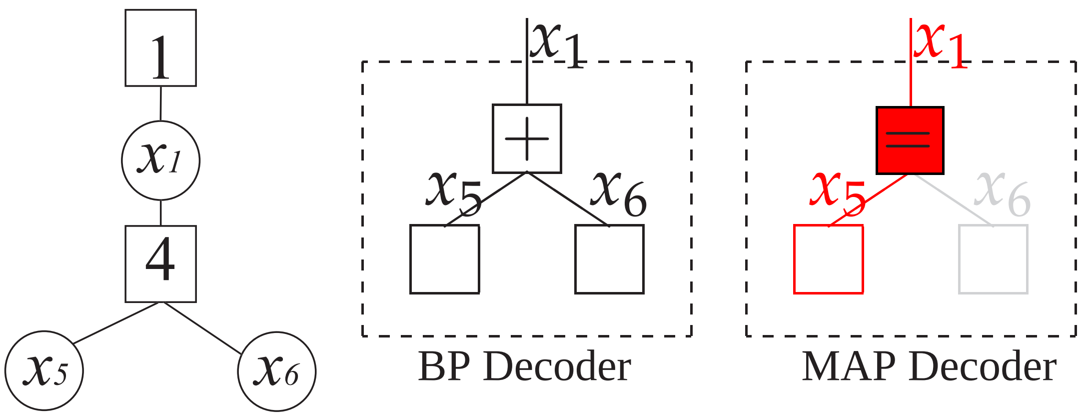

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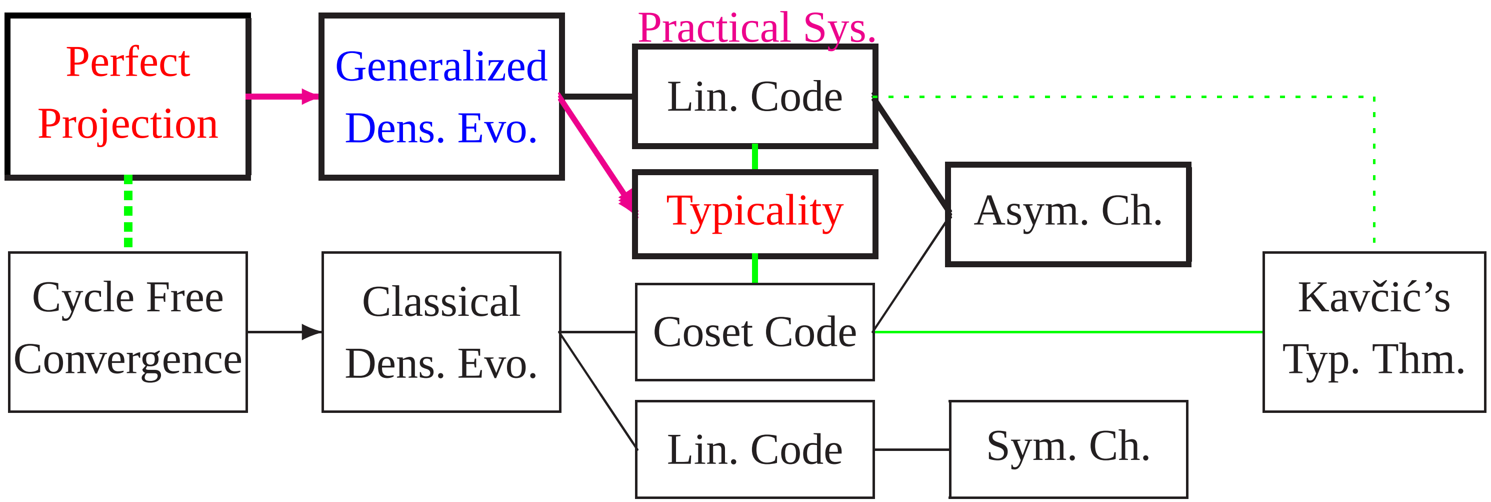

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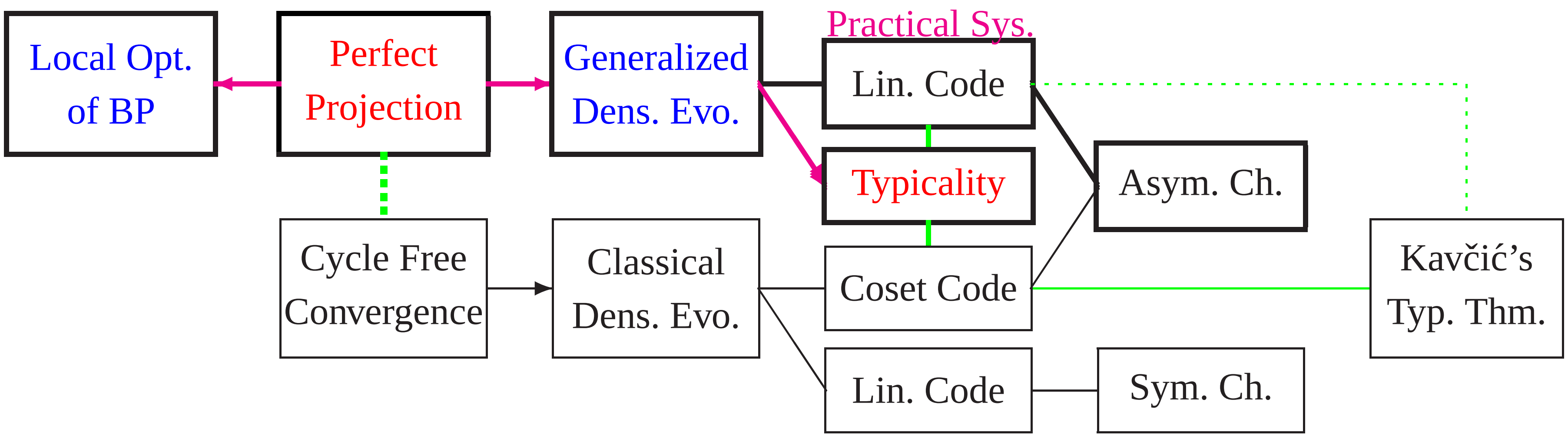

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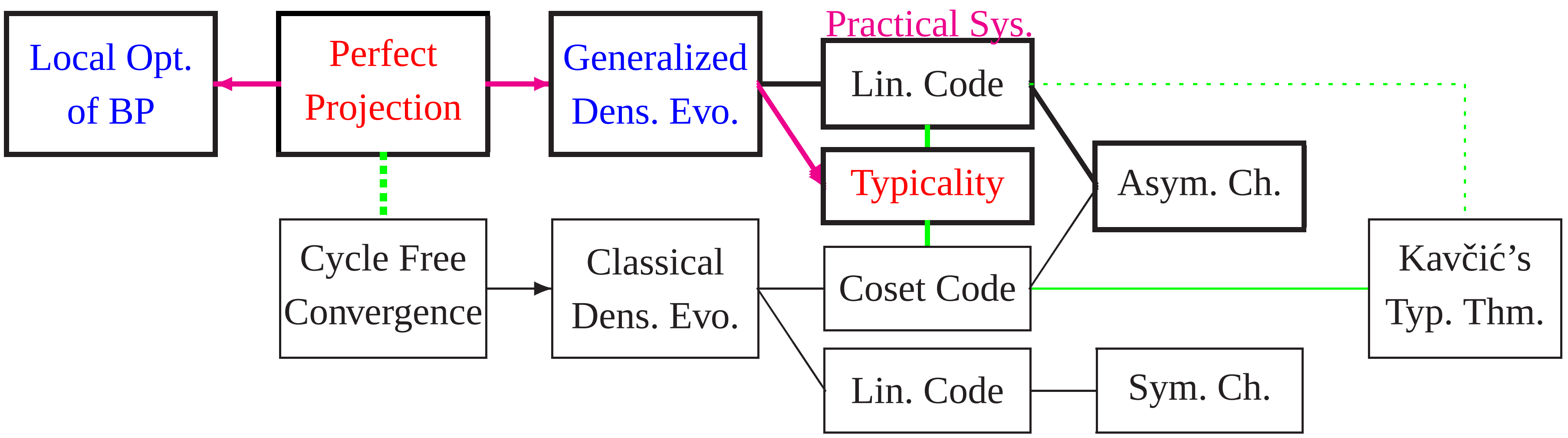

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Reference: [Burshtein, Miller 01], [Khandekar 02], [Land *et al.* 03], and [Sutskover, Shamai 03].

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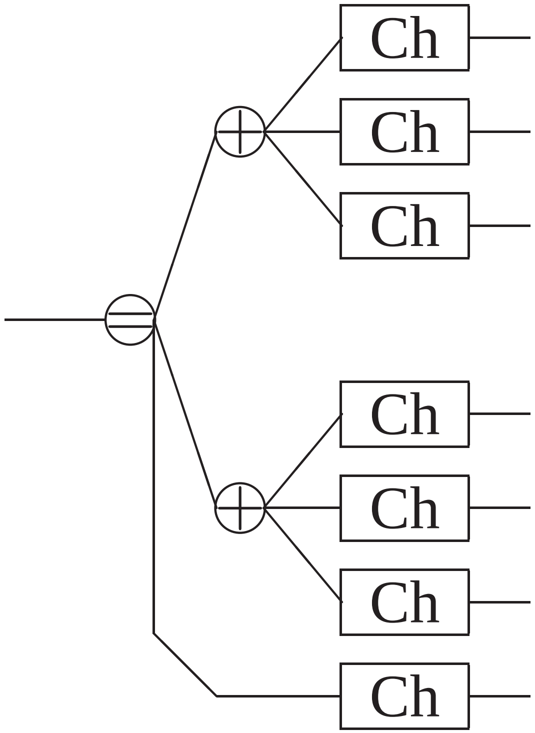

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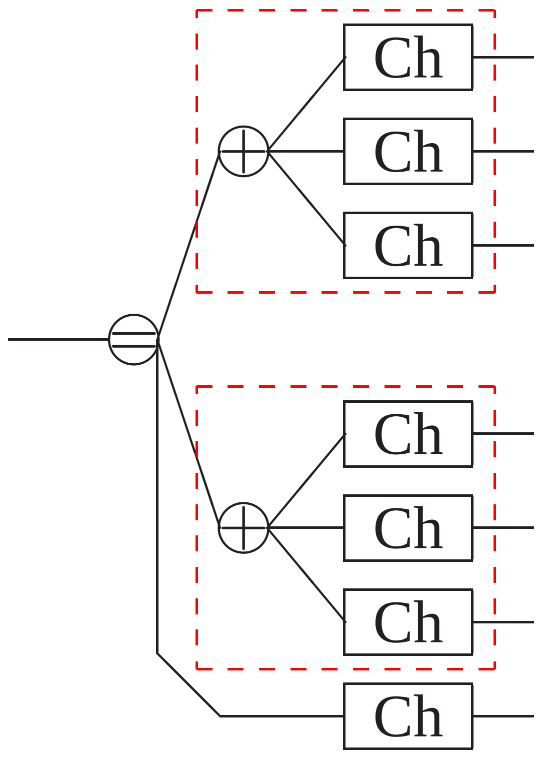

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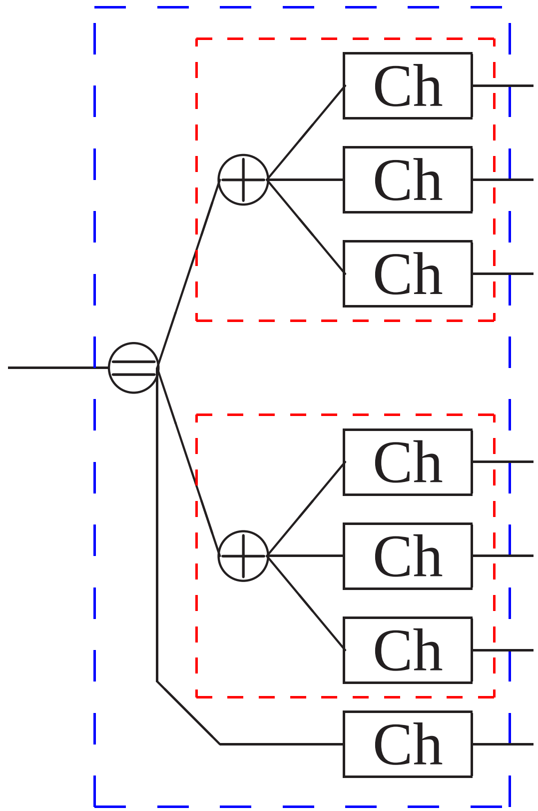

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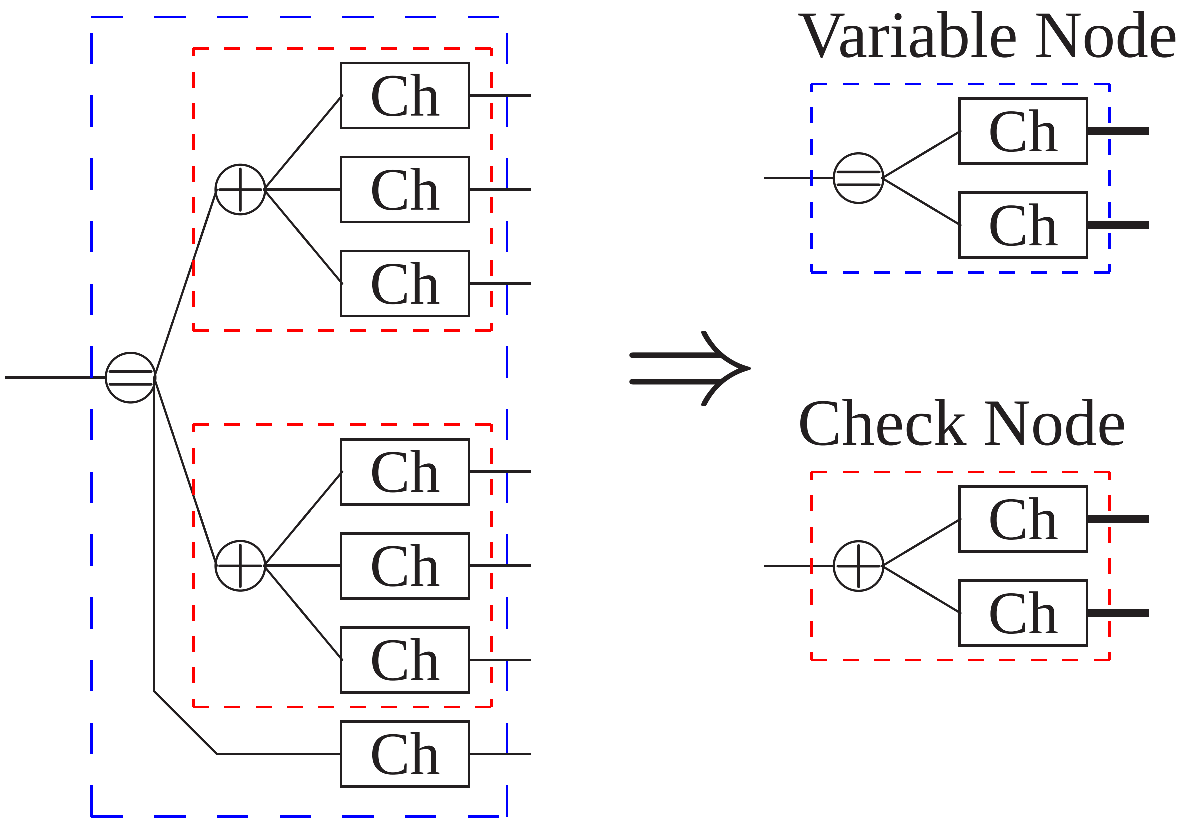
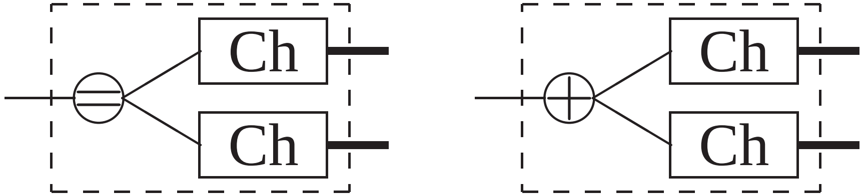


Existing Finite Dim. Bnds.

- Assumption: binary-input/symmetric-output channels.
- Iteratively trace the evolution of a one-dimensional index.



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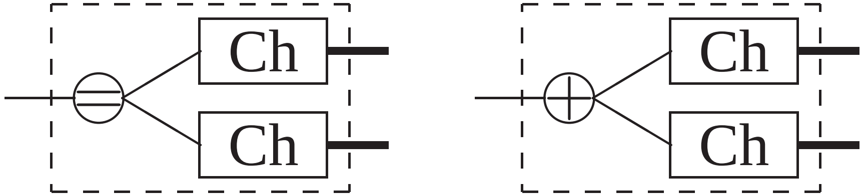

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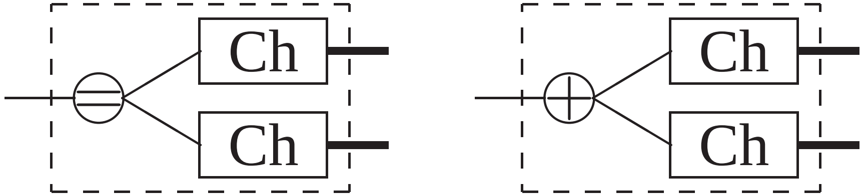

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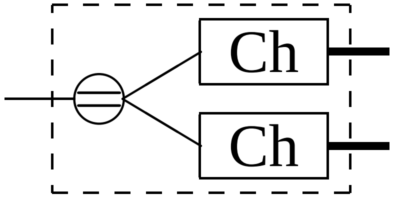
- All are based on the convexity/concavity of their transfer functions.

Discovery



$$\text{INFO}_{out} \geq V_{BSC}(\text{INFO}_{in,1}, \text{INFO}_{in,1})$$

$$\text{INFO}_{out} \leq V_{BEC}(\text{INFO}_{in,1}, \text{INFO}_{in,1})$$

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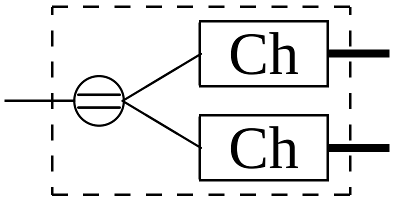

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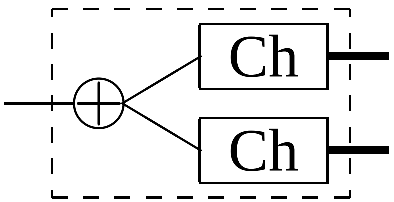

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Iterative Bound: $\text{INFO}^{(l+1)} \geq V_{BSC}\left(\text{INFO}^{(0)}, C_{BEC}\left(\text{INFO}^{(l)}\right)\right)$

Reference: [Land *et al.* 03], and [Sutskover, Shamai 03]

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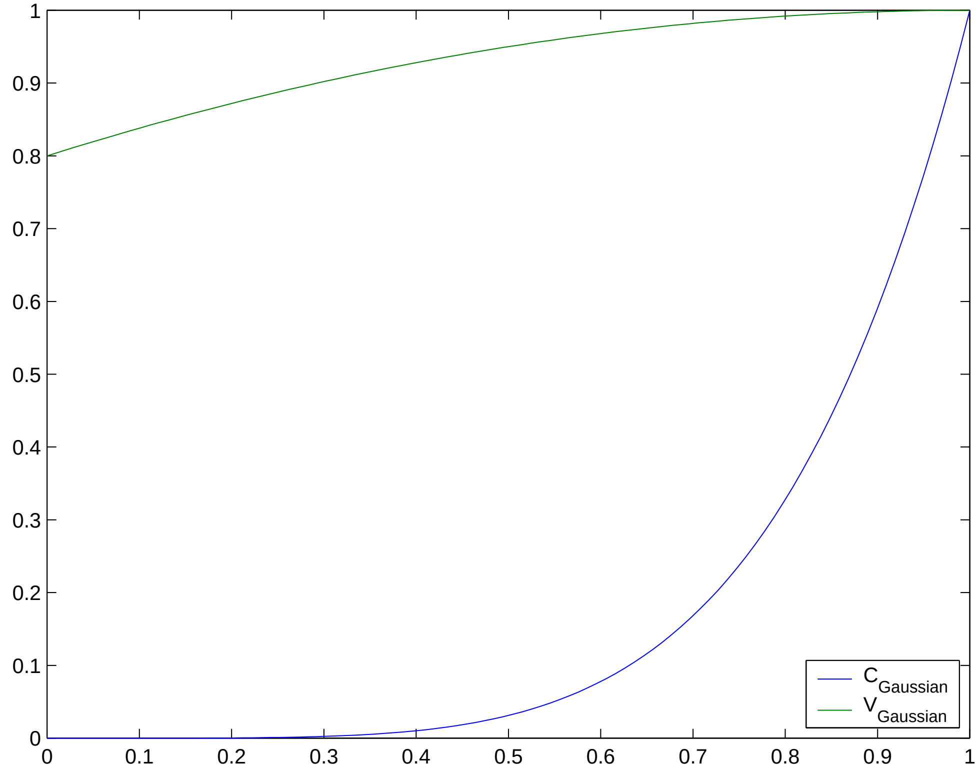

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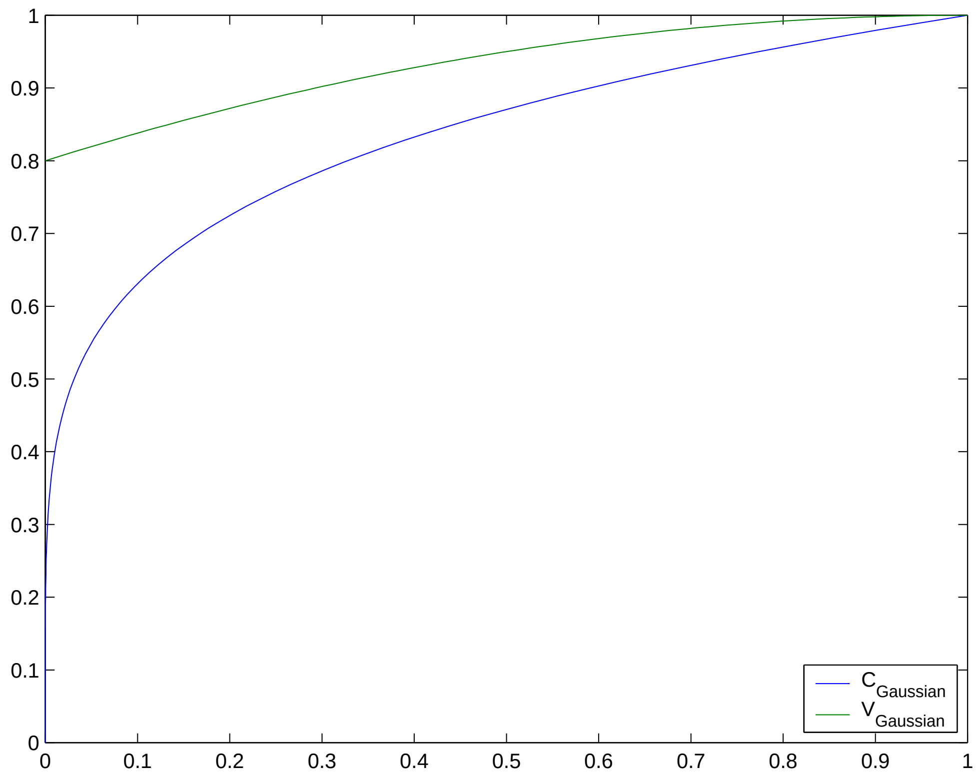

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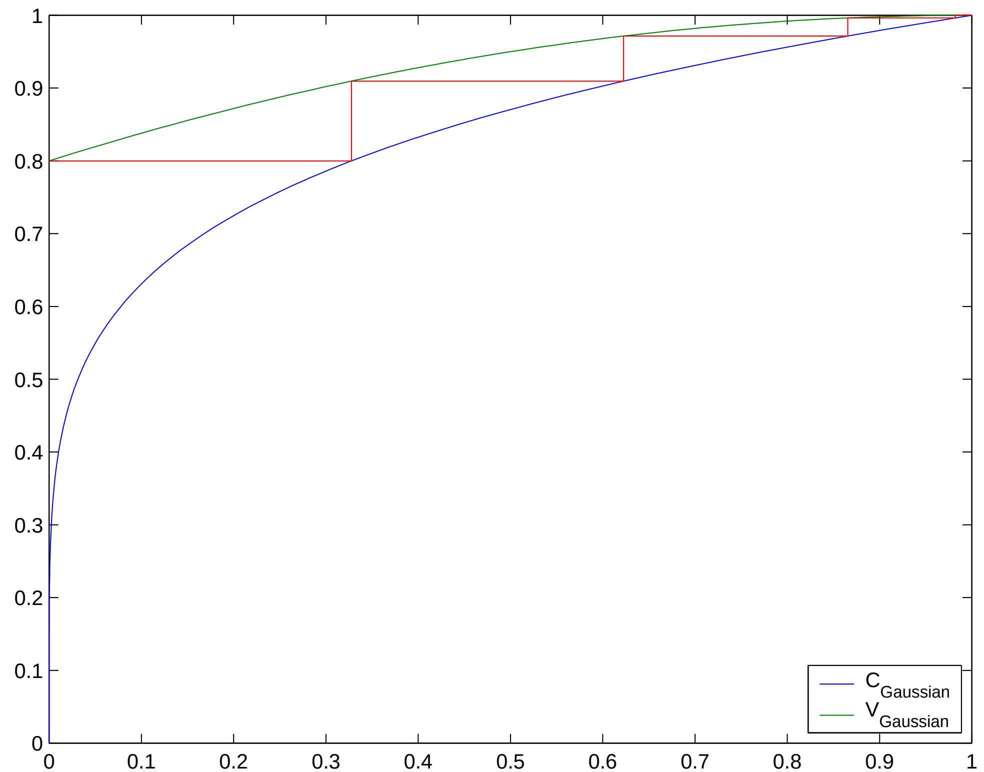

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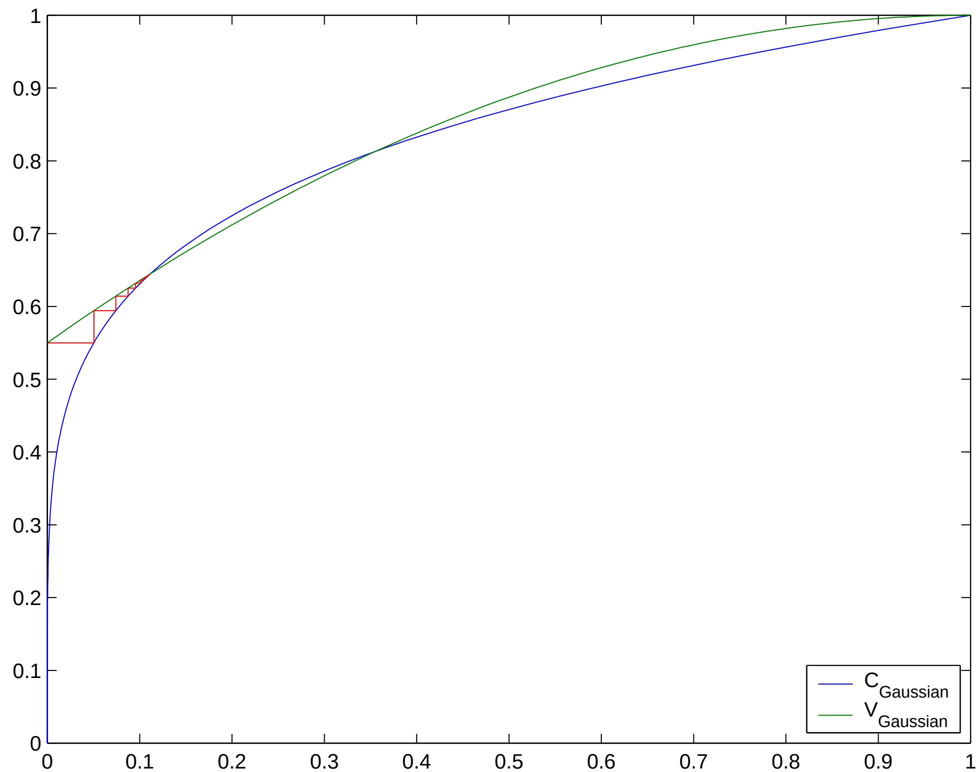

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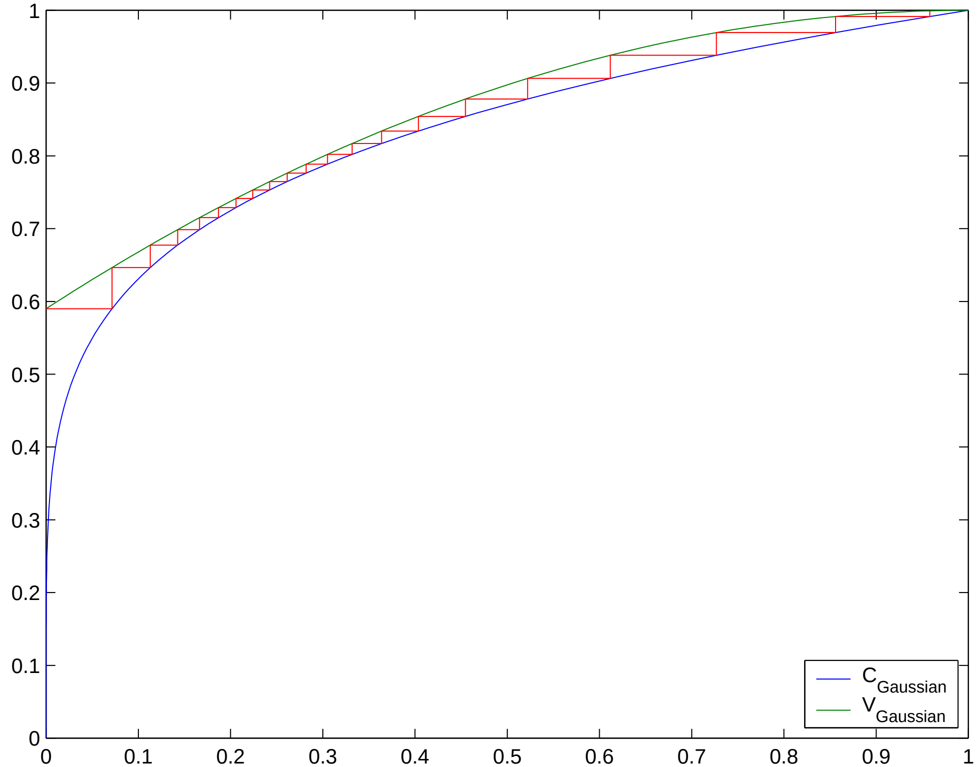
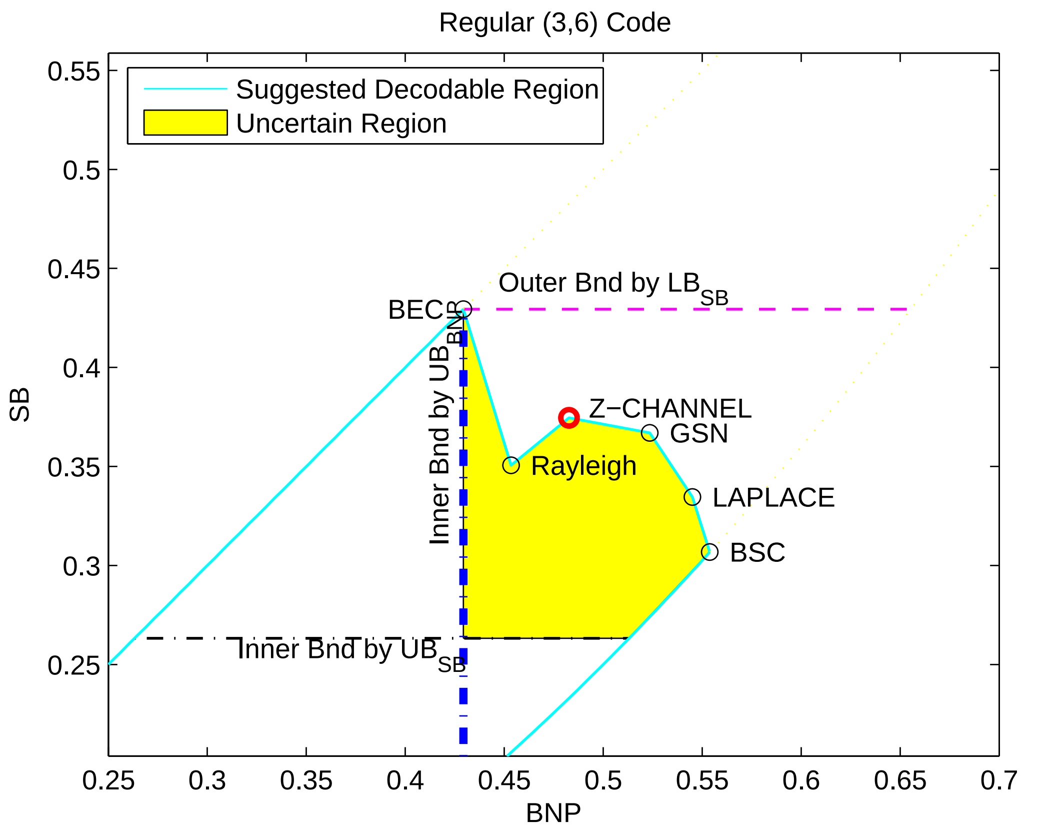


The First Bound for Non-Symmetric Channels

$$\mathrm{BNP}^{(l+1)} \leq \mathrm{BNP}^{(0)} \left(1 - \left(1 - \mathrm{BNP}^{(l)}\right)^{d_c - 1}\right)^{d_v - 1}$$



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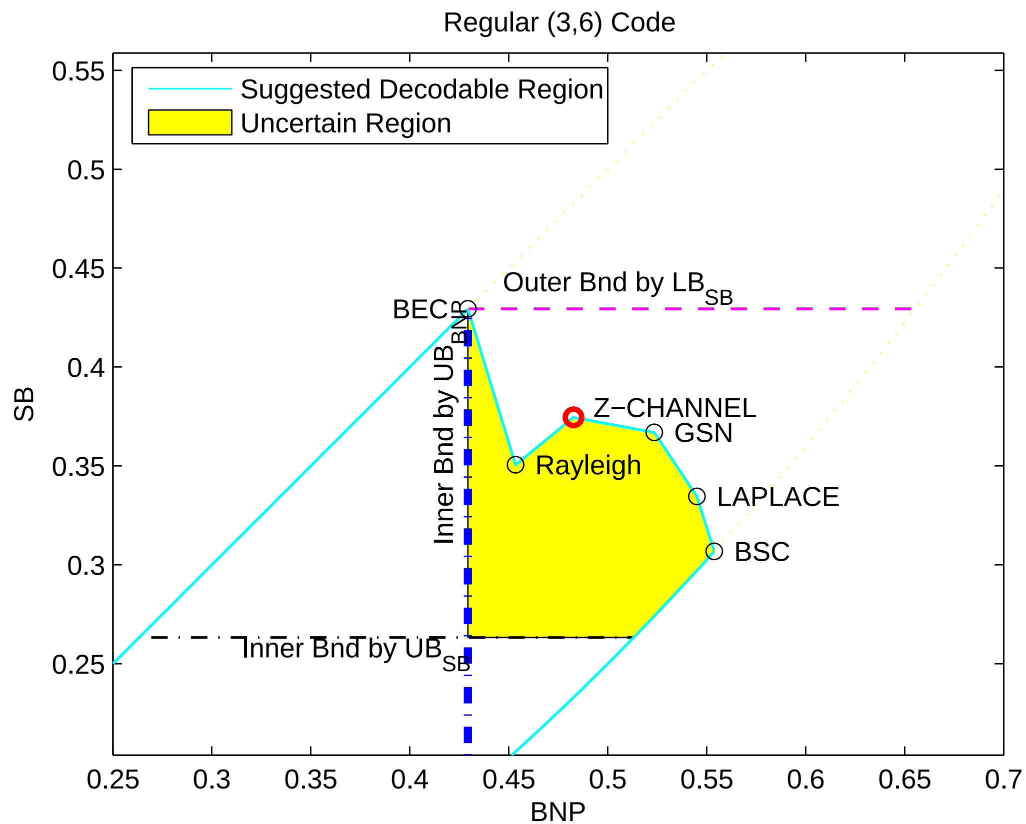
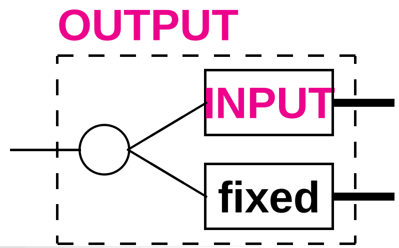


- Uniformly good performance for general fading channels.

$UB_{\text{BNP},SB}$ **given** $(\text{BNP}_{in}, SB_{in})$



Equivalent to finding the universal optimizer for all $p_2 \in [0, 1/2]$:

$$\forall p_2 \in [0, 1/2], \max \quad \int \text{BNP}_{out}(p_1, p_2) dP_1(p_1)$$

$$\text{and} \int SB_{out}(p_1, p_2) dP_1(p_1)$$

$$\text{subject to} \quad \int \text{BNP}(p_1) dP_1(p_1) \leq \text{BNP}_{in,1}$$

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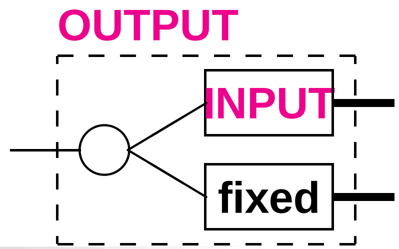

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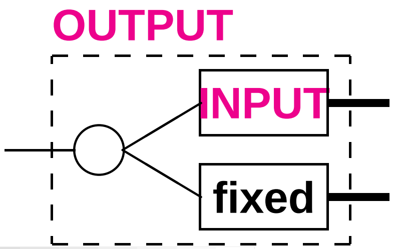

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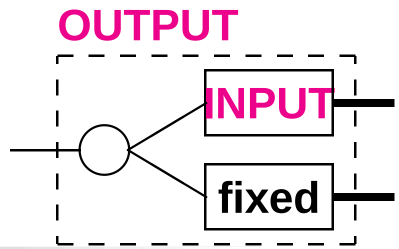

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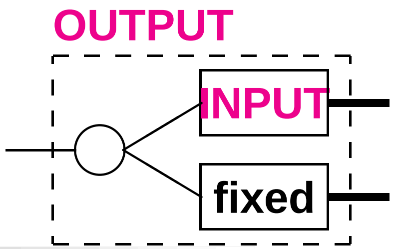

Equivalent to finding the universal optimizer for all $p_2 \in [0, 1/2]$:

$$\int \mathrm{BNP}_{out}(p_1, p_2) dP_1^{\dagger}(p_1) \geq \int \mathrm{BNP}_{out}(p_1, p_2) dP_1(p_1), \text{ and}$$

$$\int SB_{out}(p_1, p_2) dP_1^{\dagger}(p_1) \geq \int SB_{out}(p_1, p_2) dP_1(p_1), \forall p_2 \in [0, 1/2]$$

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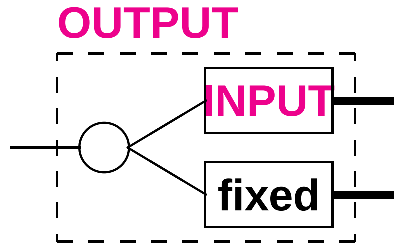
$dP^*(p)$ and $dP^\dagger(p)$



$$dP_1^*(p_1) = \begin{cases} 1 - \frac{\text{BNP}_{in,1}}{t} & \text{if } p_1 = 0 \\ \frac{\text{BNP}_{in,1}}{t} & \text{if } 2\sqrt{p_1(1-p_1)} = t \\ 0 & \text{otherwise} \end{cases},$$

$$dP_1^\dagger(p_1) = \begin{cases} (1-f_{SB})\frac{t}{t+\text{BNP}_{in,1}} & \text{if } 2\sqrt{p_1(1-p_1)} = \text{BNP}_{in,1} \\ f_{SB} & \text{if } 2\sqrt{p_1(1-p_1)} = \sqrt{SB_{in,1}} \\ (1-f_{SB})\frac{\text{BNP}_{in,1}}{t+\text{BNP}_{in,1}} & \text{if } 2\sqrt{p_1(1-p_1)} = t \\ 0 & \text{otherwise} \end{cases},$$

$$t = \frac{SB_{in,1}}{\text{BNP}_{in,1}}$$

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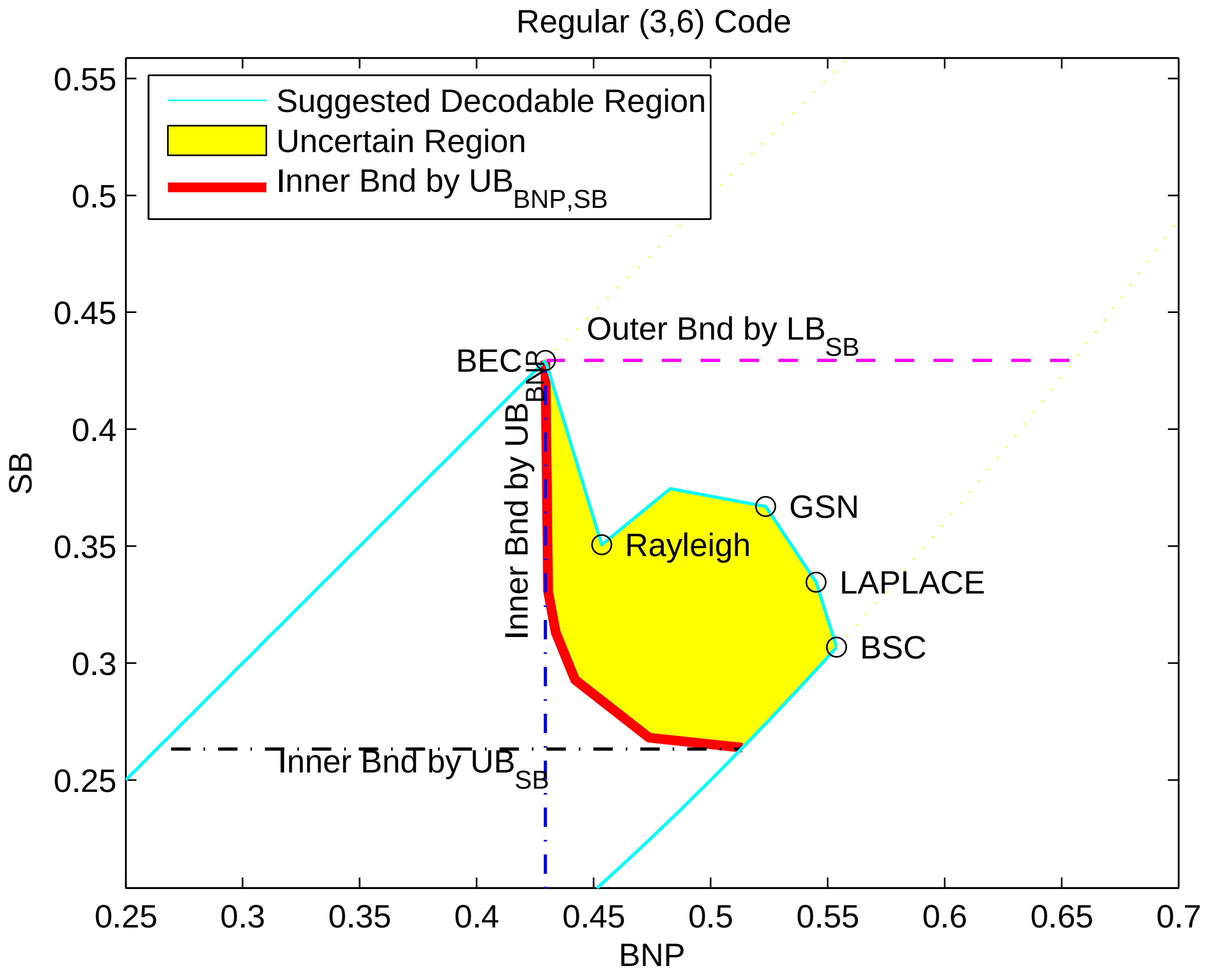
$$\eta(w) = w^3 - 2tw^2 + (t - \text{BNP}_{in,1})^2 w$$

$$w^* = \begin{cases} 2\sqrt{t\text{BNP}_{in,1}} & \text{if } \eta'(2\sqrt{t\text{BNP}_{in,1}}) \leq 0 \\ \frac{2t - \sqrt{4t^2 - 3(t-\text{BNP}_{in,1})^2}}{3} & \text{otherwise.} \end{cases}$$

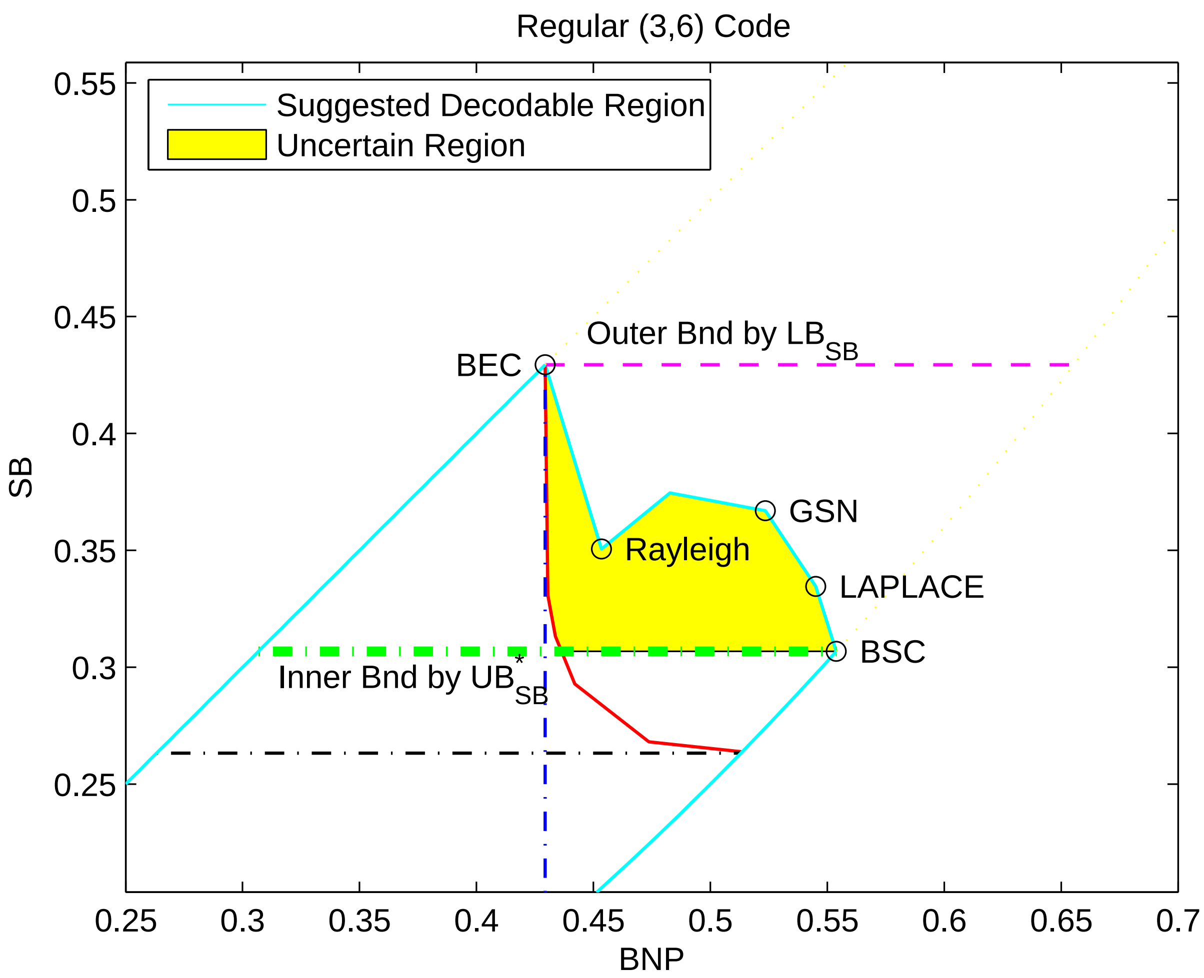
$UB_{\mathrm{BNP},SB}$

- Strict improvements over existing one-dimensional bounds.



A Non-iterative Tight SB Bound

- Iteration-based approach vs. Non-iteration-based approach



$\mathbb{Z}_m$-based LDPC Codes

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C.-C. Wang, S.R. Kulkarni, and H.V. Poor, "On Finite-Dimensional Bounds for LDPC-like Codes with Iterative Decoding," in *Proc. Int'l Symp. Inform. Theory & its Applications.*, Parma, Italy, Oct. 2004.

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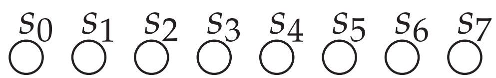

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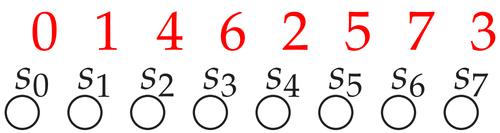

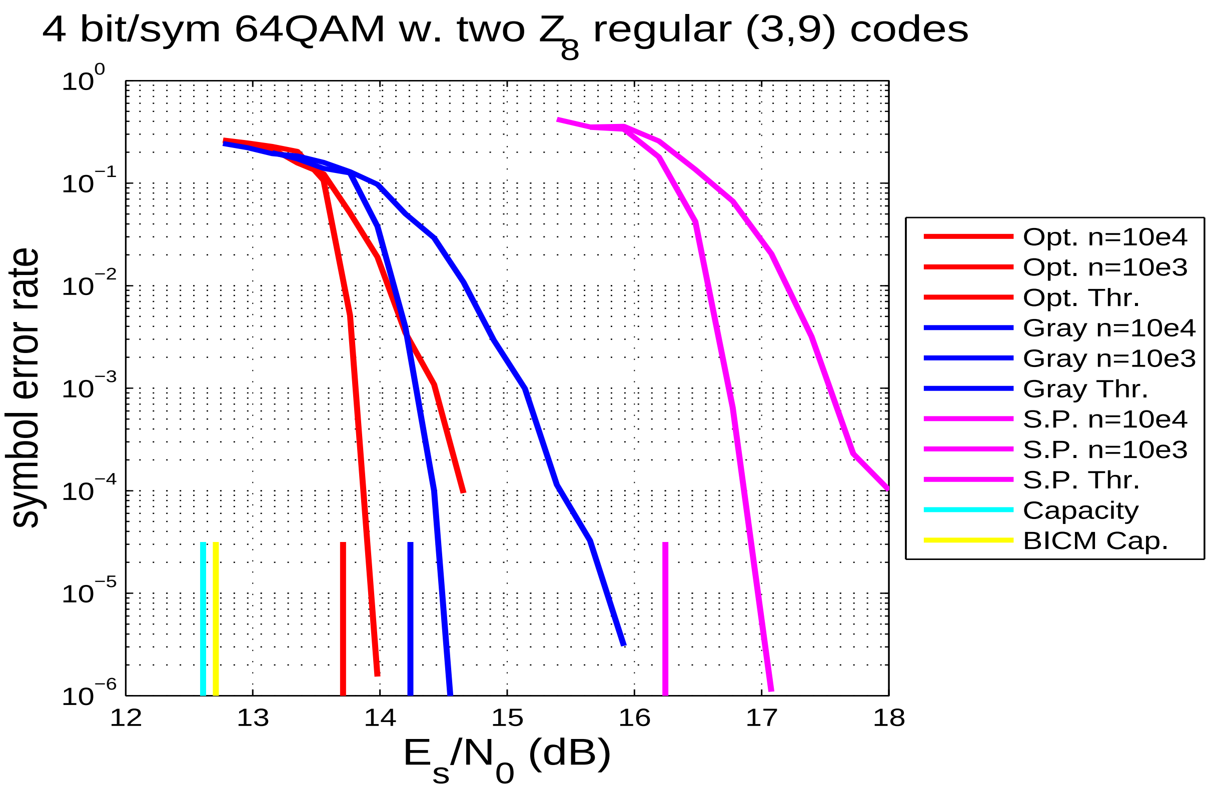

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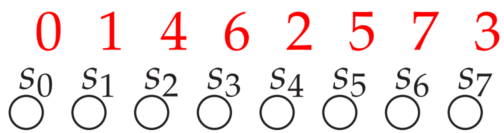

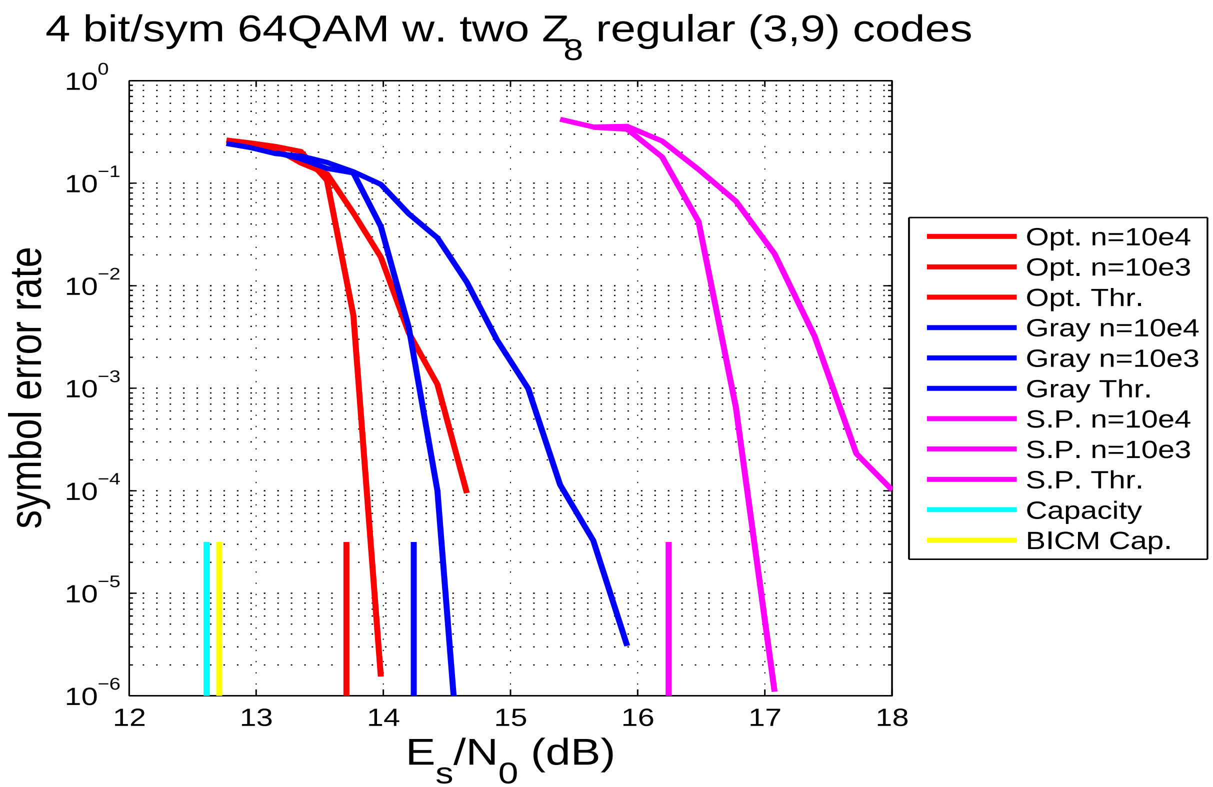

- 0.3dB better than BICM w. bin. regular codes
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- Further code optimization is the next step.
- Compatible with almost all equalization schemes

Conclusion

- Part I — Linear LDPC codes on symmetric & non-symmetric channels

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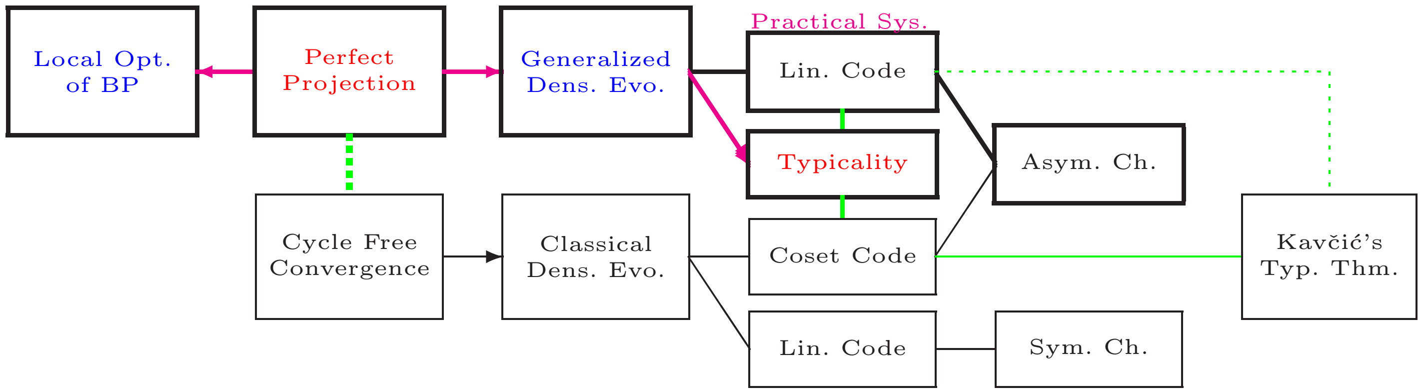

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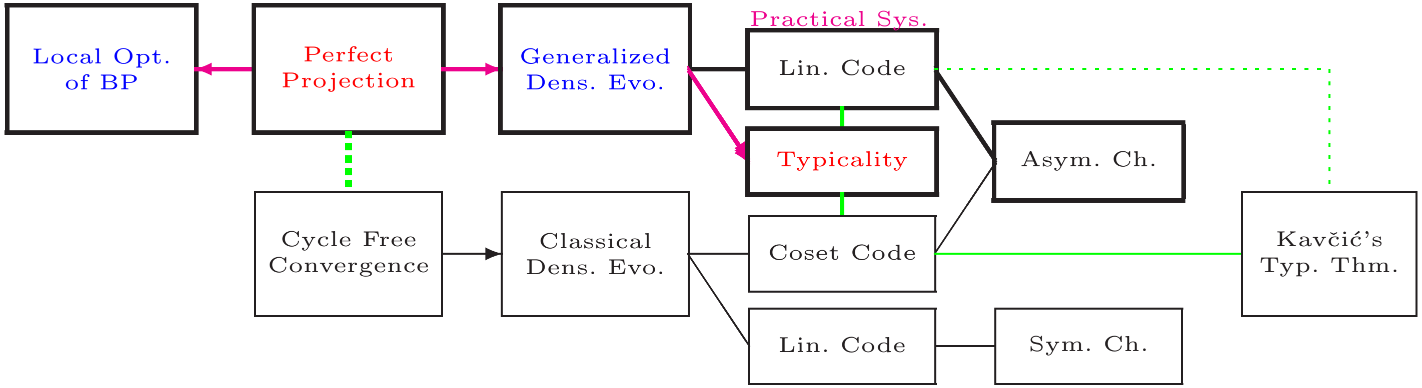

- Part II — EXIT chart & finite-dim. bounds for LDPC codes

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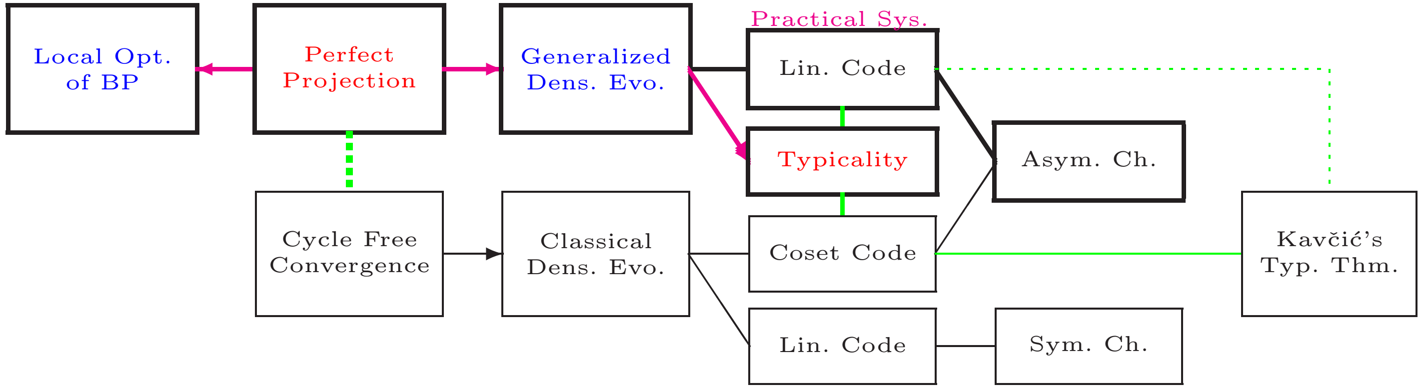

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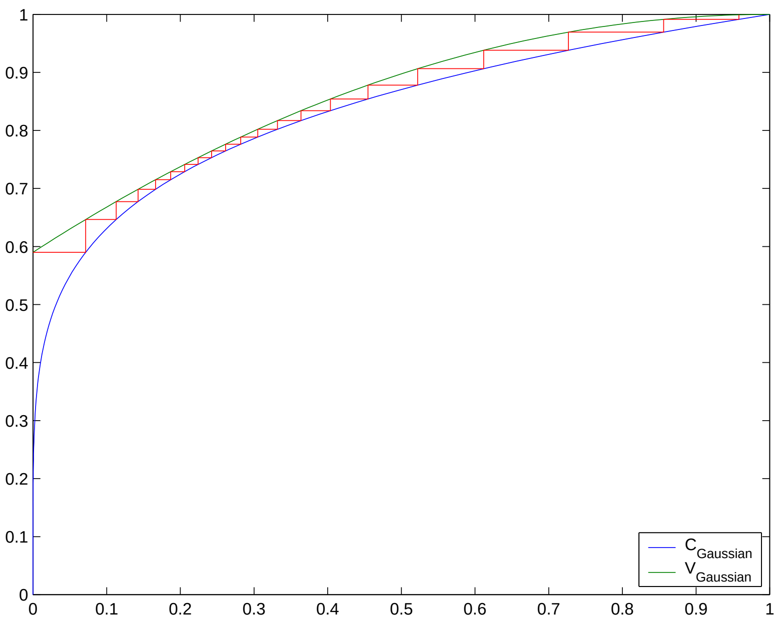

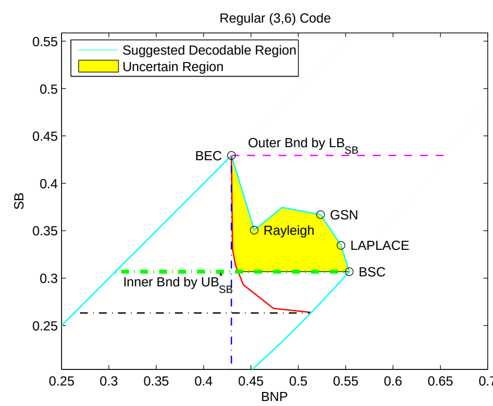

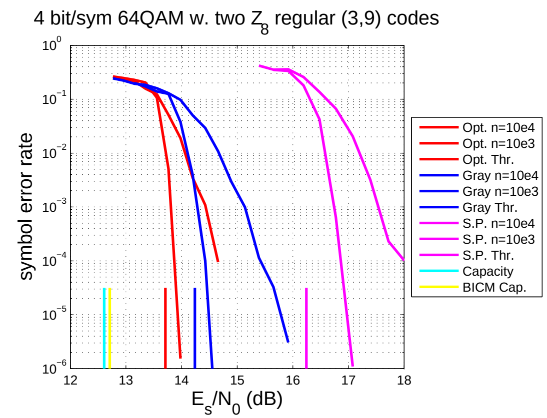

Future Work

LDPC Codes

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- BP Algorithms
- Practical schemes

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  - Coded modulation, dirty paper codes, lossless data compression, etc.