Beyond the Butterfly — A Graph-Theoretic Characterization for Network Coding with Two Simple Unicast Sessions

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Two Simple Unicast Sessions

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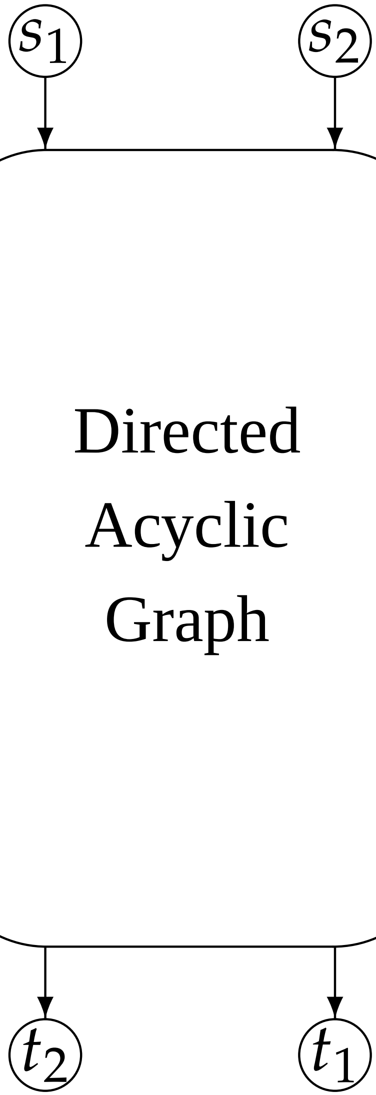

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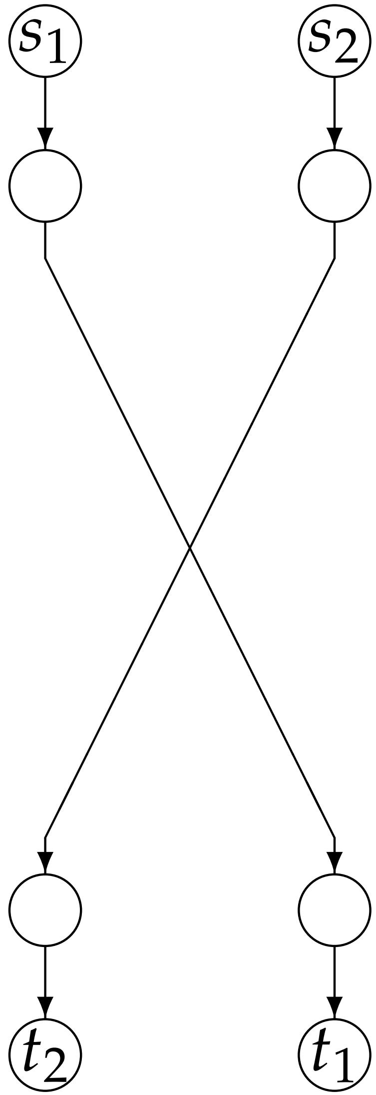
$\Longleftrightarrow$ Edge disjoint paths



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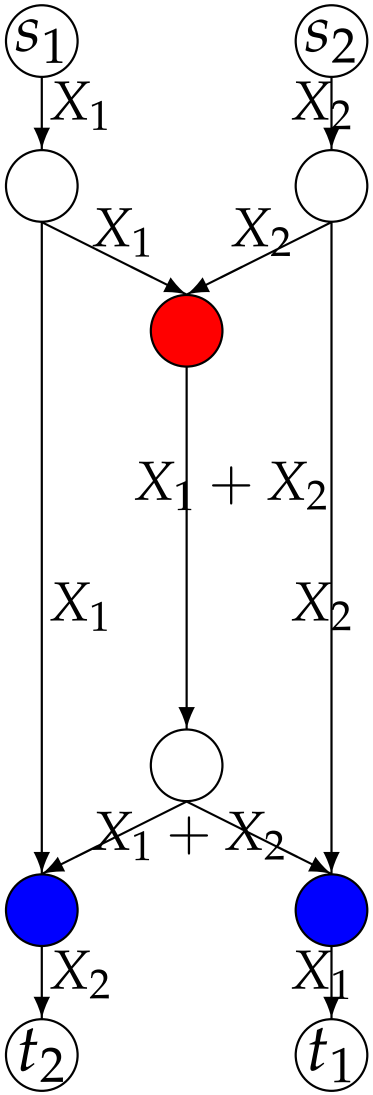

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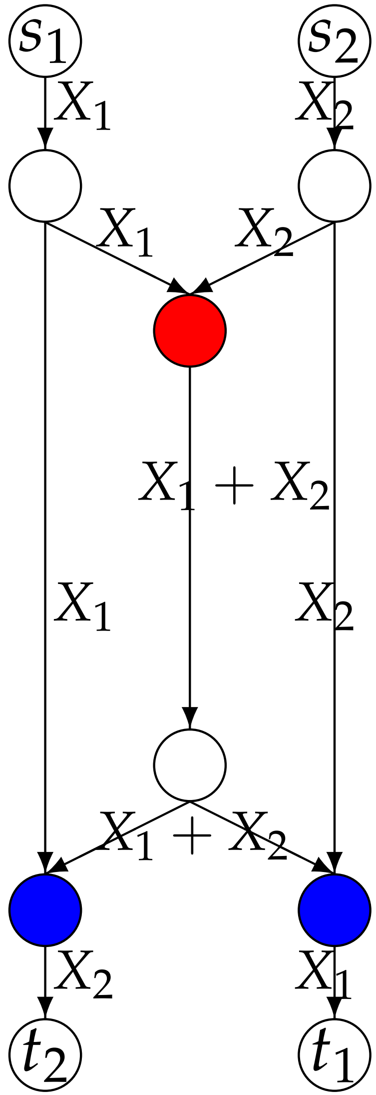

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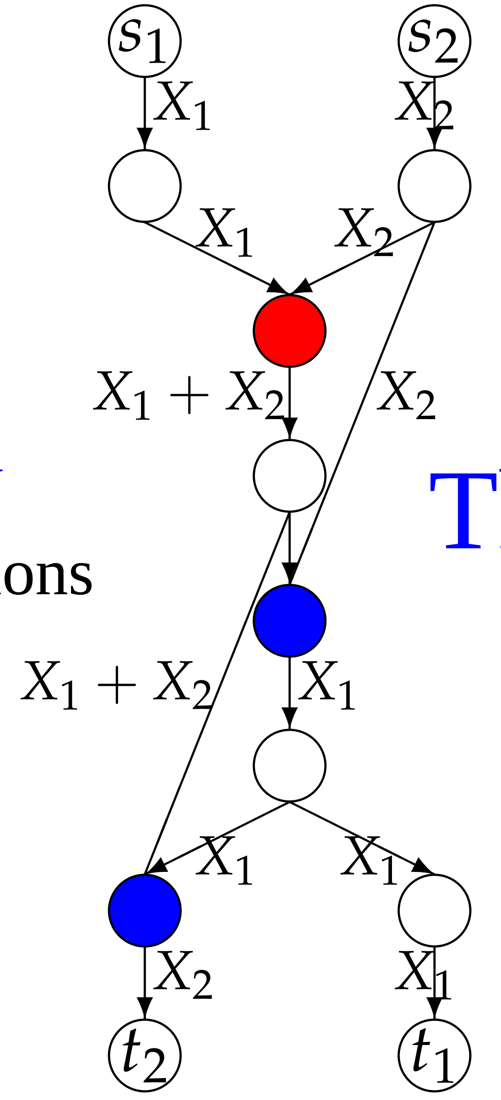

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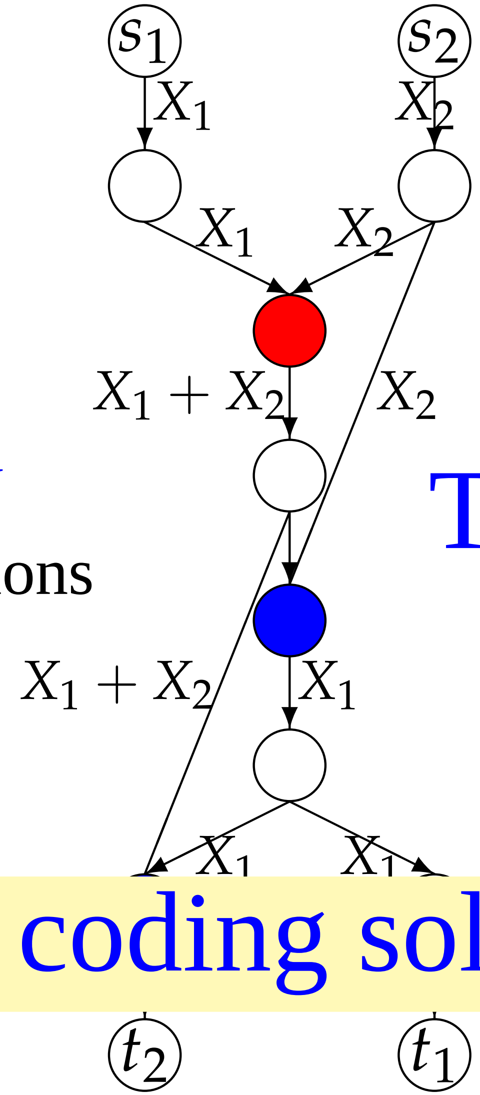

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Q: Network coding solutions $\Leftrightarrow$???

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- Applications on distributed rate control algorithms.

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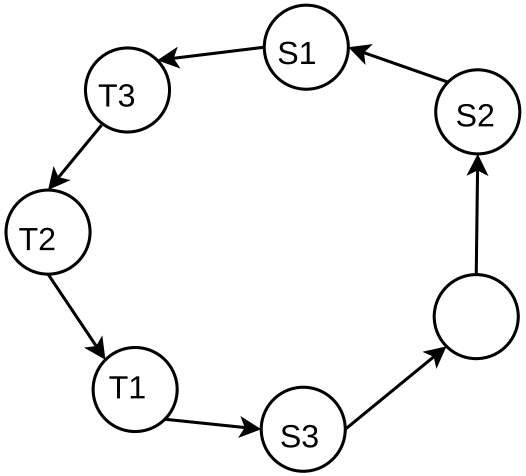
- Directed Cycles [1]

$$\sum_{i \text{ separated by } e} r_i \leq c(e)$$



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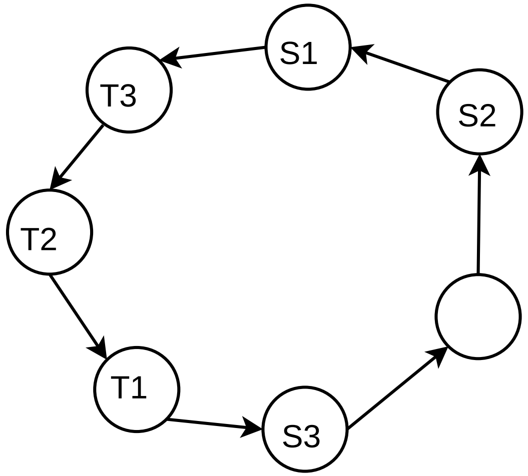

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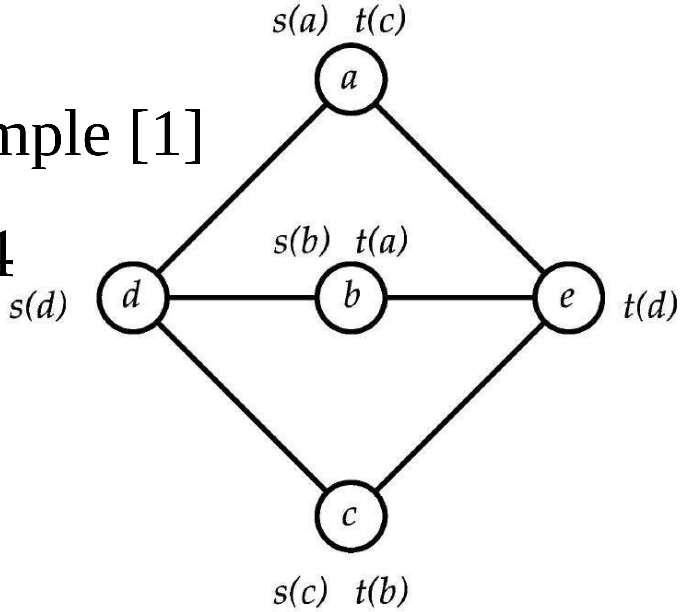

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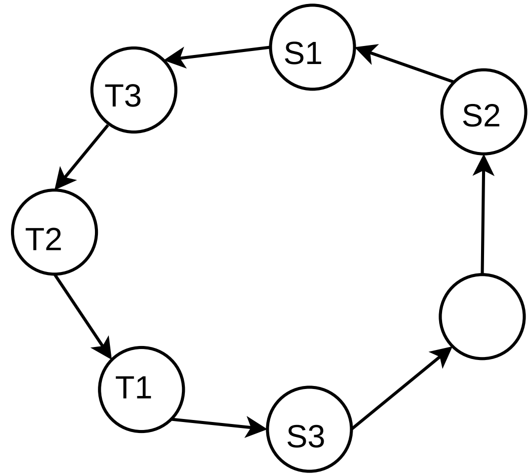

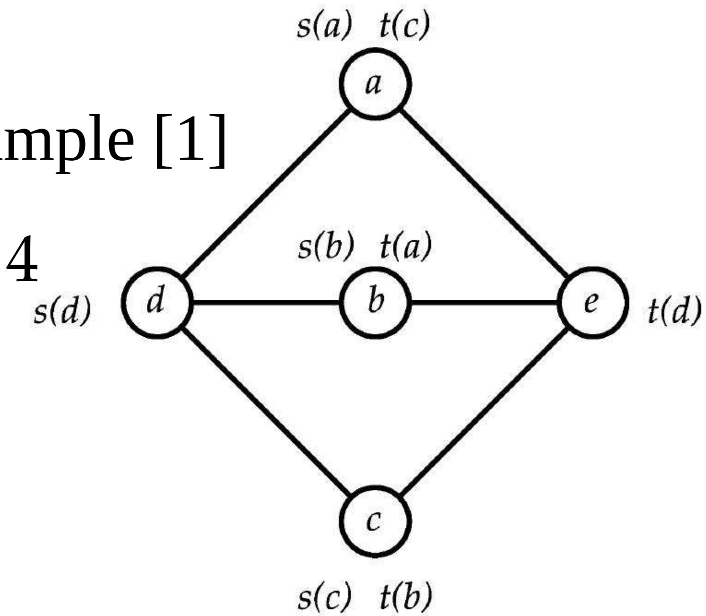

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- Directed, acyclic, degree 2, three-layer networks [2]

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  - Butterfly-based construction [Traskov *et al.* 06], pollution-treatment [Wu 06].

The Main Theorem

- Setting: General finite directed acyclic graphs, unit edge capacity, (s_1, t_1) & (s_2, t_2), two integer symbols X_1 and X_2.
- Number of Coinciding Paths of edge e: $\mathcal{P} = \{P_1, \cdots, P_k\}$, and $\mathsf{ncp}_{\mathcal{P}}(e) = |\{P \in \mathcal{P} : e \in P\}|$.

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1. $\exists \mathcal{P} = \{P_{s_1,t_1}, P_{s_2,t_2}\}$, *such that*

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2. $\exists \mathcal{P} = \{P_{s_1,t_1}, P_{s_2,t_2}, P_{s_2,t_1}\}$ *and* $\mathcal{Q} = \{Q_{s_1,t_1}, Q_{s_2,t_2}, Q_{s_1,t_2}\}$ *s.t.*

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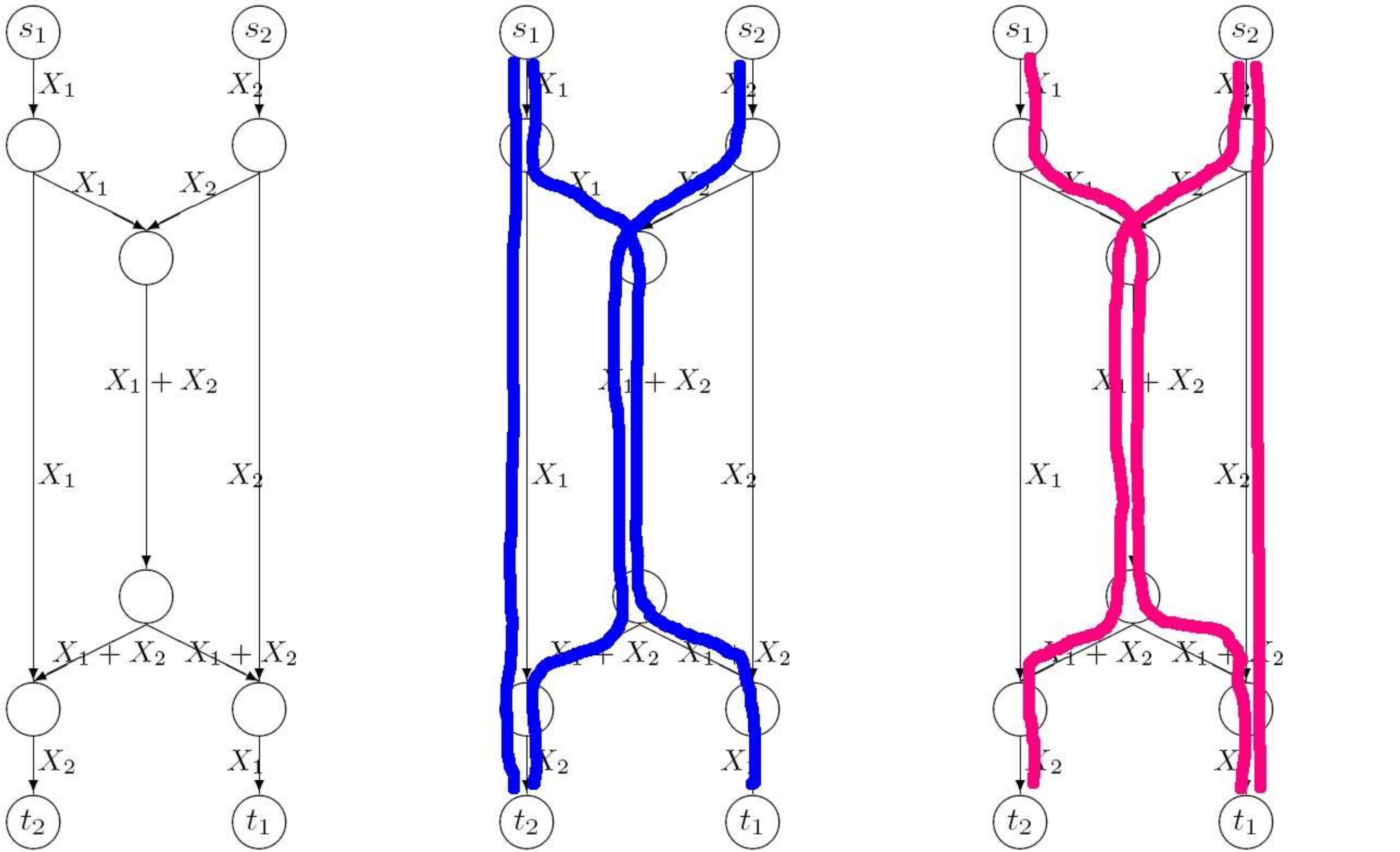
Routing: edge disjointness vs. Network coding: controlled overlaps.

Feasible Example: The Butterfly



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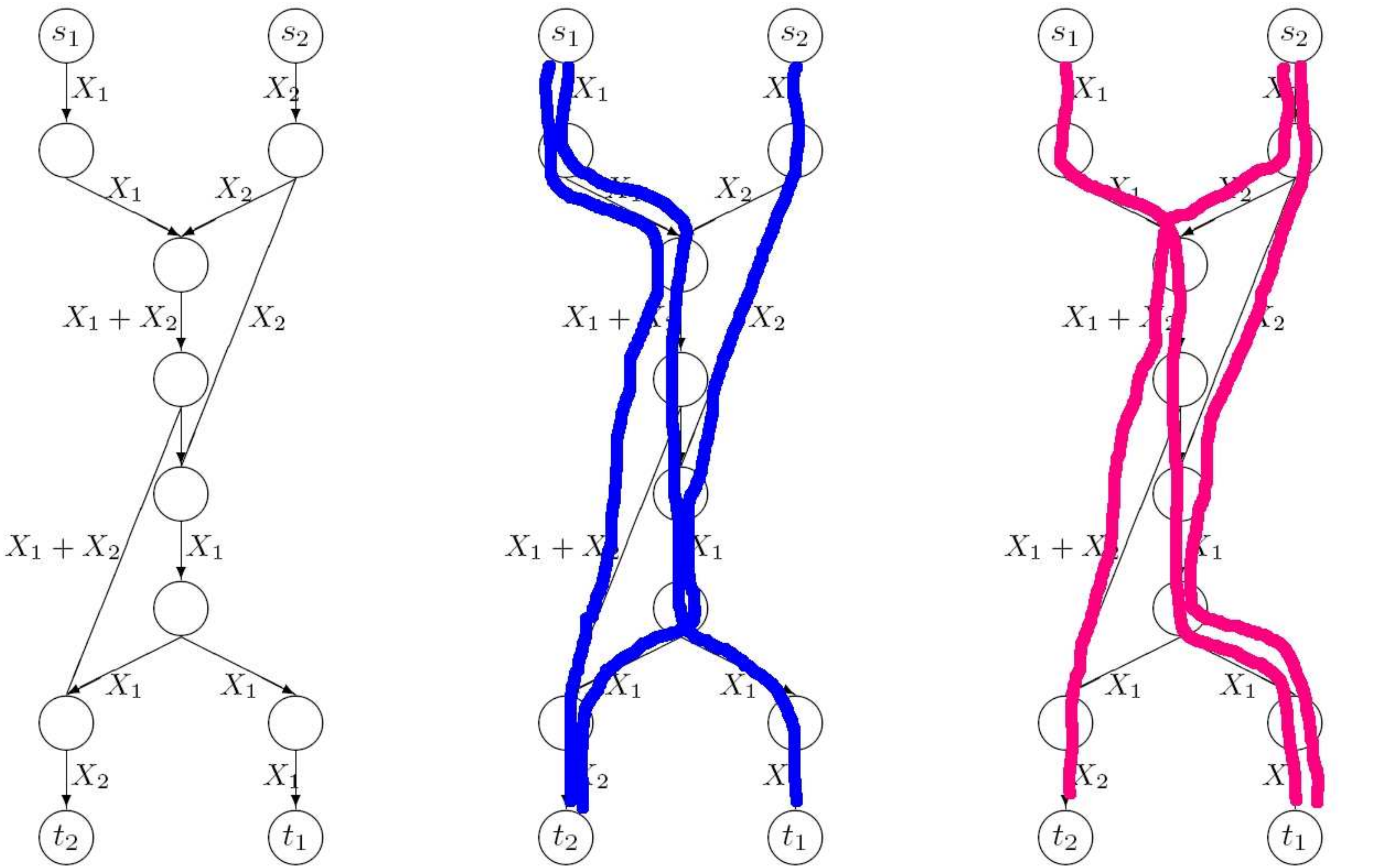
Feasible Example 2: The Grail



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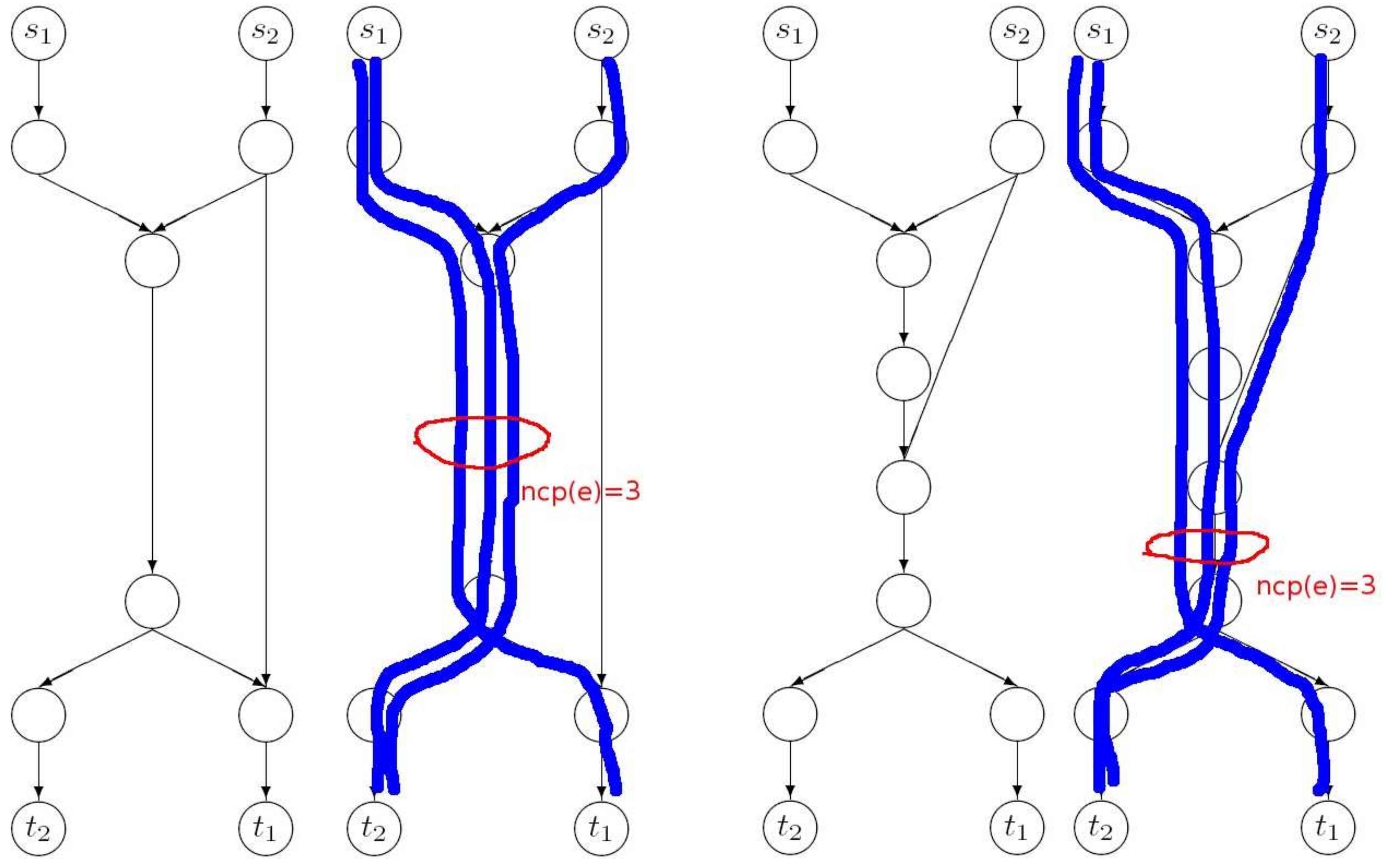

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- Linear network coding is sufficient, a byproduct of the proof.

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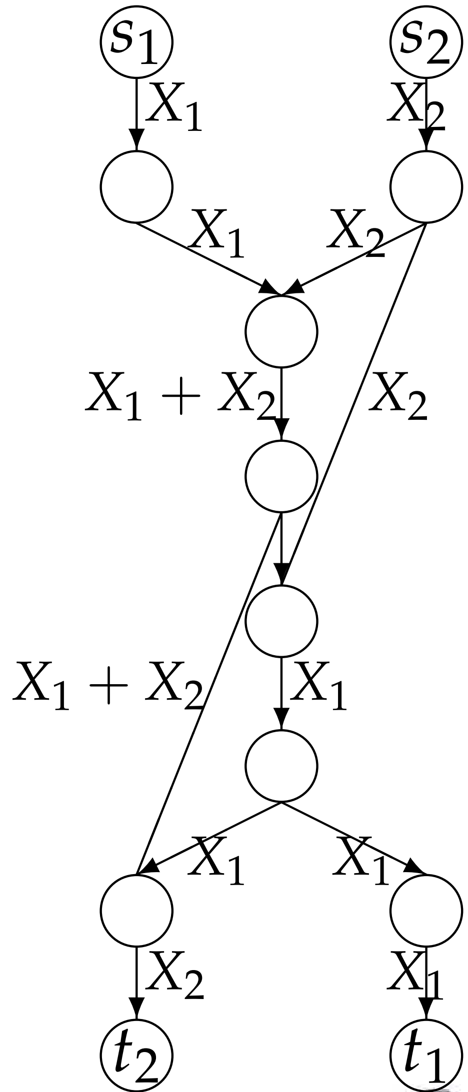
A network coding solution exists but not a routing one.



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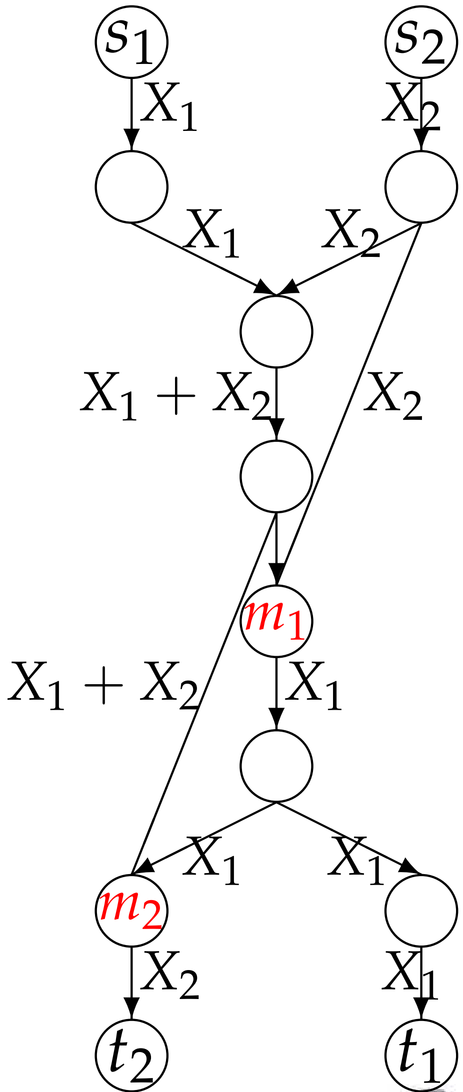

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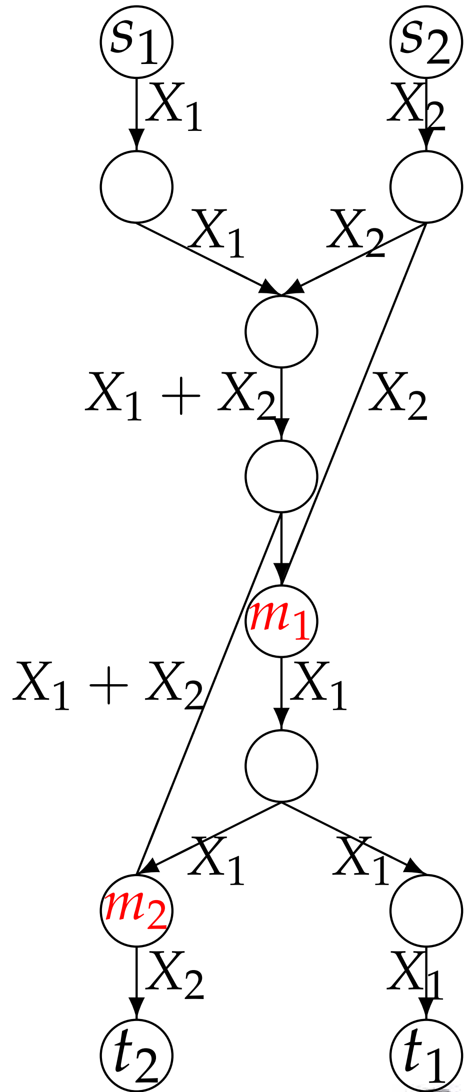

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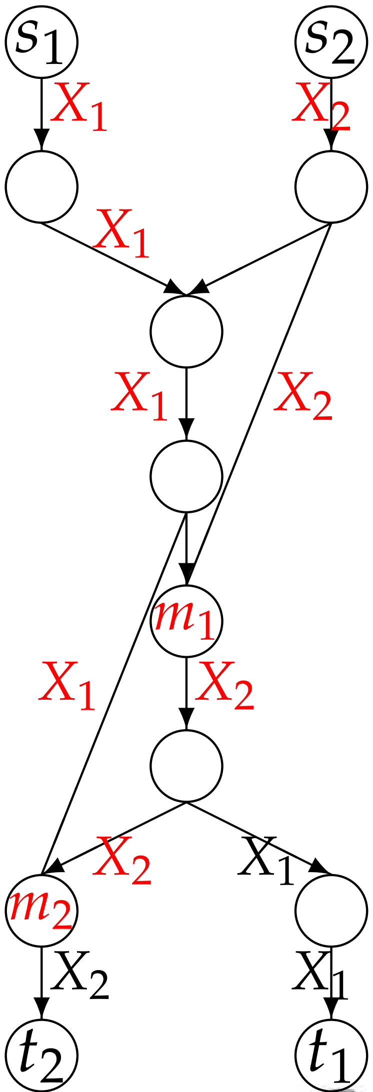

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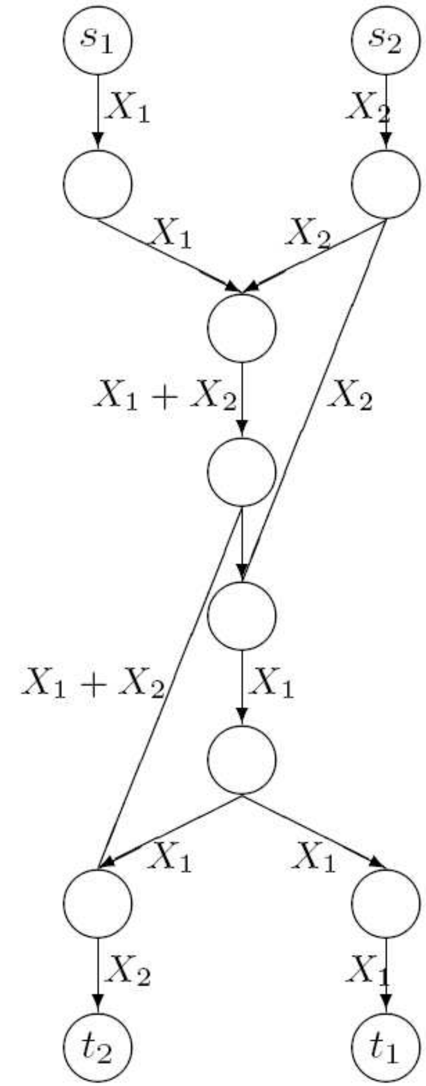

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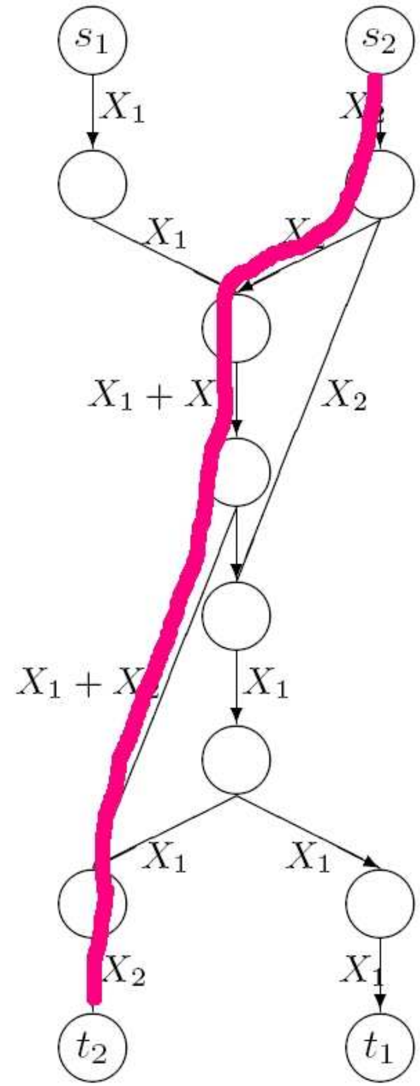

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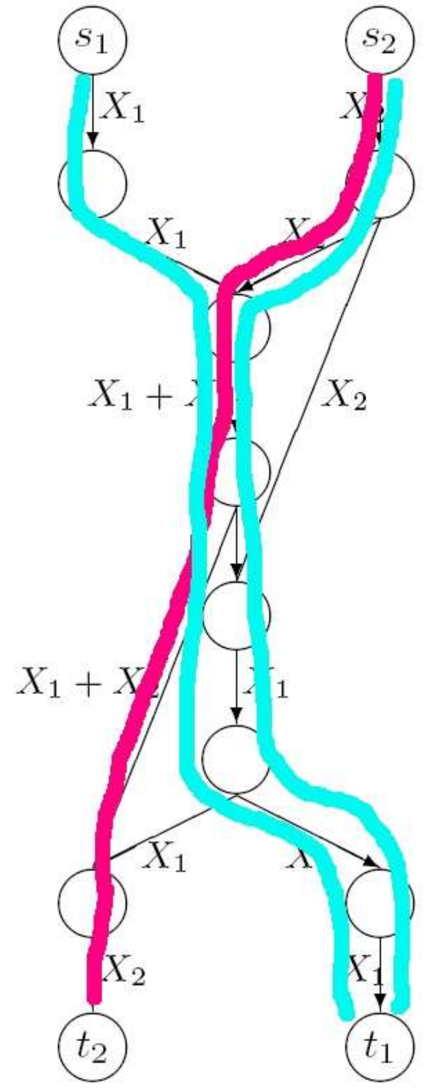

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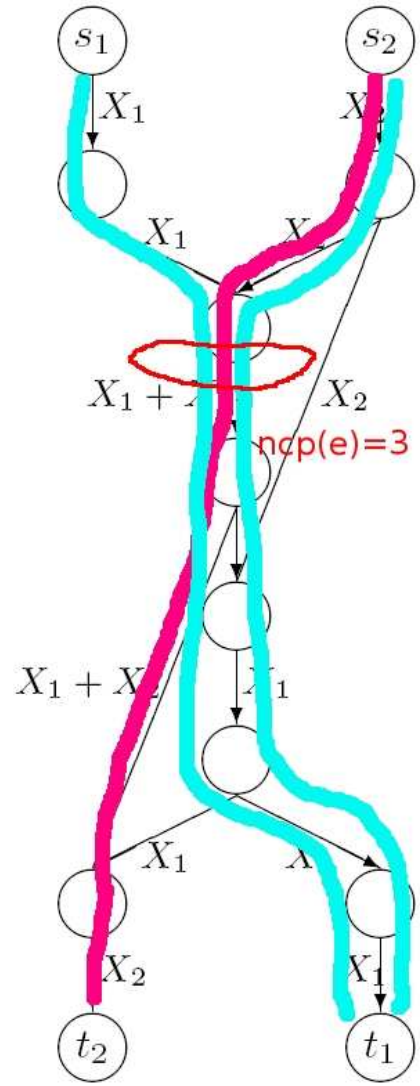

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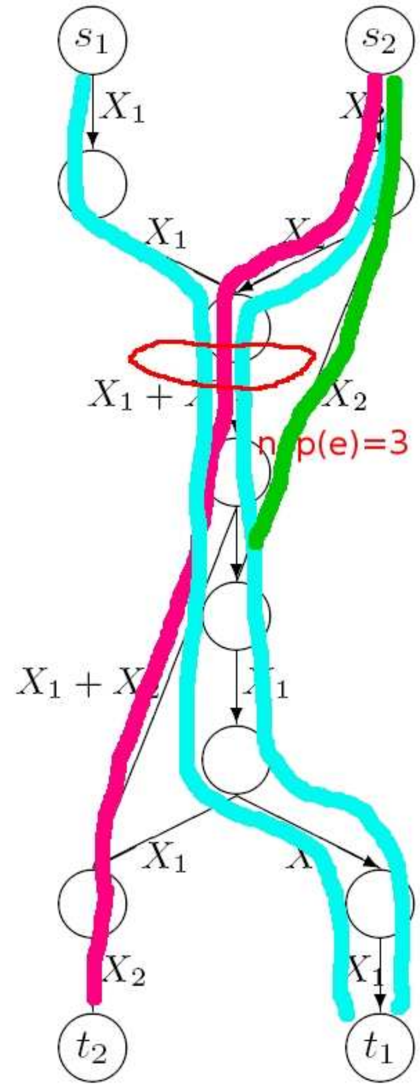

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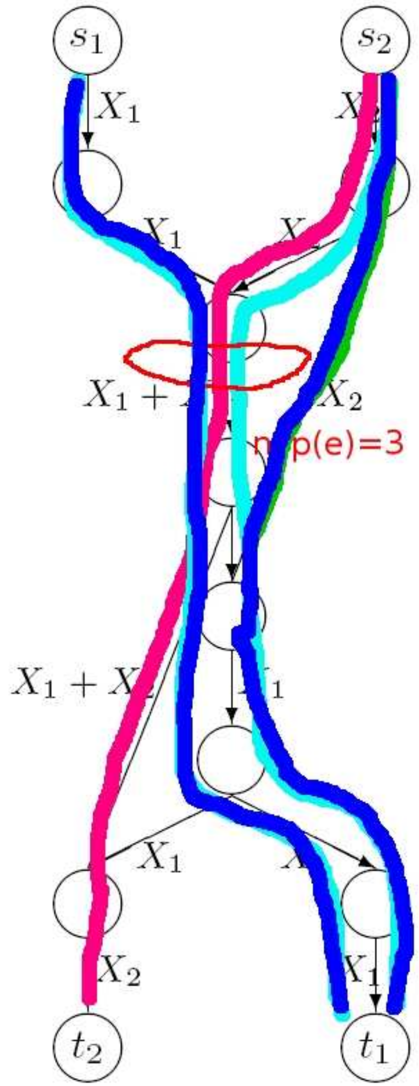

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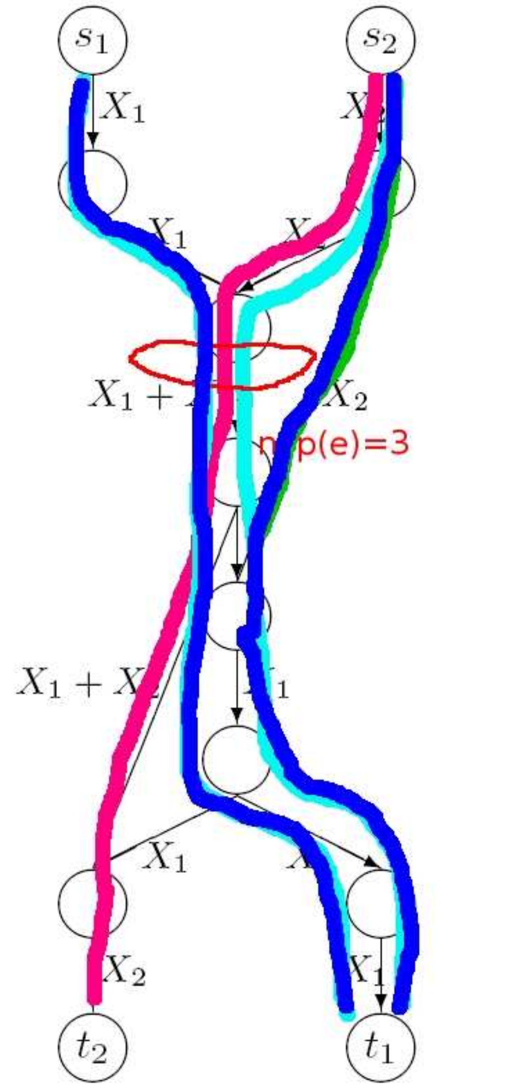

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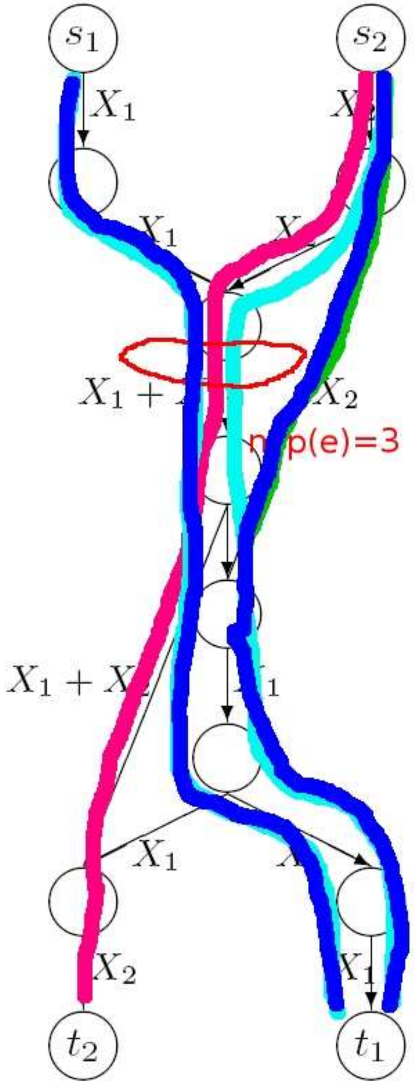

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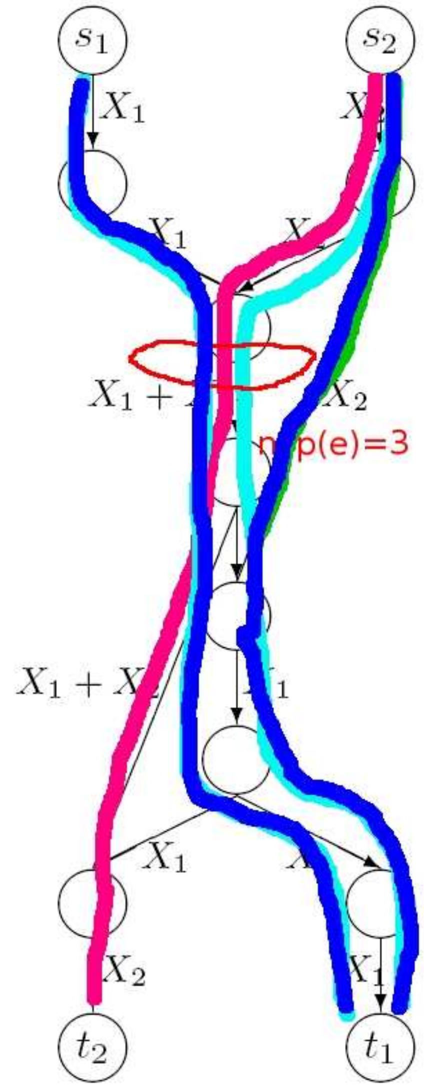

A proof of the sufficiency

Assume $\{P_{s_1,t_1}, P_{s_2,t_2}, P_{s_2,t_1}\}$ and $\{Q_{s_1,t_1}, Q_{s_2,t_2}, Q_{s_1,t_2}\}$
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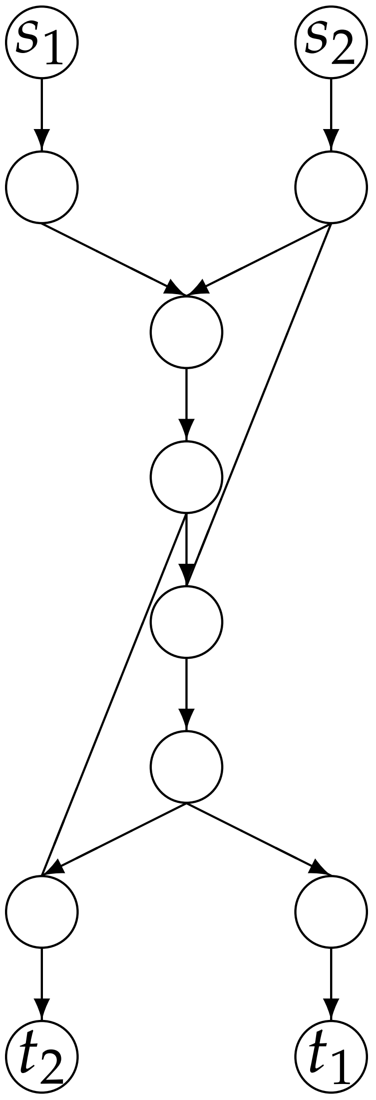

A two-staged, add-up-&-reset construction

1. The random add-up stage:
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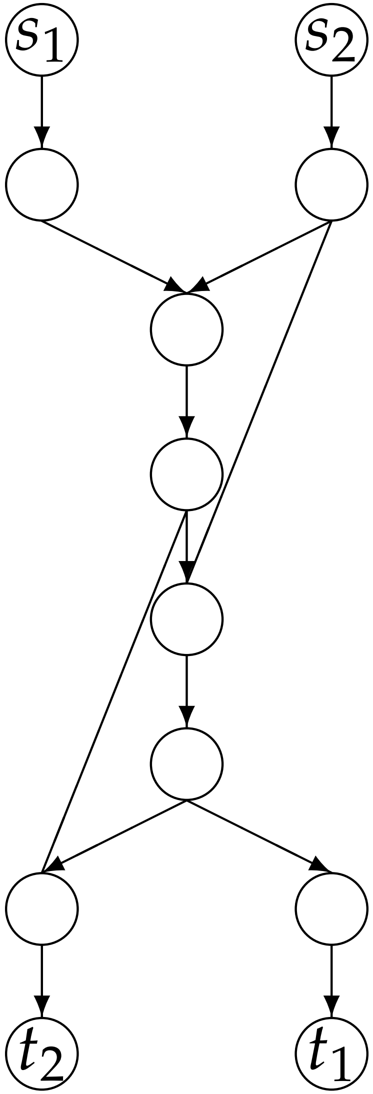
Random add-up:

(a) If all M_{IN} messages are identical, then $M_e = M_{\text{IN}}$.

(b) Otherwise $M_e = a_1 M_1 + \cdots + a_m M_m$ for $a_i > 0$, such that M_e is linearly indep. of any other messages $M_{e'}$ for those e' not in the downstream of e.



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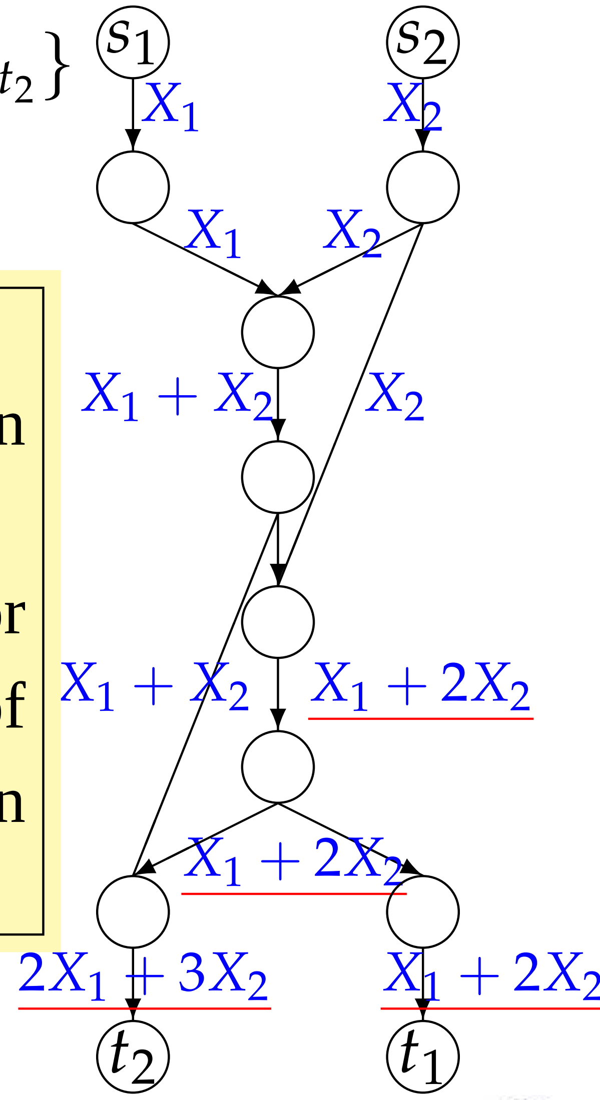

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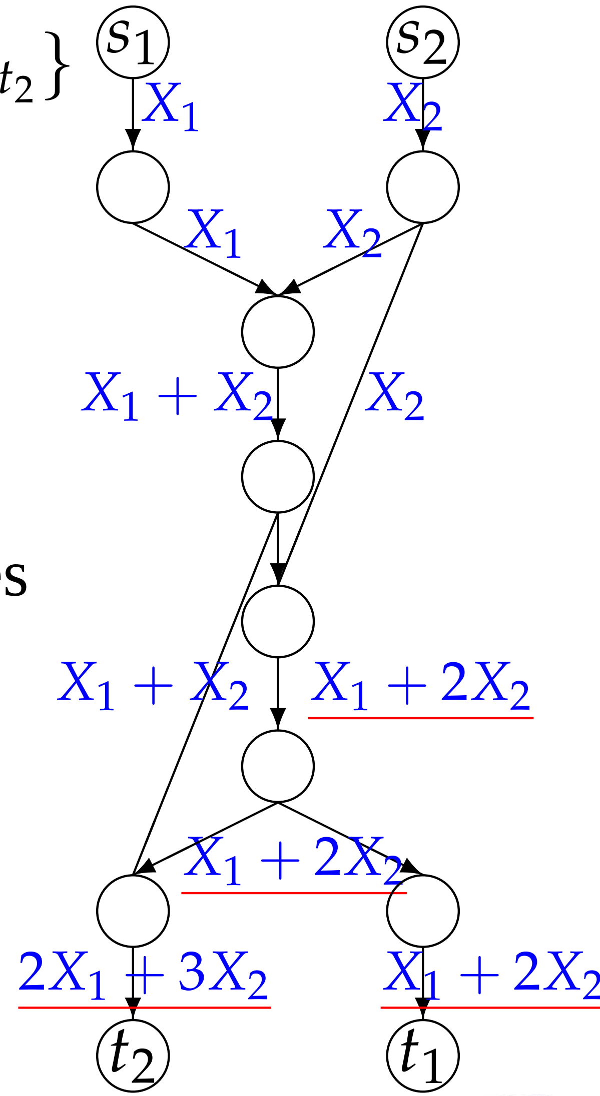

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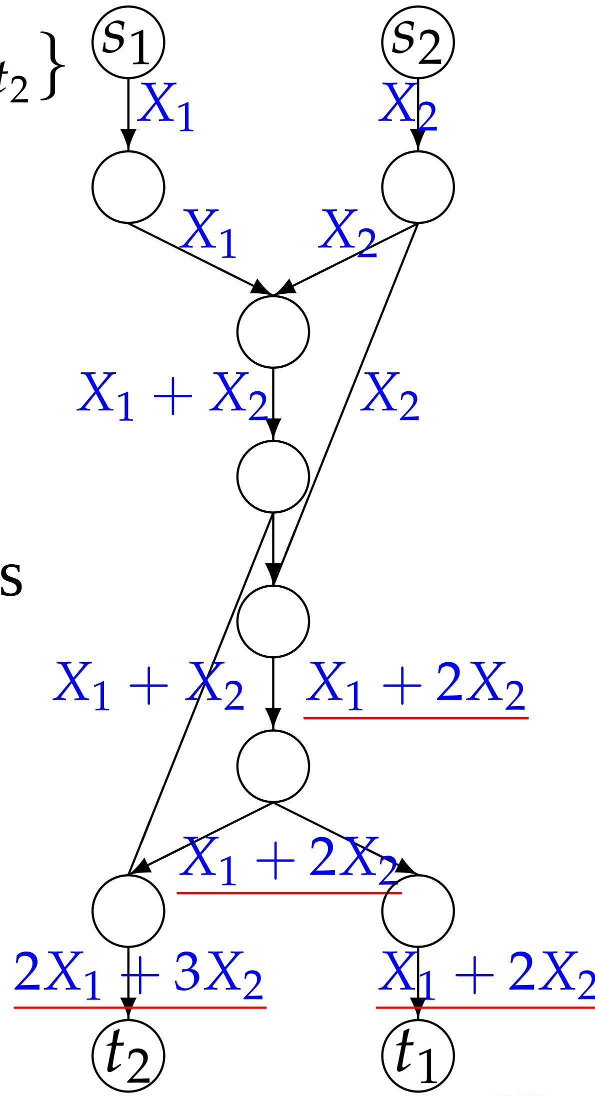

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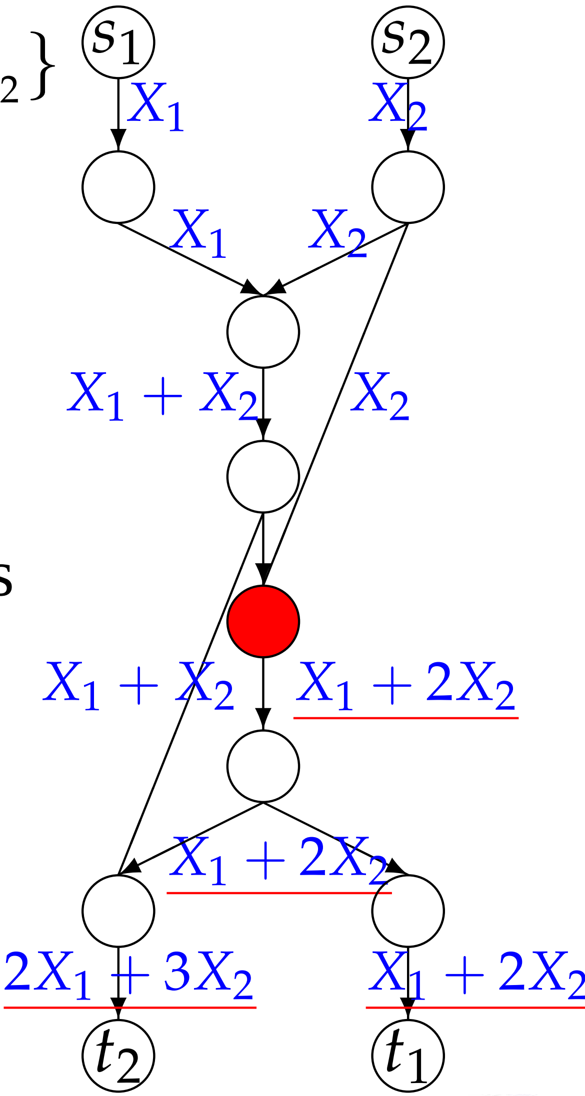

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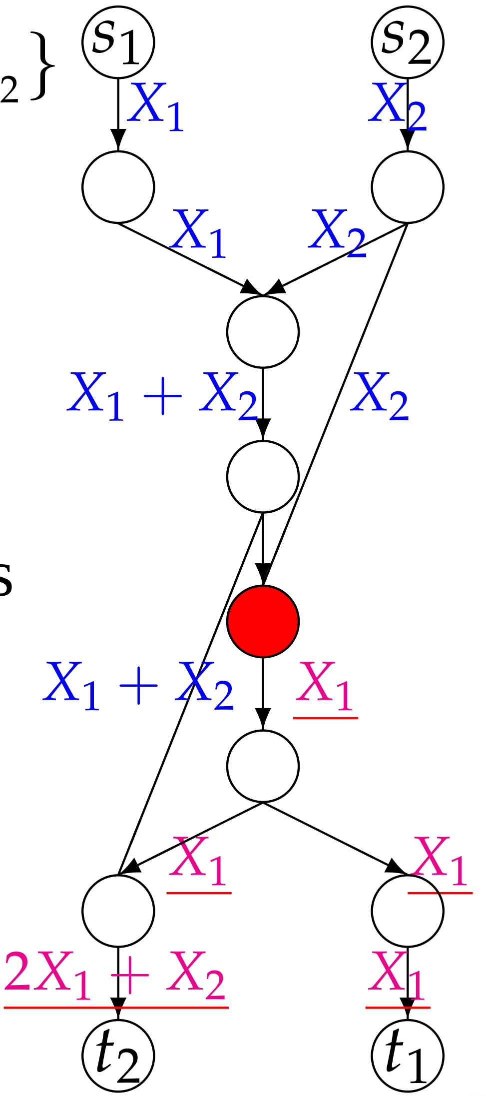

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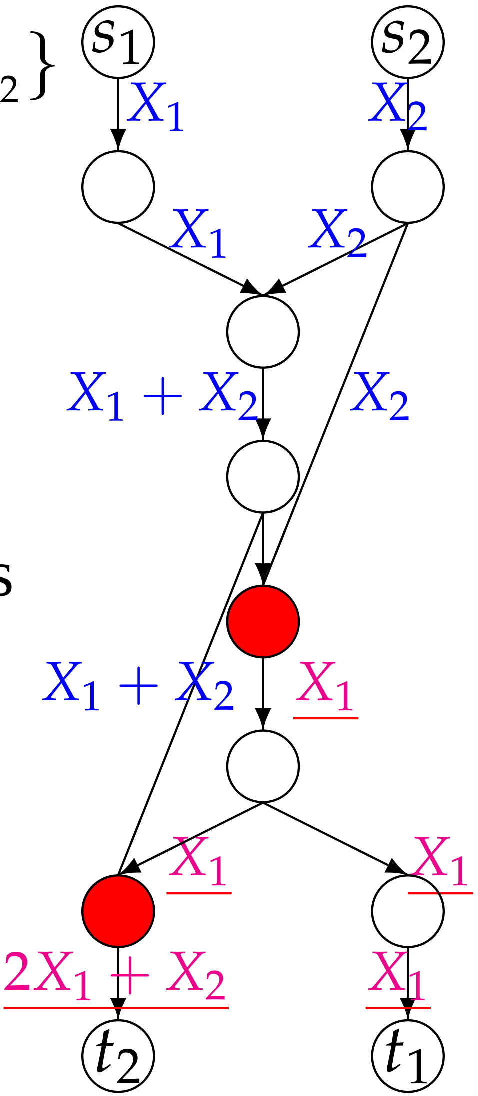

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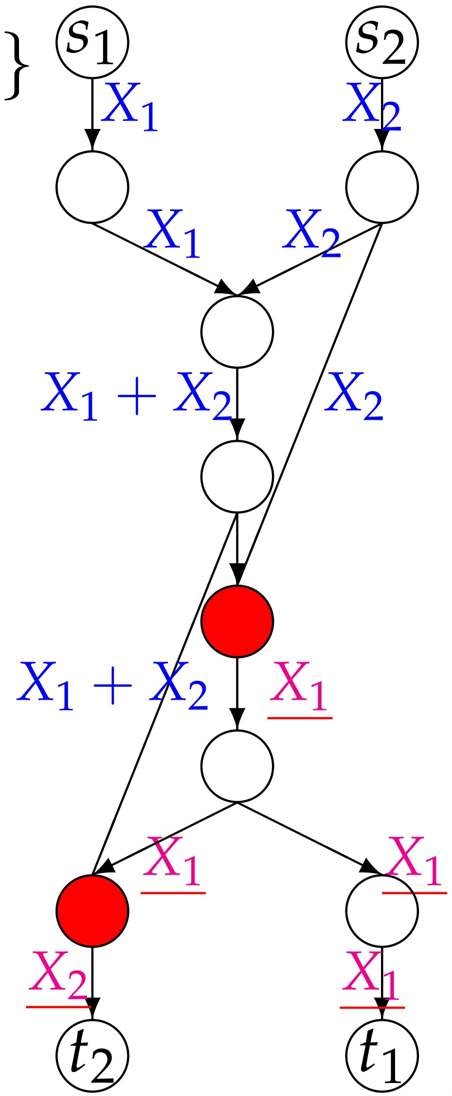

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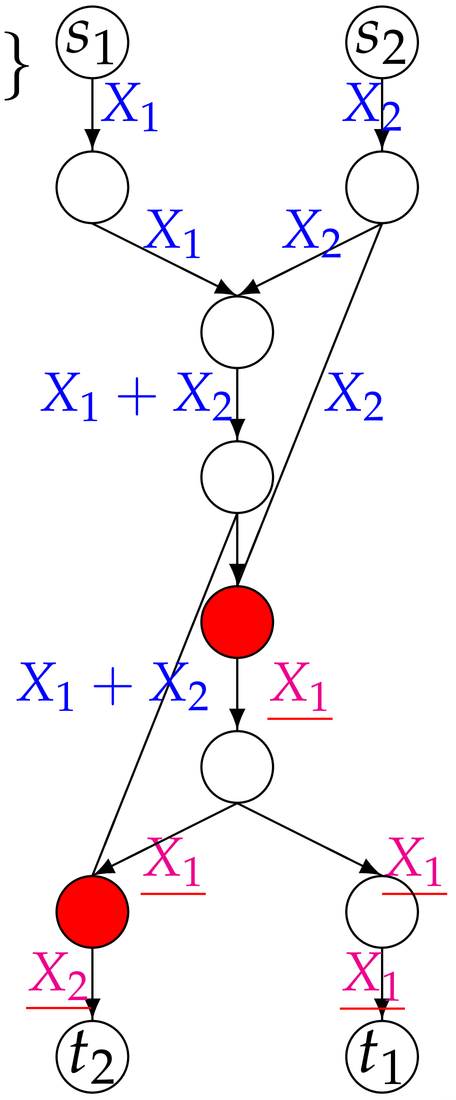

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   - Controlled overlap condition $\Rightarrow$ the feasibility.

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A two-staged, add-up-&-reset construction

$$\mathsf{ncp}_{\{P_{s_1,t_1}, P_{s_2,t_1}, Q_{s_1,t_1}\}}(e) = 3,$$

$$\mathsf{ncp}_{\{P_{s_1,t_1}, P_{s_2,t_1}, P_{s_2,t_2}\}}(e) = 2,$$

$\Longrightarrow e \notin P_{s_2,t_2}$

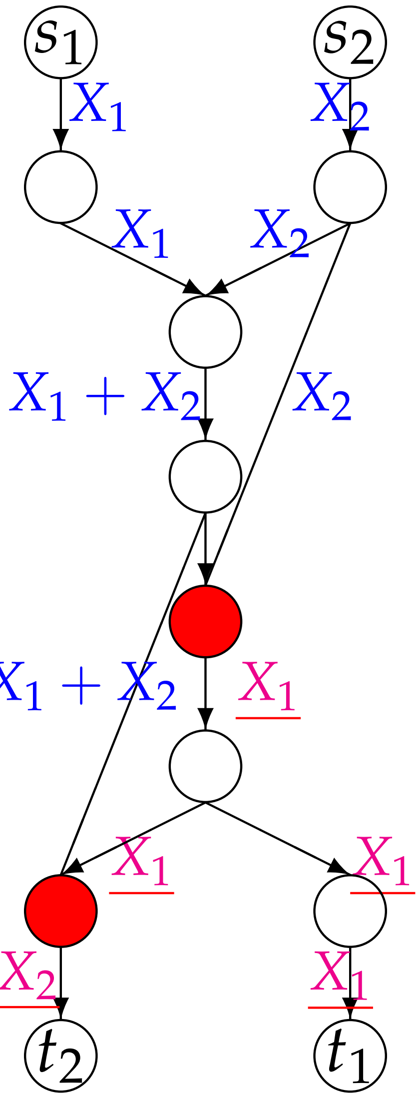
$\Longrightarrow$ Messages along P_{s_2,t_2} are not affected.

ges

a

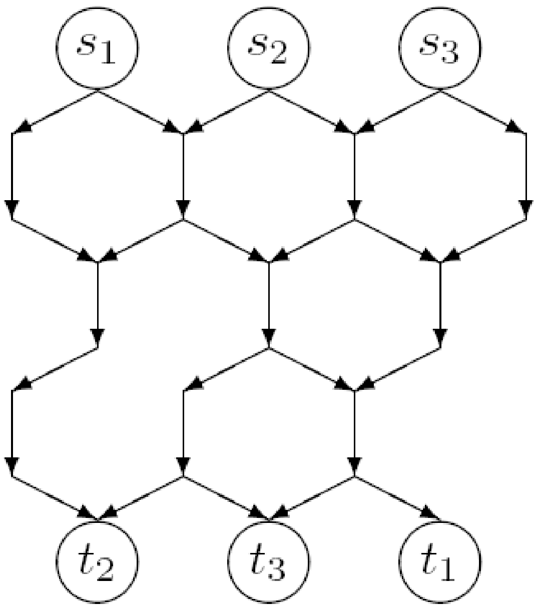
need basis.

- Controlled overlap condition $\Rightarrow$ the feasibility.



Improved Capacity Region

- Existing results: Search for butterfly coding opportunities via linear/integer programming. [Traskov *et al.* 06]



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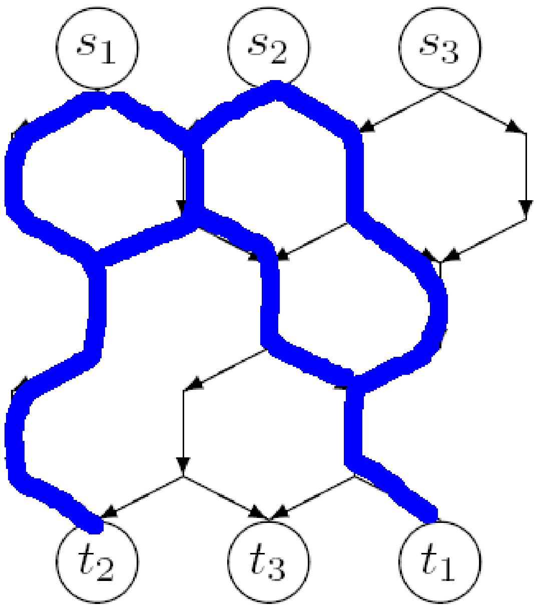

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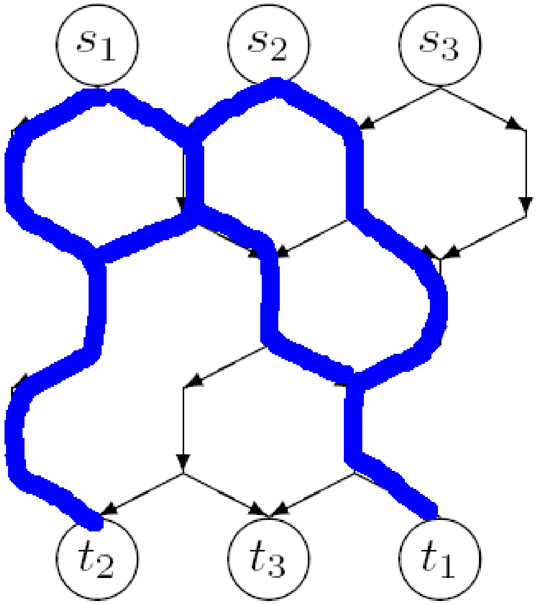

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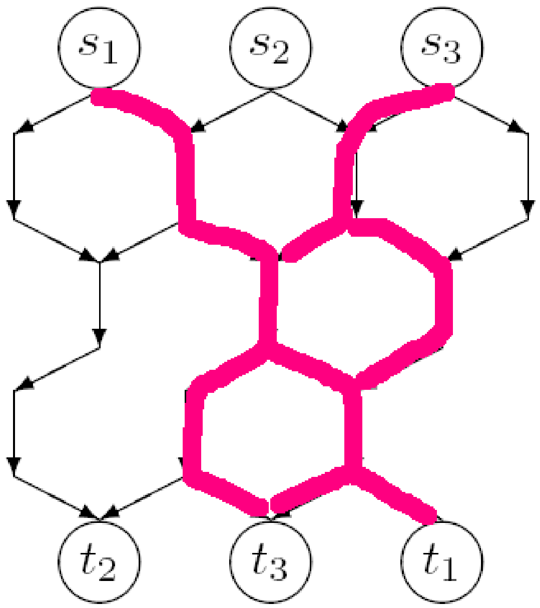

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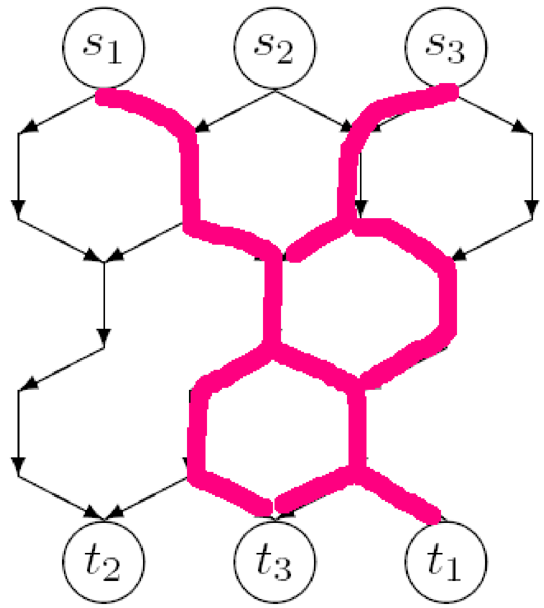

- The capacity region is strictly improved.

Capacity Region (Cont'd)

- Pattern-based construction vs. path-based construction

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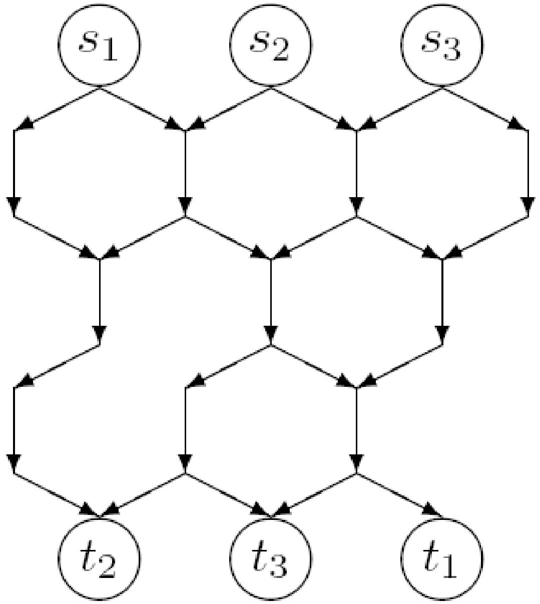
- Pattern-based construction vs. path-based construction



- (s_1, t_1): 1 path, (s_2, t_2): 3 paths, (s_3, t_3): 4 paths,
  (s_1, t_2): 3 paths, (s_2, t_3): 5 paths, (s_1, t_3): 2 paths
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Capacity Region (Cont'd)

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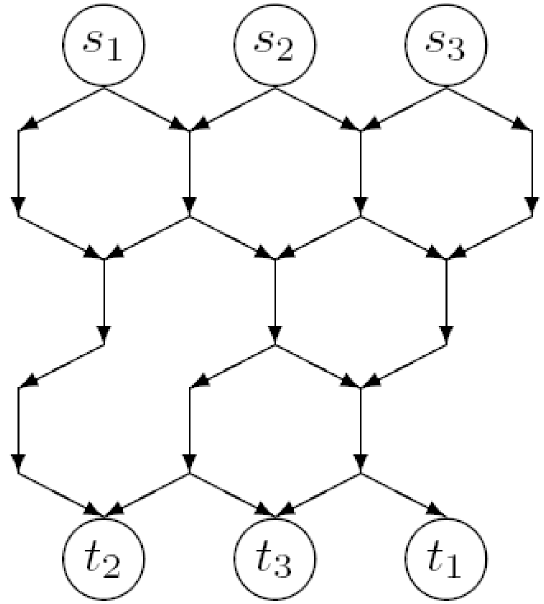

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- Bottleneck identification for all path combinations.

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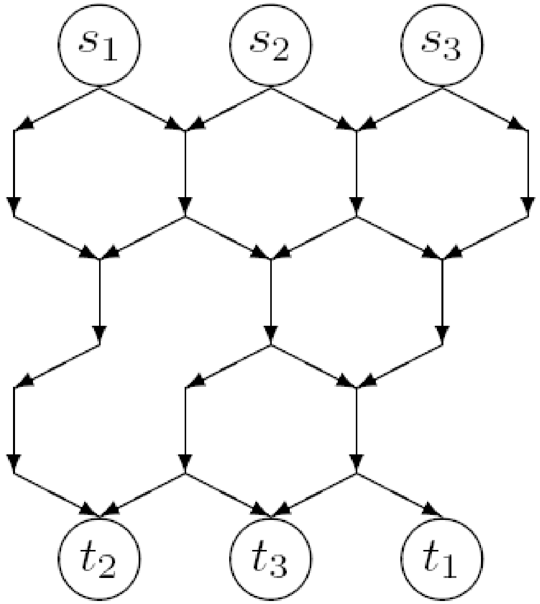

- Distributed path-based network optimization
  with arbitrary utility function. [Submitted to Infocom 08]

- Bottleneck identification for all path combinations.

Other implications

- The network-sharing bound in [Yan *et al.* 06]
  - Cut-based outer bound for K-pair unicasts.
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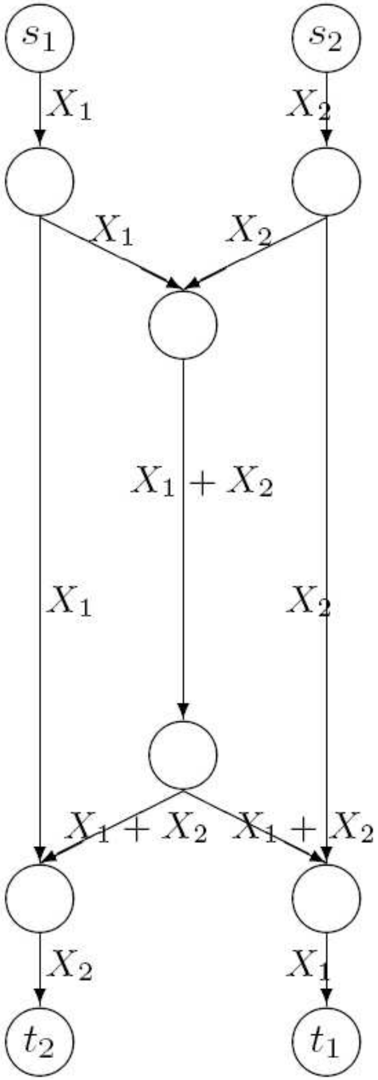

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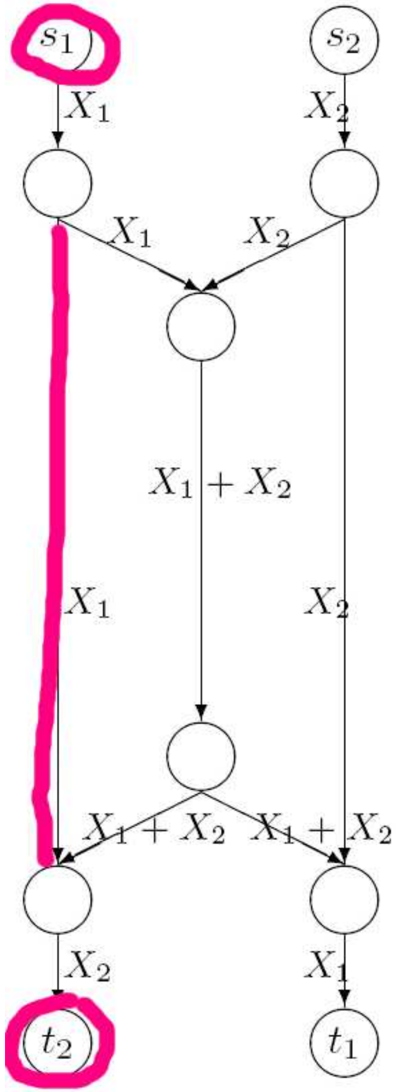

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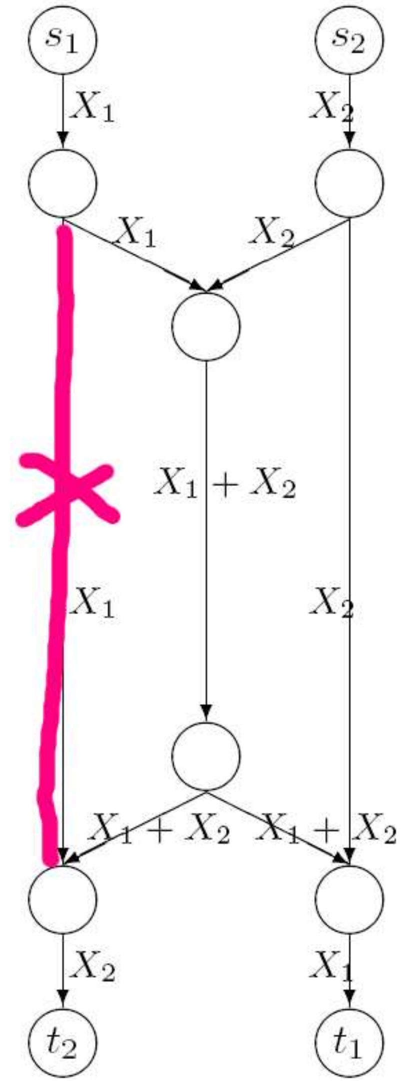

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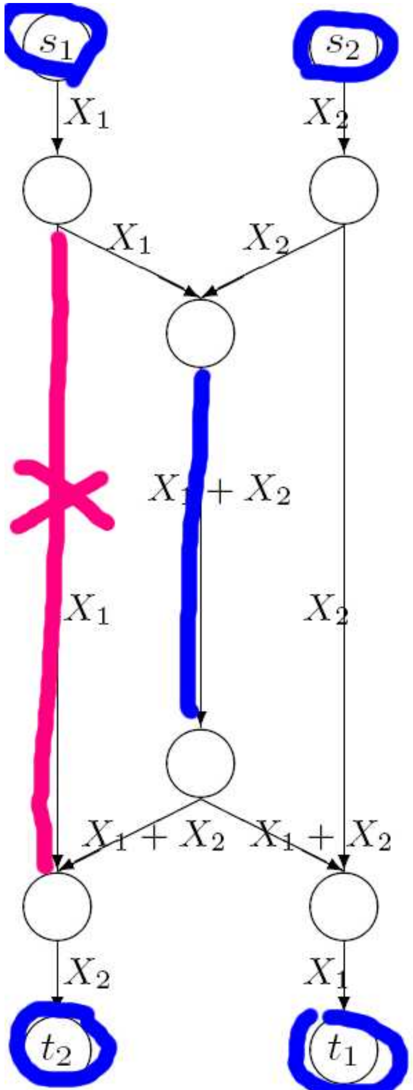

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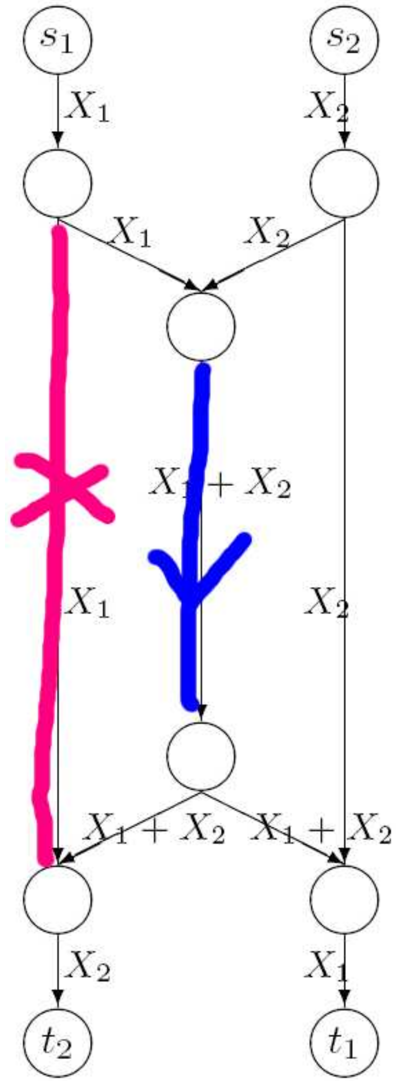

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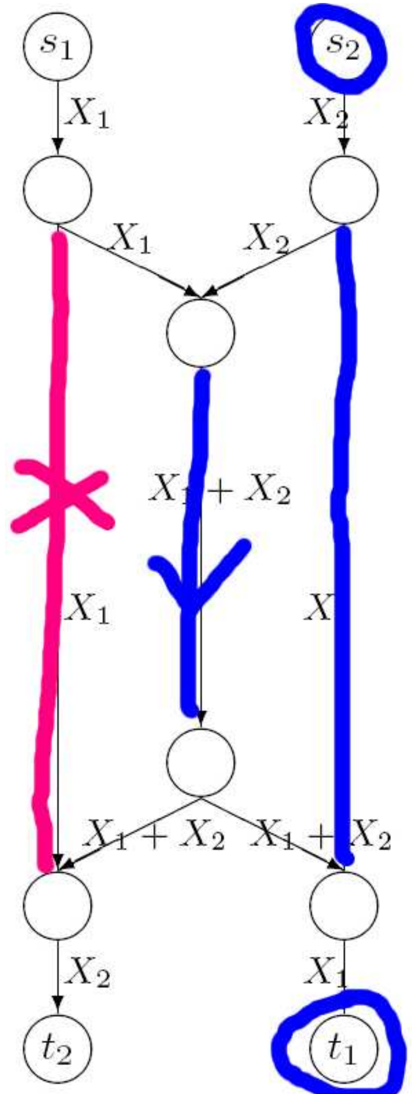

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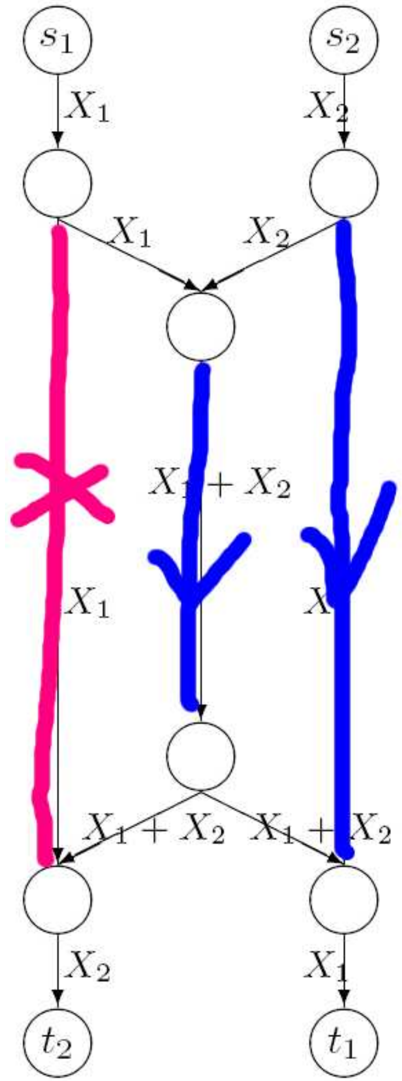

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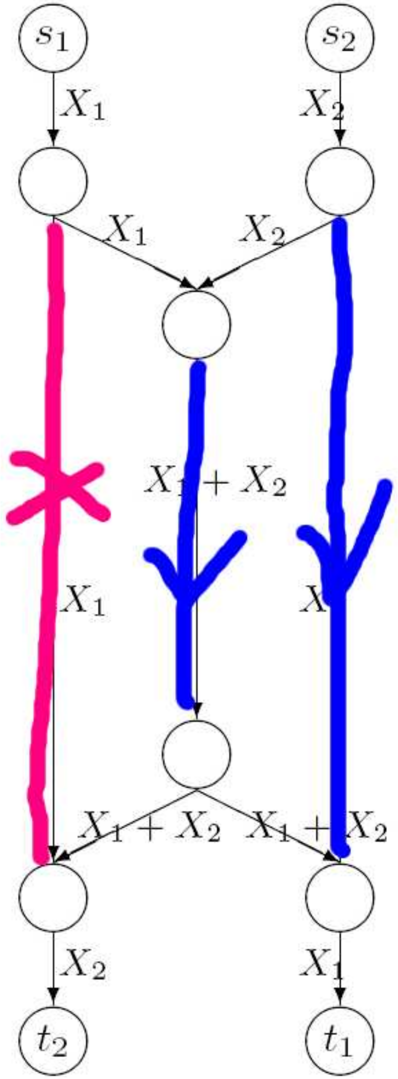

Corollary 1 *The network-sharing bound is tight. Namely, if the network-sharing bound is ≥ 2 for all permutation and for all cuts, then network coding is feasible.*

Discussion

Network coding w. two simple unicasts

$\Longleftrightarrow$ Path selections $\mathcal{P}$ and $\mathcal{Q}$ w. controlled overlap

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  - Of a form similar to the min-cut max-flow theorem,
  - It can be generalized to two simple multicast sessions [submitted to Allerton 07]

    Send X_1 and X_2 along $(s_1, \{t_{1,i}\})$ and $(s_2, \{t_{2,j}\})$ where $\{t_{1,i}\} \cap \{t_{2,j}\} \neq \emptyset$.