# Intersession Network Coding for Two Simple Multicast Sessions

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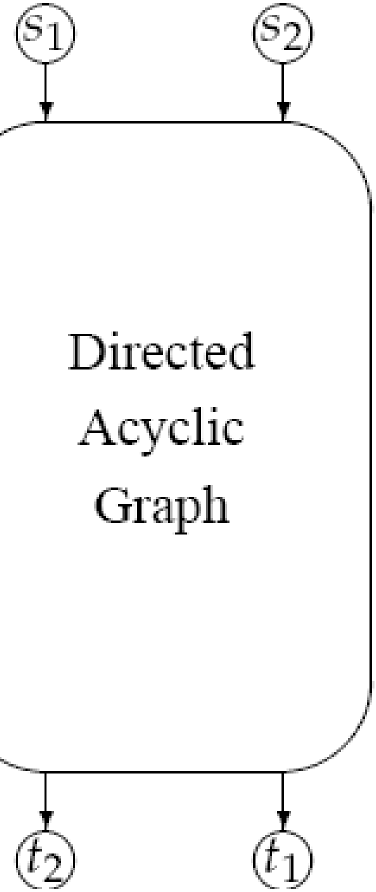
Presented by Abdallah Khreishah

# Motivations

When can we send $X_1$ and $X_2$ simultaneously?



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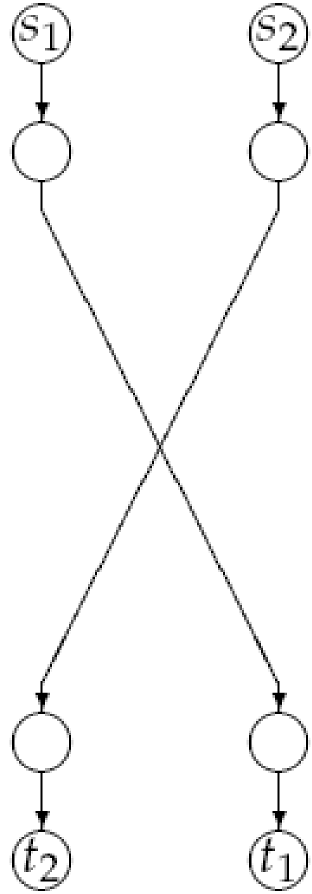
$\Longleftrightarrow$ Edge disjoint paths



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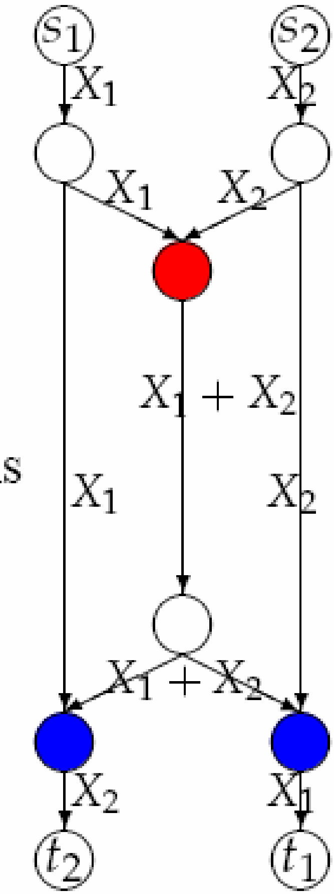

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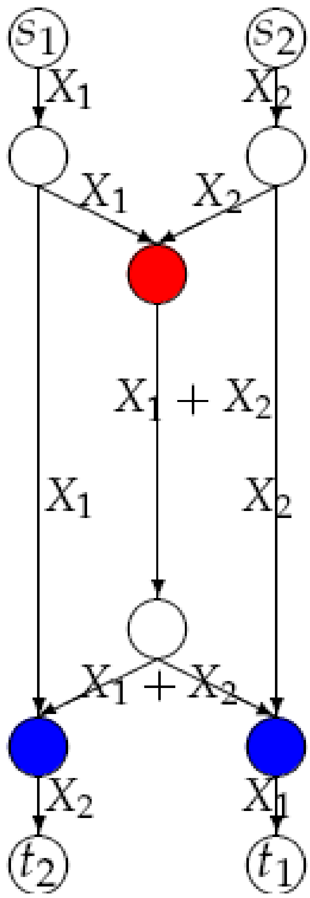

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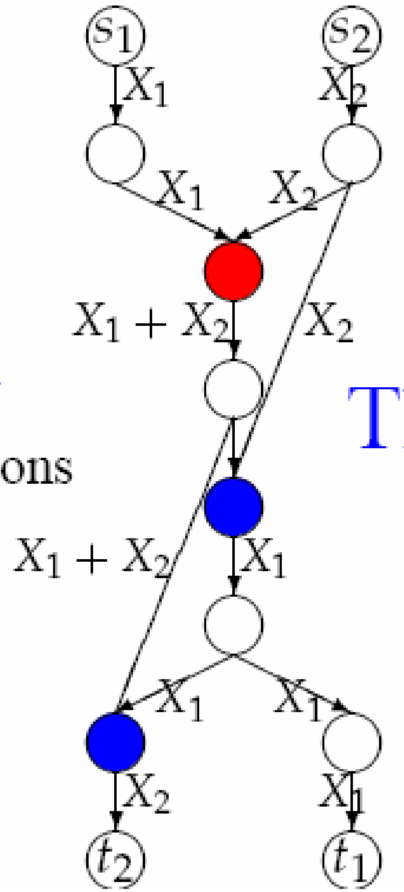


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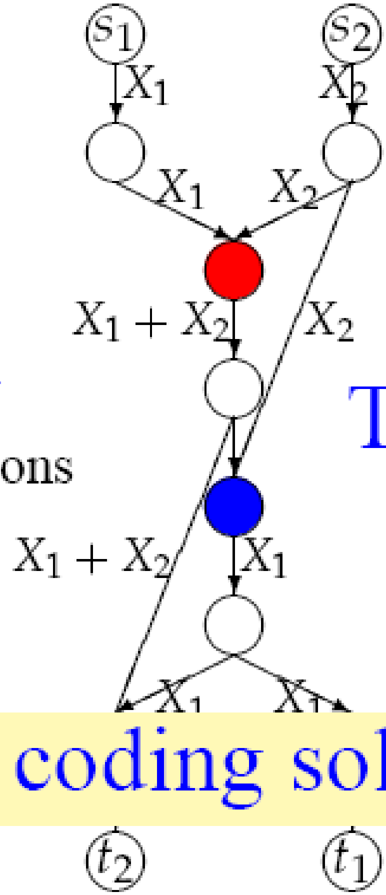


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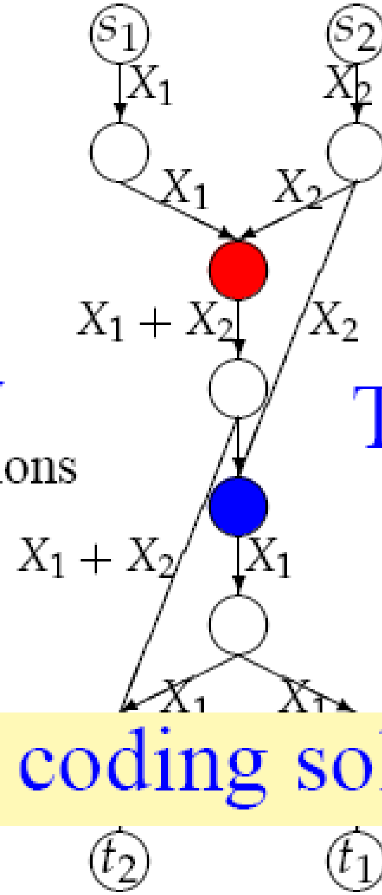


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**ISIT07**

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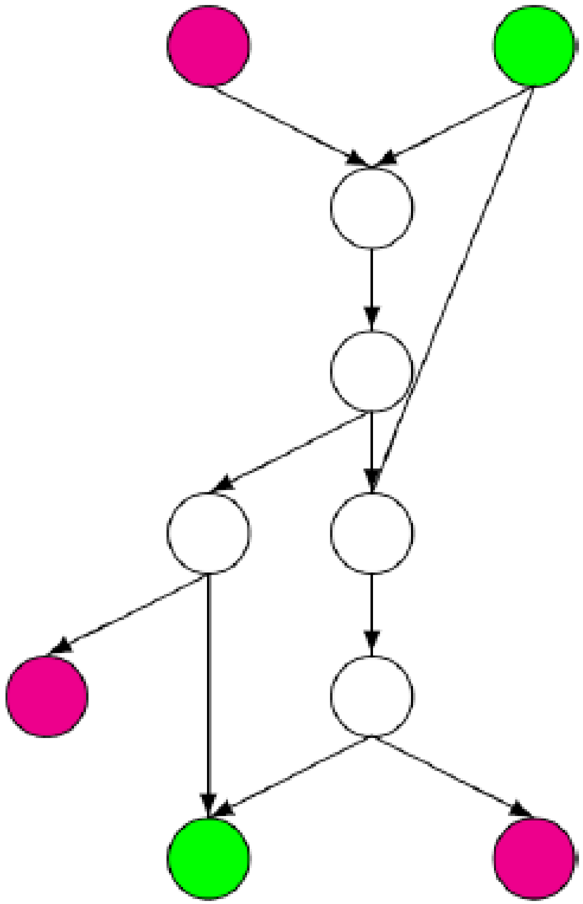

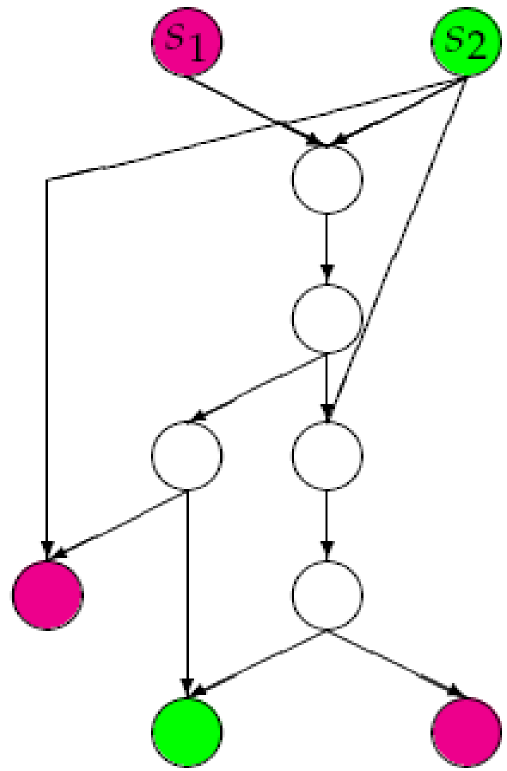

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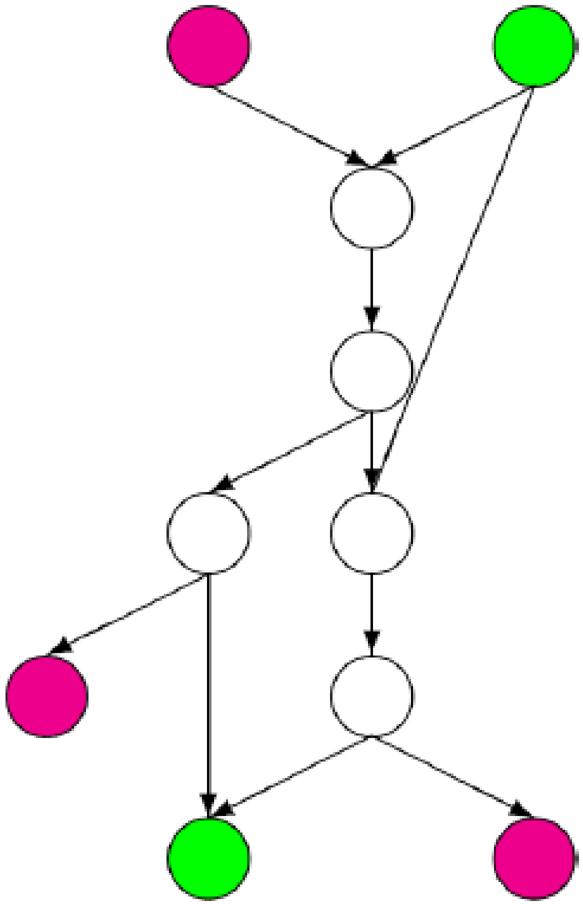

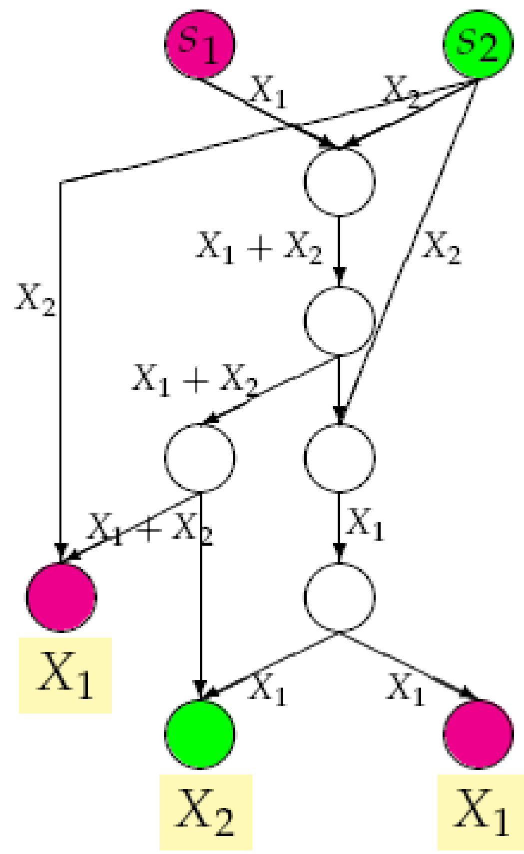

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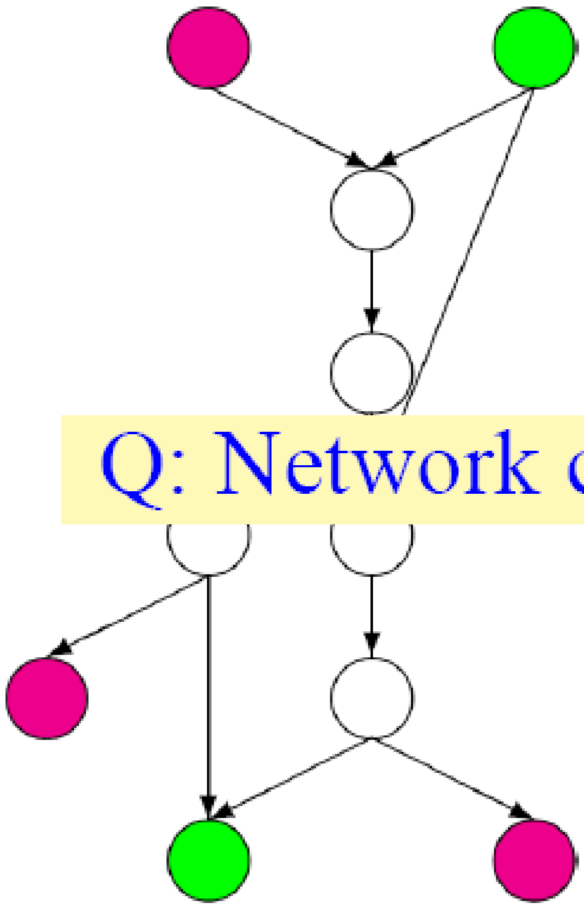

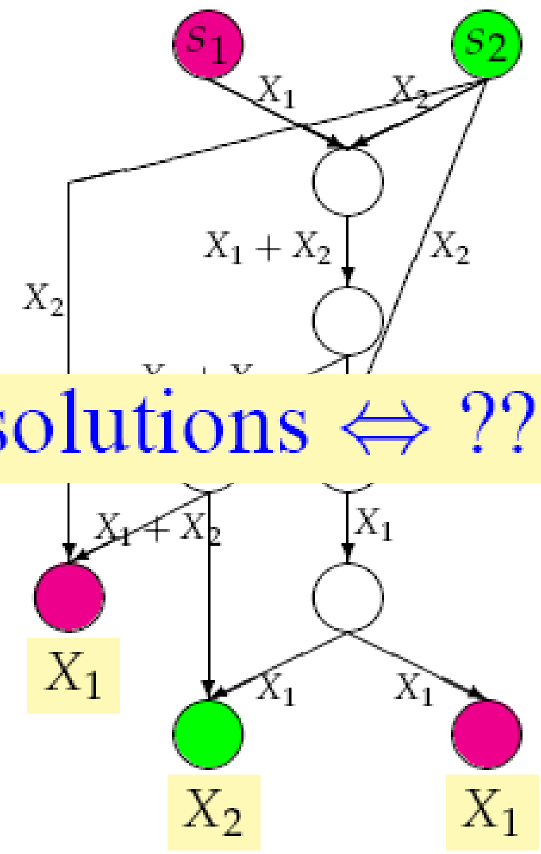


Q: Network coding solutions ⇔ ???

# Content

- Review the existing capacity results
- Settings and Main theorems for 2-m case.
- Corollaries
  - Complexity of deciding the feasibility
  - Bandwidth requirement
  - Sufficiency of Linear network coding
- Strengthened characterization of the 2-2 case.
  - Corollary: reciprocity
- Rate-control algorithms.
- Future work

# Review

**Theorem 2** *[Ahlswede et al. 00] For a single multicast session, rate $r$ is achievable if for all dest. $t_i$, the min-cut/max-flow $\rho_G(s, t_i)$ between $s$ and $t_i$ satisfies*

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Intra-session network coding only [Chen *et al.* 07]

$$\max_{r_i} \quad \sum_i U(r_i)$$

$$\text{subject to} \quad \sum_i f_{i,e} \leq c_e, \ \forall e \in E$$

$$\forall i, \ \{f_{i,e}\}_{e \in E} \text{ and } r_i \text{ satisfy the flow conditions.}$$

# Review

Rate region for single source and two receivers $t_1$ and $t_2$ with common information rate $R_0$ and private information rates $R_1$ to $t_1$ and $R_2$ to $t_2$. [ErezFedar03], [NgaiYeung04].

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- Pure inform.-theoretic approaches: Fundamental regions: [Song et al. 03], [Yan et al. 07], entropy calculus [Jain et al. 06].
- The network-sharing bound [Yan et al. 06], the information dominance condition [Harvey et al. 06], and the edge-cut bounds [Kramer et al. 06].

# Settings

Setting: General finite directed acyclic graphs, unit edge capacity,
$(s_1, \{t_{1,i}\}_i)$ & $(s_2, \{t_{2,j}\}_j)$, two integer symbols $X_1$ and $X_2$.

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Number of Coinciding Paths of edge $e$: $\mathcal{P} = \{P_1, \cdots, P_k\}$, and $\mathrm{ncp}_{\mathcal{P}}(e) = |\{P \in \mathcal{P} : e \in P\}|$.

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Application: multi-resolution multicast, as in $s_1 = s_2$, $\{t_{1,i}\} \subseteq \{t_{2,j}\}$.

# Main Theorem

**Theorem 3** *The existence of intersession network coding* $\Leftrightarrow$

$$\exists \mathcal{P} = \{P_{s_1,t_{1,i}}, P_{s_2,t_{1,i}} : \forall i\} \cup \{P_{s_2,t_{2,j}} : \forall j\},$$
$$\exists \mathcal{Q} = \{Q_{s_2,t_{2,j}}, Q_{s_1,t_{2,j}} : \forall j\} \cup \{Q_{s_1,t_{1,i}} : \forall i\},$$

*such that*

$$\max_{e \in E} \mathsf{ncp}_{\{P_{s_1,t_{1,i}}, P_{s_2,t_{1,i}}, P_{s_2,t_{2,j}}\}}(e) \leq 2, \ \ \forall i, j,$$

*and* $$\max_{e \in E} \mathsf{ncp}_{\{Q_{s_2,t_{2,j}}, Q_{s_1,t_{2,j}}, Q_{s_1,t_{1,i}}\}}(e) \leq 2, \ \ \forall i, j.$$

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$$\exists \mathcal{P} = \{\overset{1\longrightarrow 1}{P_{s_1,t_{1,i}}}, \overset{2\longrightarrow 1}{P_{s_2,t_{1,i}}} : \forall i\} \cup \{\overset{2\longrightarrow 2}{P_{s_2,t_{2,j}}} : \forall j\},$$

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$2 \longrightarrow 2 \quad 1 \longrightarrow 2 \qquad\qquad 1 \longrightarrow 1$

*such that*

Choose paths for $i$ and $j$ separately.

$$\max_{e \in E} \mathsf{ncp}_{\{P_{s_1,t_{1,i}}, P_{s_2,t_{1,i}}, P_{s_2,t_{2,j}}\}}(e) \leq 2, \quad \forall i, j,$$

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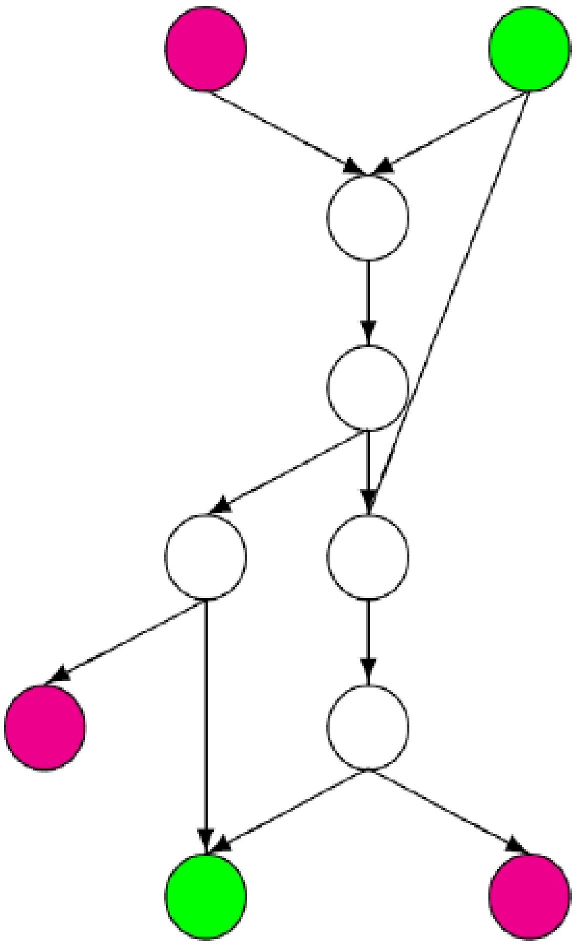
Routing: edge disjointness vs. Network coding: controlled overlaps.

# Infeasible Example



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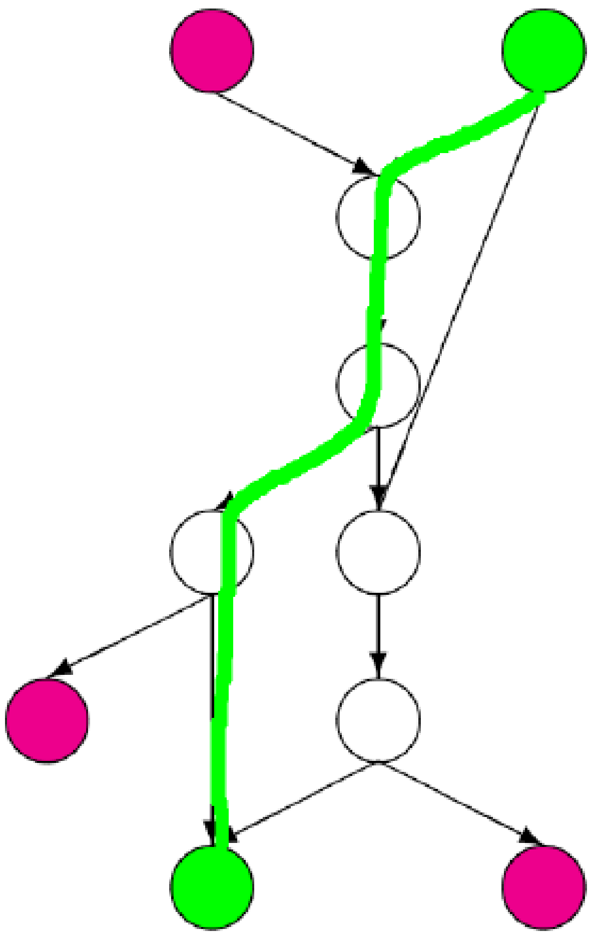

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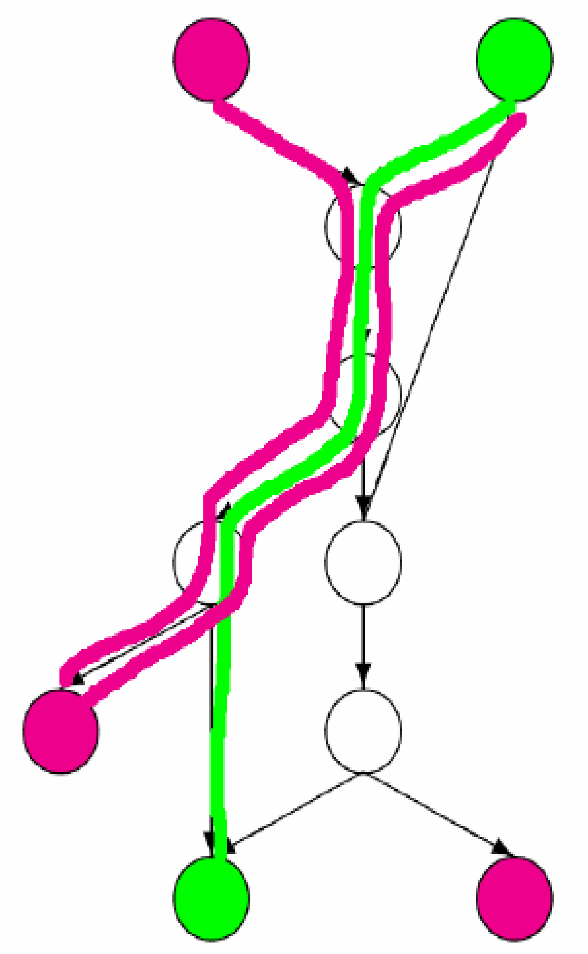

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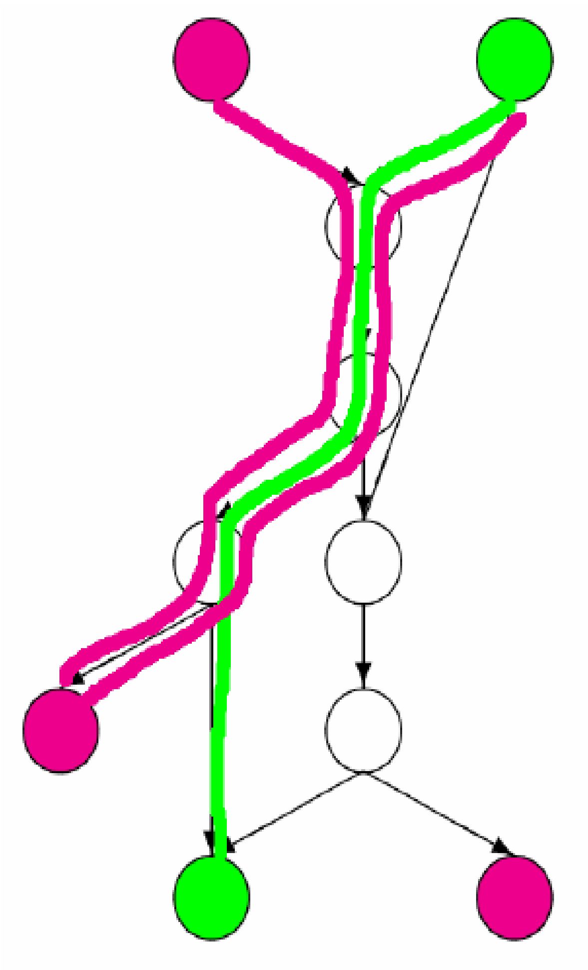

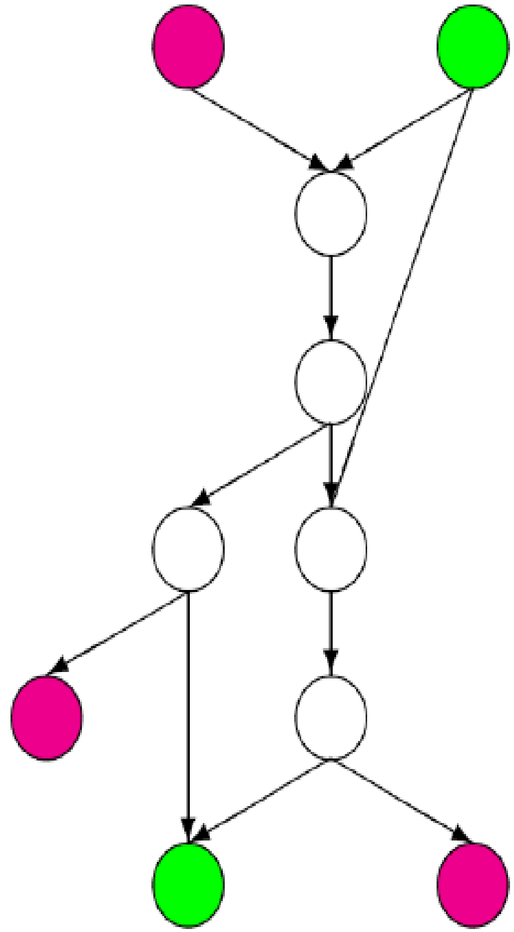

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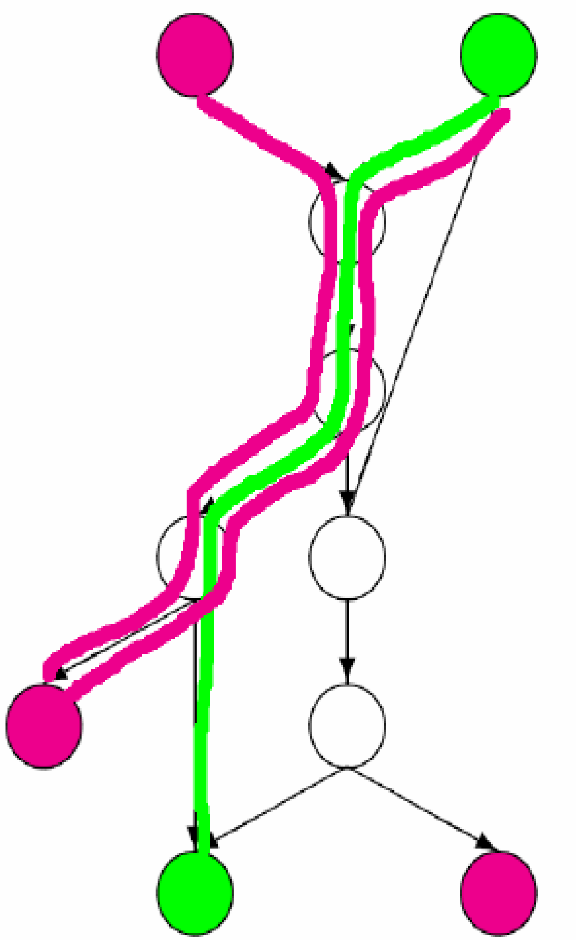

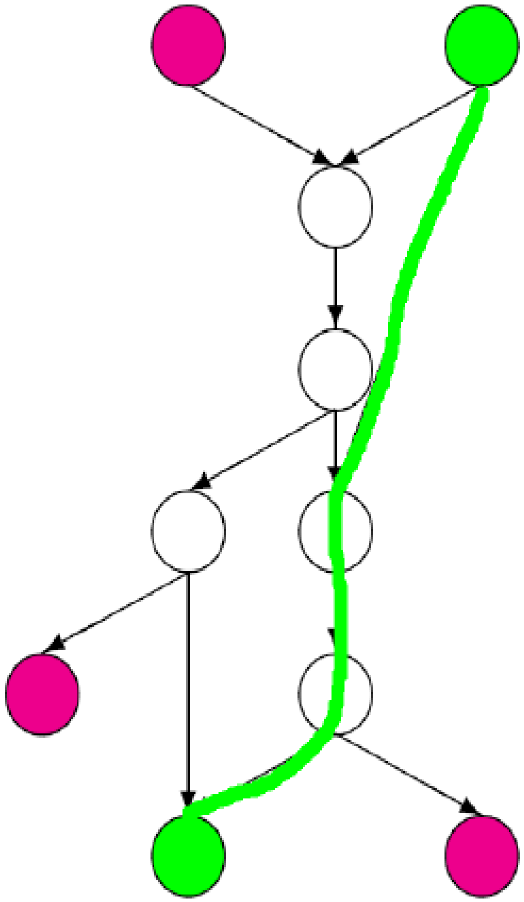

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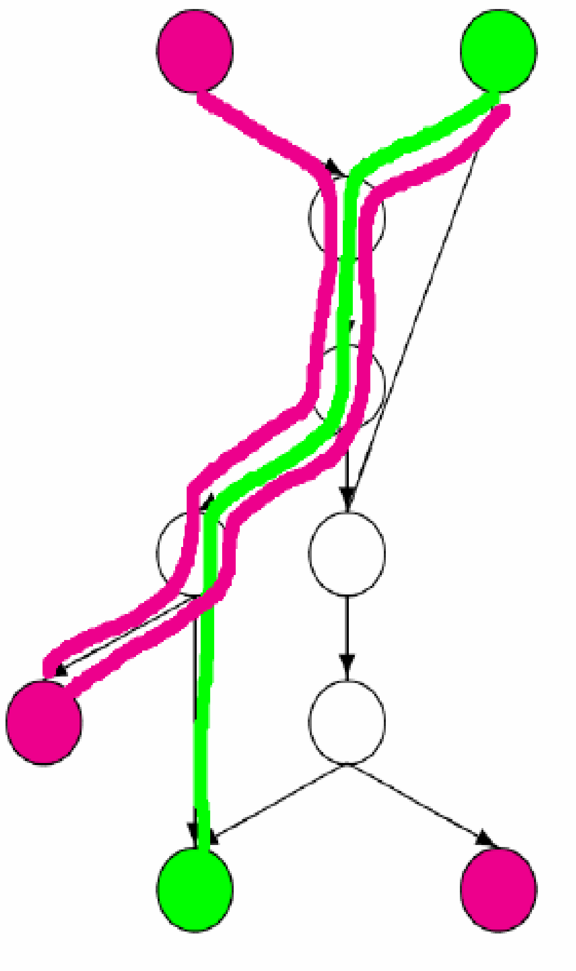

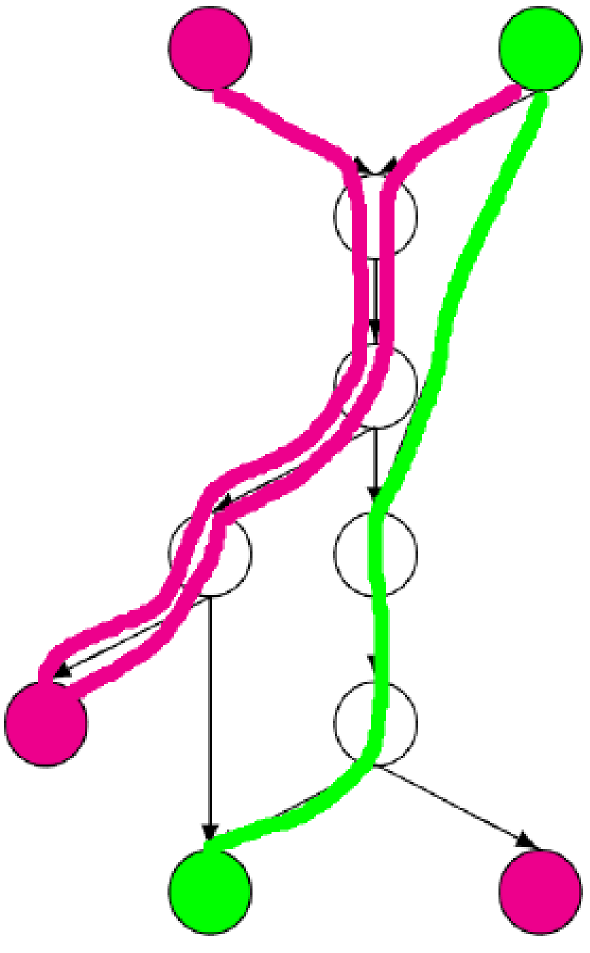

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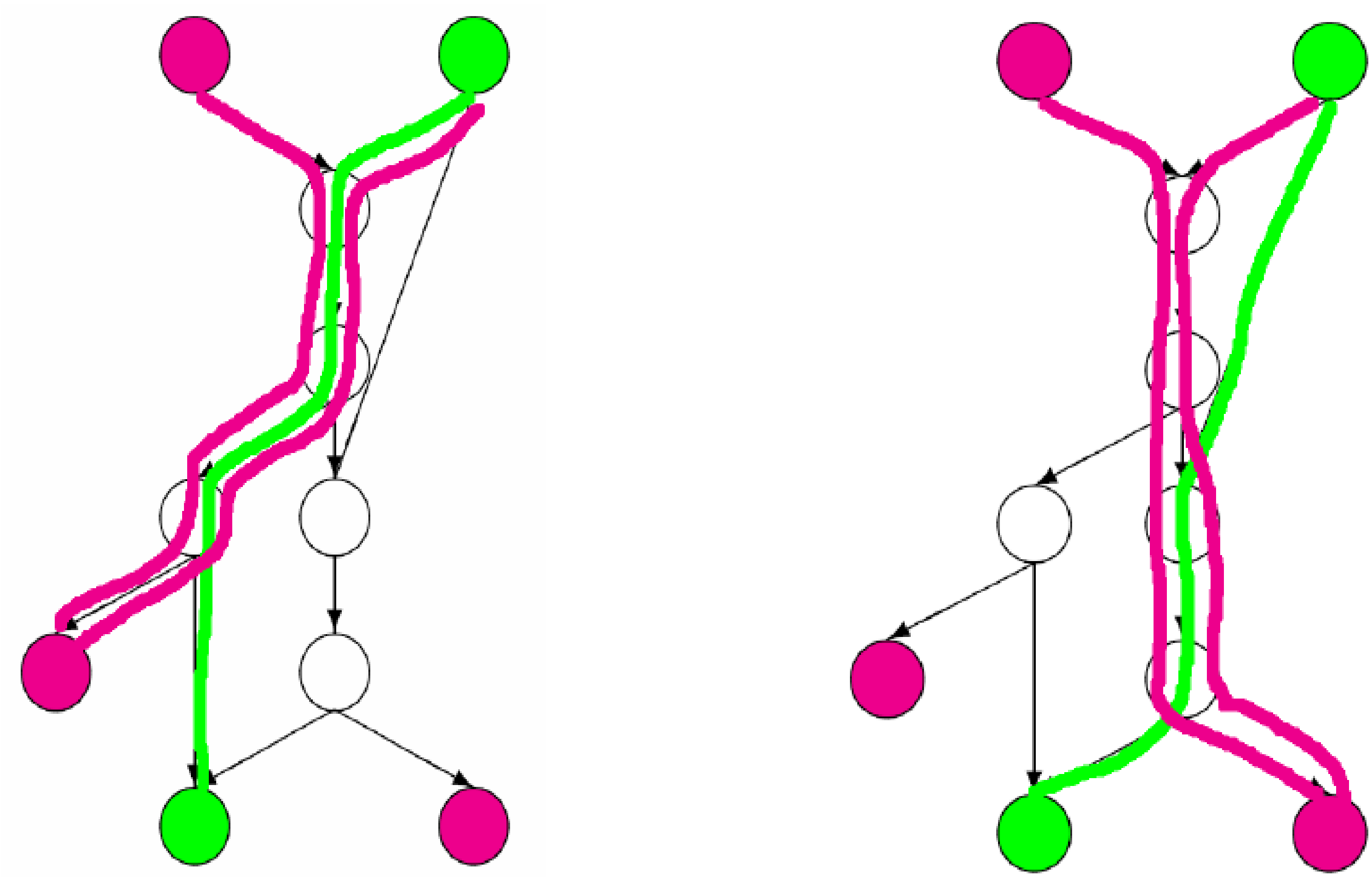

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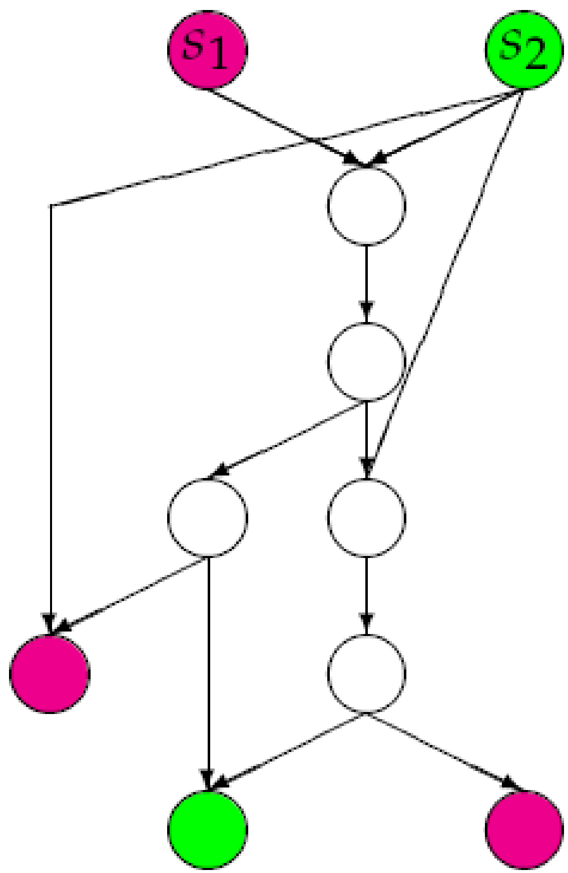

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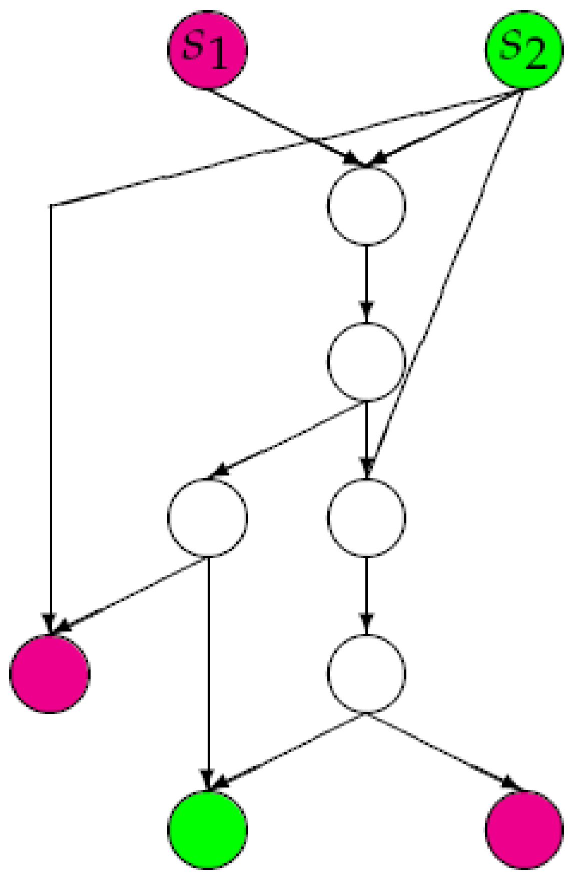

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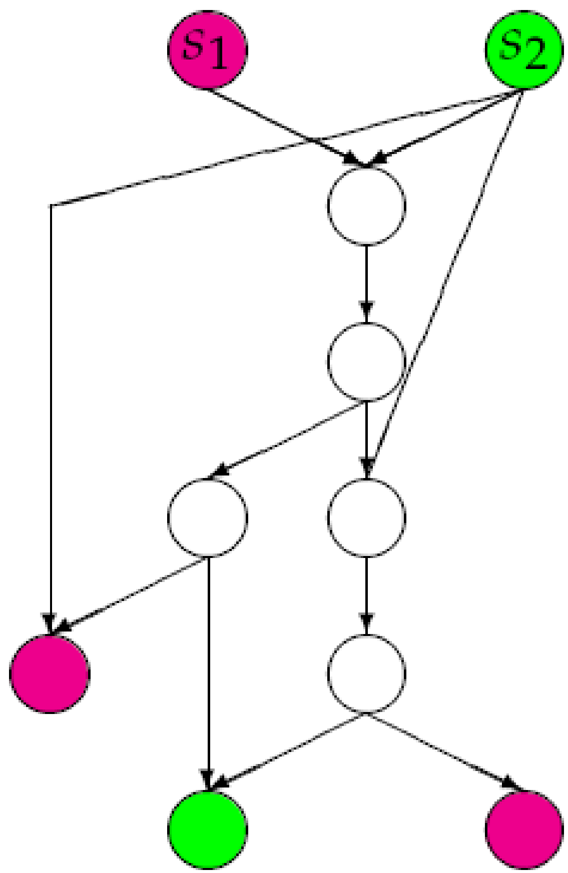

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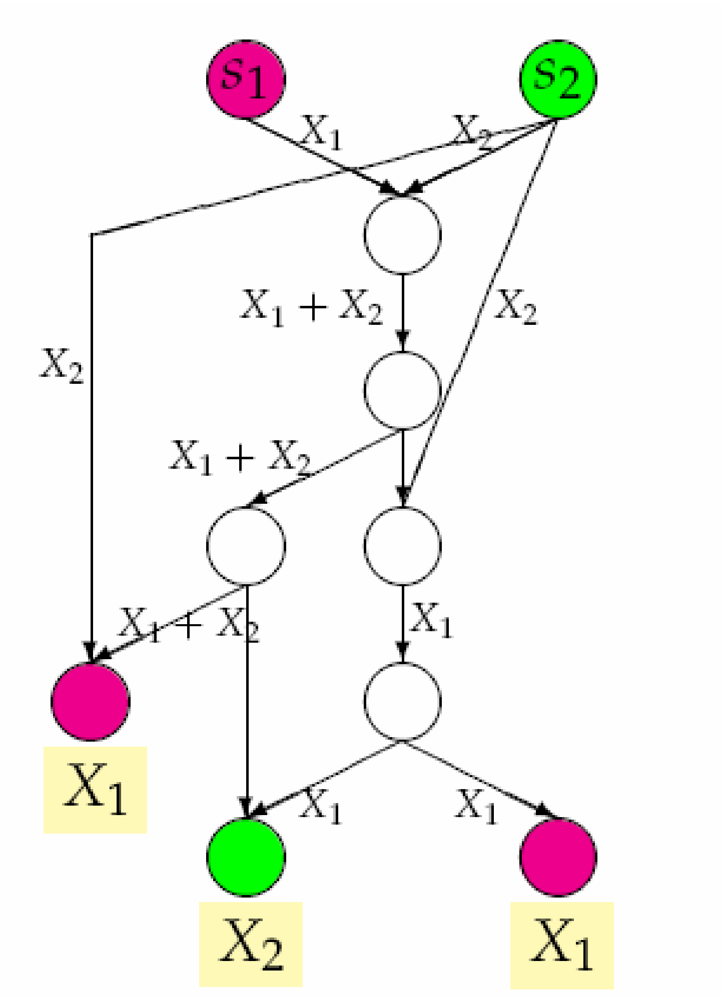

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**Theorem 2** *Network coding* $\Longleftrightarrow$ *one of the following two holds.*

1. $\exists \mathcal{P} = \{P_{s_1,t_1}, P_{s_2,t_2}\}$, *such that*

$$\max_{e \in E} \mathsf{ncp}_{\mathcal{P}}(e) \leq 1.$$

2. $\exists \mathcal{P} = \{P_{s_1,t_1}, P_{s_2,t_2}, P_{s_2,t_1}\}$ *and* $\mathcal{Q} = \{Q_{s_1,t_1}, Q_{s_2,t_2}, Q_{s_1,t_2}\}$ *s.t.*

$$\max_{e \in E} \mathsf{ncp}_{\mathcal{P}}(e) \leq 2 \text{ and } \max_{e \in E} \mathsf{ncp}_{\mathcal{Q}}(e) \leq 2.$$

# Degenerate Case(2)

If $\{t_{1,i}\} = \{t_{2,j}\}$, then it becomes the min-cut max-flow condition.

**Theorem 3** *The existence of intersession network coding* $\Leftrightarrow$

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# Corollaries

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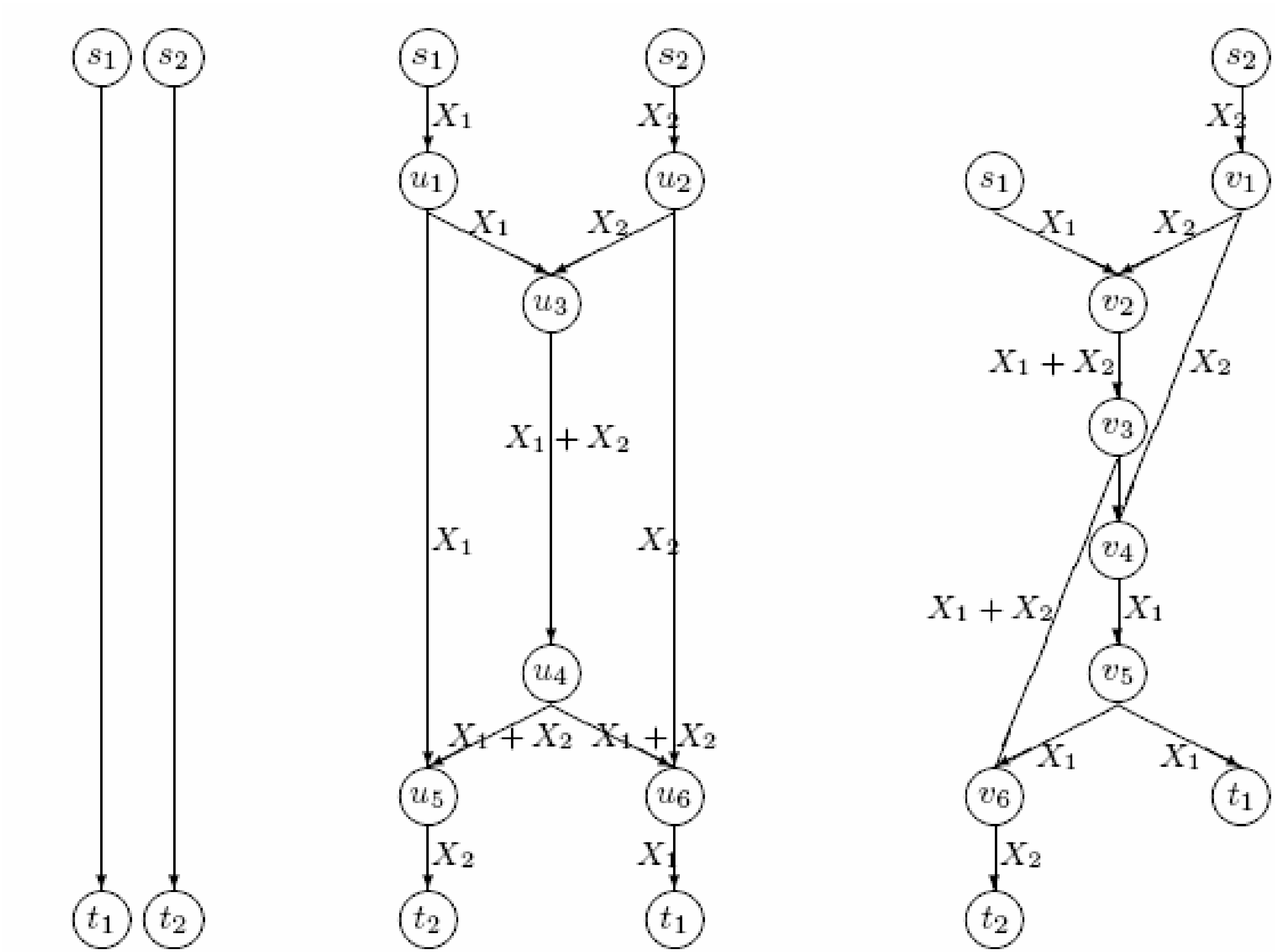
*Corollary 3 (Sufficiency of Linear Network Codes):* The existence of a non-linear network coding solution implies the existence of a linear network coding solution.

# Strengthened 2-2 case.




(a) 2-EDPs (b) The full butterfly (c) The full grail

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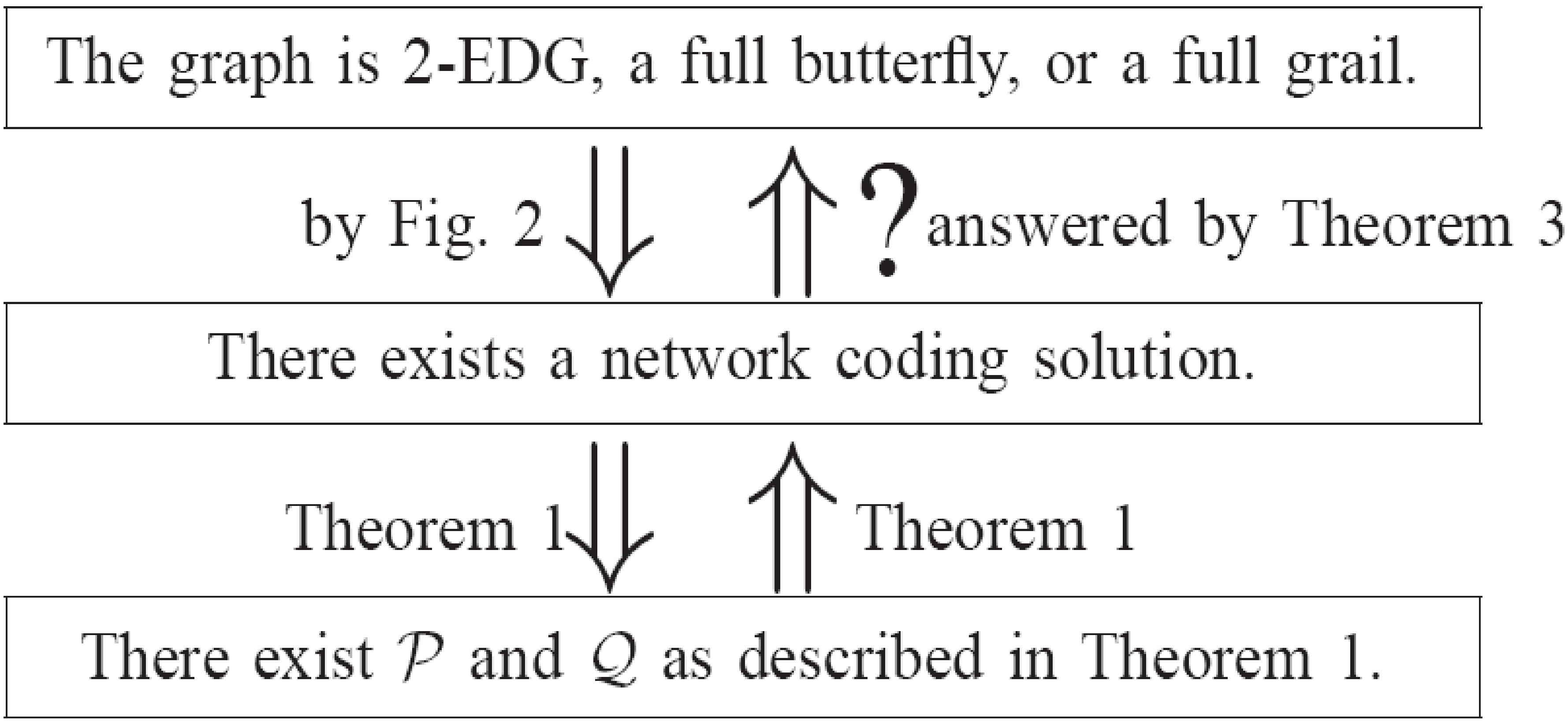

# Strengthened 2-2 case.

*Theorem 4:* If a network coding solution exists, then it is either the case that a routing solution also exists (with 2-EDPs) or the case in which the graph contains a subgraph that is topologically similar to a butterfly or a grail.

# Strengthened 2-2 case.

*Theorem 4:* If a network coding solution exists, then it is either the case that a routing solution also exists (with 2-EDPs) or the case in which the graph contains a subgraph that is topologically similar to a butterfly or a grail.

Different from the information decomposition arguments in [Fragouli and Soljanin 06].

- For two sources instead of one
- Uses purely graph theoretic arguments.
- No need to consider the graph's line graph.

# Sketch of the proof

$\max_{e \in E} \mathrm{ncp}_{\mathcal{P}}(e) \leq 2$

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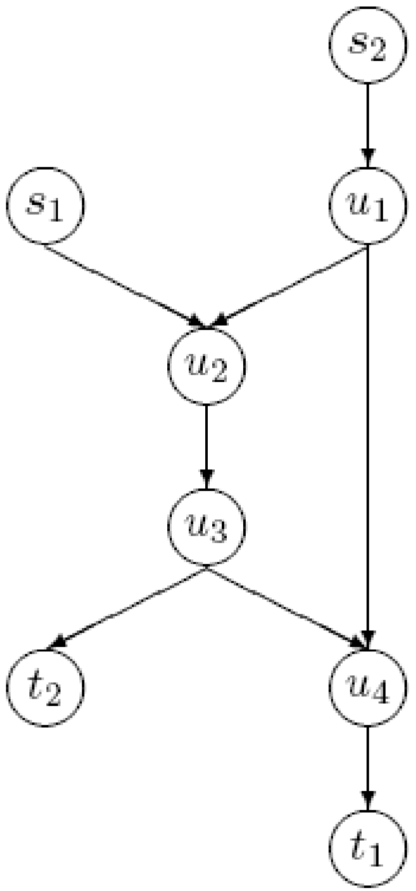

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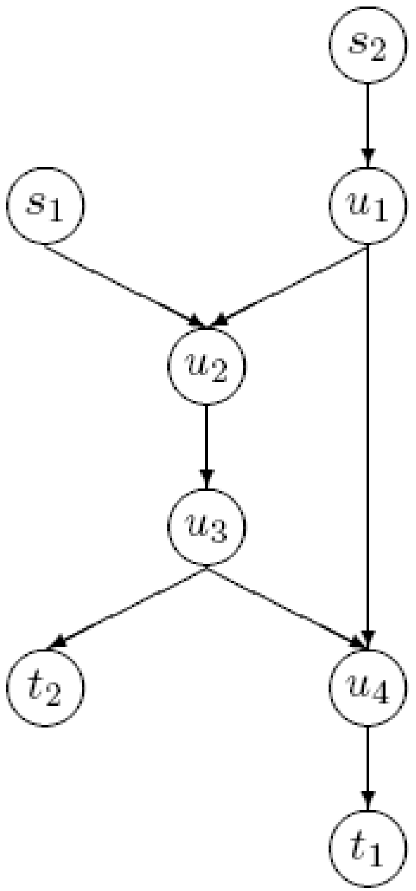

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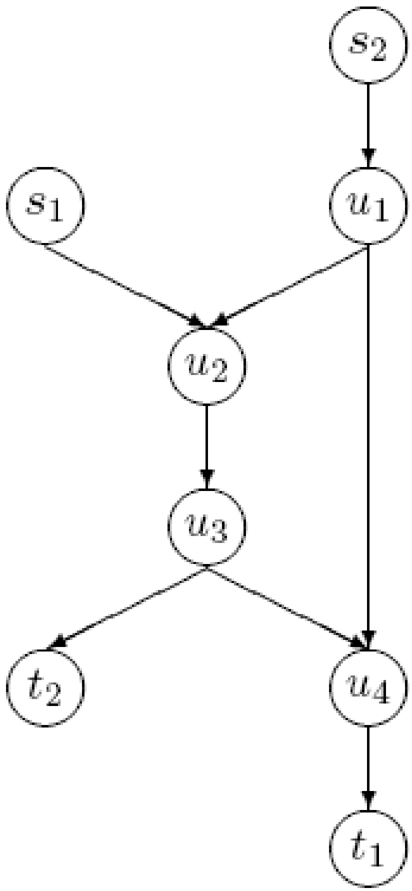
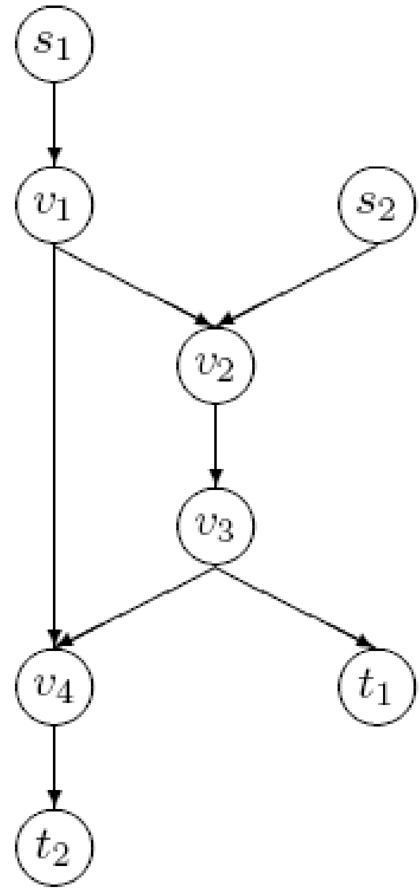



$\max_{e \in E} \mathsf{ncp}_{\mathcal{Q}}(e) \leq 2.$



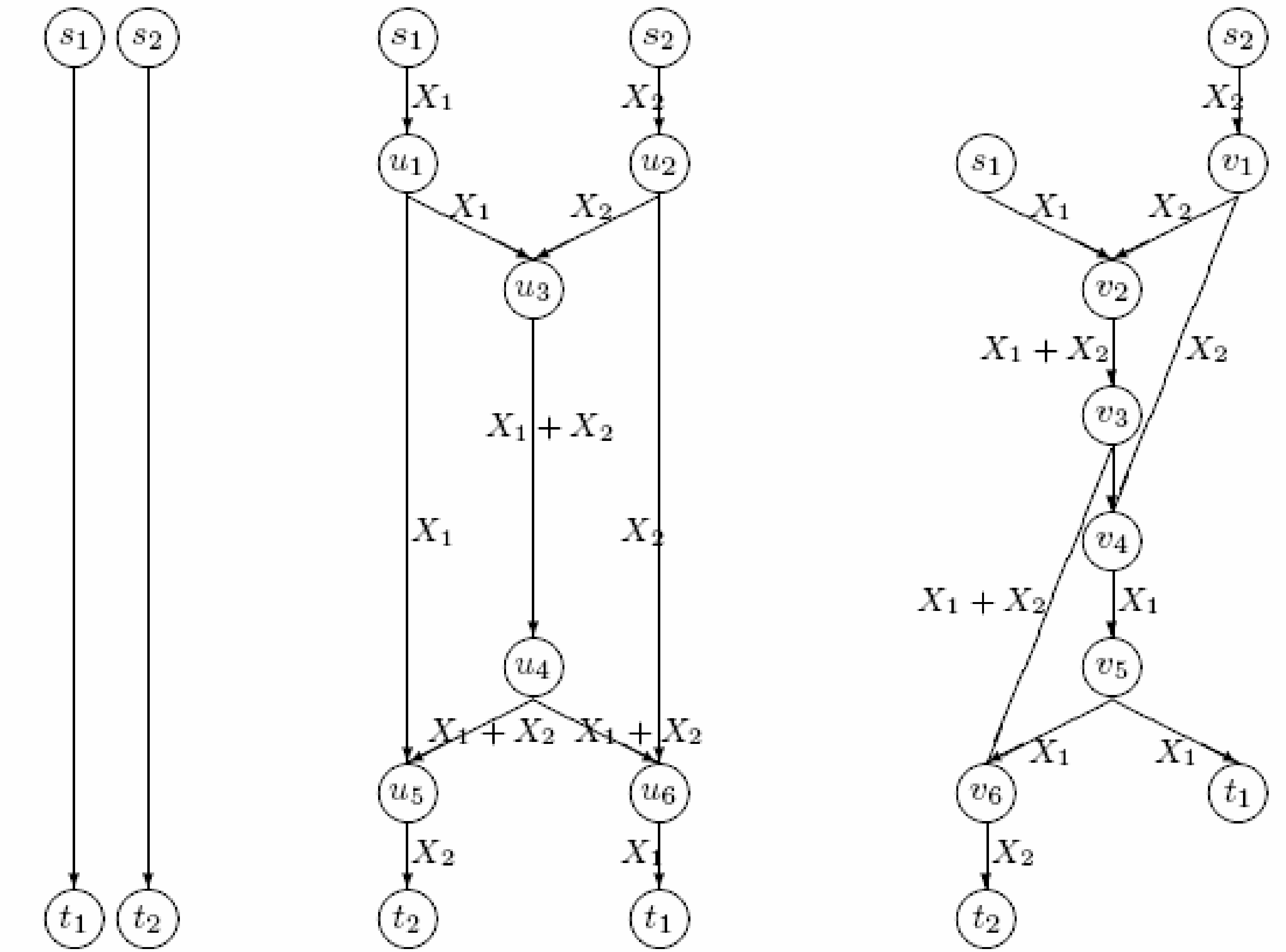
# Corollaries

*Corollary 4:* If there exists any network coding solution for the two-unicast problem, there exists a binary linear network coding solution for the same problem. Furthermore, either a routing-based solution exists (with no encoding/decoding) or one needs exactly three binary exclusive OR (XOR) operations for the entire network encoding/decoding solution.

# Corollaries

*Corollary 5: (Bandwidth and alphabet optimality)* For any scheme that is not one of the three cases and uses higher than binary alphabets, then there exists a scheme that is strictly bandwidth and alphabet more efficient.

# Corollaries (Reciprocity)




(a) 2-EDPs (b) The full butterfly (c) The full grail

# Applications

$I$: the no. coexisting unicast sessions $(s_i, t_i)$

$\mathcal{P}(i)$: the set of all $(s_i, t_i)$ paths

$\mathbb{P}(i,j)$: the set of all $(P_{s_i,t_i}, P_{s_j,t_i}, P_{s_j,t_j})$ tuples

$E^k_{e,i}$ : $= 1$, if link $e$ uses the $k$-th path in $\mathcal{P}(i)$

$= 0$, otherwise

$H^l_{e,ij}$ : $= 2$, if for the $l$-th tuple in $\mathbb{P}(i,j)$, $\mathrm{ncp}(e) = 3$

$= 1$, if for the $l$-th tuple in $\mathbb{P}(i,j)$, $\mathrm{ncp}(e) = 1, 2$

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$$\max_{\vec{x}, \vec{g}} \sum_{i=1}^{I} U_i \left( \sum_{k=1}^{|\mathcal{P}(i)|} x_i^k + \sum_{j \neq i} \sum_{l=1}^{|\mathbb{P}(i,j)|} \sum_{m=1}^{|\mathbb{P}(j,i)|} g_{ij}^{lm} \right)$$

$$\text{s.t.} \sum_{i=1}^{I} \sum_{k=1}^{|\mathcal{P}(i)|} E^k_{e,i} x_i^k + \sum_{i=1}^{I} \sum_{i<j} \sum_{l=1}^{|\mathbb{P}(i,j)|} \sum_{m=1}^{|\mathbb{P}(j,i)|} \max(H^l_{e,ij}, H^m_{e,ji}) g_{ij}^{lm} \leq C_e, \forall e$$

$$x_i^k \geq 0, \quad g_{ij}^{lm} = g_{ji}^{ml} \geq 0, \quad \forall i \neq j, l, m$$

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# Applications

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Distributed path-based network optimization

with arbitrary utility function. [Submitted to Infocom 08]

ma

$\vec{x}$,

$$\text{s.t.} \sum_{i=1}^{I} \sum_{k=1}^{|\mathcal{P}(i)|} E^k_{e,i} x^k_i + \sum_{i=1}^{I} \sum_{i<j} \sum_{l=1}^{|\mathbb{P}(i,j)|} \sum_{m=1}^{|\mathbb{P}(j,i)|} \max(H^l_{e,ij}, H^m_{e,ji}) g^{lm}_{ij} \leq C_e, \forall e$$

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- We have adopted the path-based conditions for stronger rate control algorithms.

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- Characterization of the problem with more than two multicast sessions.