Capacity Bounds on Timing Channels with Bounded Service Times

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***Abstract*— It is well known that queues with exponentially distributed service time have the smallest Shannon capacity among all single-server queues with the same service rate. In this paper, we study the capacity of timing channels in which the service time distributions have bounded support (BSTC). We derive an upper bound and two lower bounds on the capacity of BSTC. We show that these capacity bounds are asymptotically tight for Uniform Service Timing Channels.**

I. Introduction

A timing channel is a non-conventional communication channel, in which a message is encoded in terms of the arrival times of bits. Queue-based timing channels were first studied in [1]. In this model, a sender transmits a sequence of bits that may be spaced (in time intervals) according to some distribution. These bits will go through a single server queue and then depart. The receiver observes the time intervals between the departing bits and decodes the message.

It has been shown in [1] that when the service time of the queue is exponentially distributed, the channel capacity, $e^{-1}\mu$ nats/sec, is the lowest among all the servers with the same service rate μ. Most of the existing work such as [1], [2], [3], [4], [5], [6], [7], [8] has focused on Exponential Service Timing Channels (ESTC). The discrete-time counterpart of the ESTC has been studied in [9], [10].

While ESTC has the lowest capacity among all servers with the same service rate, deterministic service timing channels have infinite capacity. In [11], we estimated the lower bounds on the capacities of single-server timing channels in which the service time distributions are uniform (Uniform BSTC), Gaussian (GSTC), and truncated Gaussian (Gaussian BSTC). The capacities of these channels is on the order of $-\log_2(\mu\sigma)$ bits/sec as $\sigma \to 0$, where μ is the service rate and σ is the standard deviation of the service time.

In many real world applications, the service time distributions have *bounded support*. By *bounded support*, we mean that there exist some constants $a, \Delta > 0$, such that the service times $S_1, S_2, \cdots$ satisfies $P(a < S_k < a+\Delta) = 1$ for all k. Such timing channels are called Bounded Service Timing Channels (BSTC), and $(a, a+\Delta)$ is called the support interval of the BSTC. We are especially interested BSTC with small relative fluctuation of the service time, i.e. $\Delta/a << 1$.

In this paper, we study the capacity of the BSTC with symmetric support interval $(1/\mu - \epsilon, 1/\mu + \epsilon)$. We derive a upper bound $C_U(\epsilon)$ on the capacity of BSTC using a feedback capacity, and two lower bounds, $C_{L,1}(\epsilon)$ and $C_{L,2}(\epsilon)$, using geometrically distributed inter-arrival times. While $C_U(\epsilon)$ is dependent upon the service time distribution, both $C_{L,1}(\epsilon)$ and $C_{L,2}(\epsilon)$ are independent of the distribution of the service time.

We further show that these bounds are asymptotically tight for the Uniform BSTC. More precisely, we show that for Uniform BSTC,

$$\begin{aligned}\Delta C_1(\epsilon) &= C_U(\epsilon) - C_{L,1}(\epsilon) < 1.5\mu \ \text{(bits/sect) for all } \epsilon,\\ \Delta C_2(\epsilon) &= C_U(\epsilon) - C_{L,2}(\epsilon) \to 0 \quad \text{as } \epsilon \to 0.\end{aligned}$$

Thus, for Uniform BSTC, both $C_{L,2}(\epsilon)$ and $C_U(\epsilon)$ are very close to its true capacity $C(\epsilon)$ when ϵ is small. Moreover, we show that $C_U(\epsilon)$ for the Uniform BSTC is the smallest among all BSTC with the same service rate and support interval $(1/\mu - \epsilon, 1/\mu + \epsilon)$. This corresponds to a min-max optimality for the capacity of BSTC. The capacity of the Uniform BSTC among all BSTC with same service rate and support interval plays a similar role of the ESTC in Anantharam and Verdu's seminal paper [1].

The rest of the paper is organized as follows: In Section II we provide an upper bound $C_U(\epsilon)$ on the capacity of BSTC using a feedback capacity. In Section III, we provide two lower bounds $C_{L,1}(\epsilon)$ and $C_{L,2}(\epsilon)$ and evaluate their performance against $C_U(\epsilon)$. We show that both $C_{L,1}(\epsilon)$ and $C_{L,2}(\epsilon)$ are asymptotically tight for the Uniform BSTC, and $C_{L,2}(\epsilon)$ is optimal. Finally, we conclude our paper in Section IV.

II. An Upper Bound on the Capacity of BSTC

Bounded Service Timing Channels (BSTC) are single-server queue based timing channels in which the service times $S_1, S_2, \cdots$ are *i.i.d.* with bounded support. In this paper, we consider the servers with symmetric distributions around the mean. $\exists \epsilon, 0 < \epsilon < 1/\mu$ such that $P(1/\mu - \epsilon < S_k < 1/\mu + \epsilon) = 1$, where $\mu = 1/E[S_k]$ is the service rate.

Uniform BSTC and *Gaussian BSTC* in [11] are special cases of *BSTC*. and we call them Uniform BSTC and Gaussian BSTC in this paper. Uniform BSTC are those channels whose service times $S_1, S_2, \cdots$ *i.i.d.* uniform random variables $U(1/\mu-\epsilon, 1/\mu+\epsilon)$. The Gaussian BSTC under our consideration are those with service times $S_1, S_2, \cdots$ to be *i.i.d* truncated Gaussian random variables $N(1/\mu, (\epsilon/3)^2) I(1/\mu-\epsilon, 1/\mu+\epsilon)$.

We will first provide an upper bound of BSTC using feedback capacity in *Proposition 1*.

Proposition 1: Consider a BSTC where the service times $S_1, S_2, \cdots$, are *i.i.d.* random variables with service rate μ and

$P[S_i \in (1/\mu - \epsilon, 1/\mu + \epsilon)] = 1$.
(a) An upper bound $C_U(\epsilon)$ on the capacity of the BSTC is

$$C_U(\epsilon) = \mu \sup_{0<\gamma<1} G(\epsilon, \gamma) \quad \text{bits/sec}$$

where

$$G(\epsilon, \gamma) = \gamma[\log_2(\epsilon\mu + \frac{1}{\gamma} - 1) + \log_2(e) - \log_2(\mu) - h(S_i)]$$

and $h(S_i)$ is the differential entropy of S_i.
(b) C_U for the Uniform BSTC with service rate μ and support interval $(1/\mu - \epsilon, 1/\mu + \epsilon)$ is the smallest among all symmetric BSTC with the same service rate and support interval.

[Proof] (a) Let S_i be the service time of the i^{th} bit, a_i and d_i be the arrival and the departure time of the i^{th} bit respectively, A_i and D_i be the inter-arrival time and the inter-departure time between the $(i-1)^{th}$ bit and the i^{th} bit respectively, and W_i is the queue's idle time before the arrival of the i^{th} bit.

An upper bound C_U is the feed-back upper bound C_{FB} such that the sender has the knowledge of d_{i-1} before deciding a_i. With the feedback, the sender has full control over W_i and can completely avoid any queueing. Thus, timing channel is reduced to a sequentially juxtaposed *i.i.d.* channel: $W_i \rightarrow W_i + S_i$. The capacity of this feedback channel is simply

$$C_{FB} = \sup_{\lambda<\mu} \lambda I(W_i; W_i + S_i)$$

where λ be the inter-departure rate, i.e. $E[D_i] = 1/\lambda$. $I(W_i, W_i + S_i) = h(W_i + S_i) - h(S_i)$, and

$$\sup_{W_i>0} [h(W_i + S_i)] = 1 + \ln(1/\lambda - 1/\mu + \epsilon) \quad \text{nats.}$$

The maximum of $h(W_i + S_i)$ is achieved when
$W_i + S_i - (1/\mu - \epsilon)$ are *i.i.d.* exponential random variables. Thus,

$$\begin{aligned} C_{FB} &= \sup_{\lambda<\mu} \{\lambda[1 + \ln(1/\lambda - 1/\mu + \epsilon) - h(S_i)]\} \\ &= \mu \sup_{\lambda<\mu} \{\frac{\lambda}{\mu}[\ln(\frac{\mu}{\lambda} - 1 + \epsilon\mu) + 1 - \ln(\mu) - h(S_i)] \end{aligned}$$

Let $\gamma = \lambda/\mu$. Define

$$G(\epsilon, \gamma) = \gamma[\log_2(\epsilon\mu + \frac{1}{\gamma} - 1) + \log_2(e) - \log_2(\mu) - h(S_i)]$$

We have an upper bound on the capacity of BSTC:

$$C_U(\epsilon) = C_{FB} = \mu \sup_{0<\gamma<1} G(\epsilon, \gamma) \text{ bits/sec}$$

(b) It follows directly from the property that the uniform random variable $U(1/\mu - \epsilon, 1/\mu + \epsilon)$ has the maximum entropy among all random variables with the support interval $(1/\mu - \epsilon, 1/\mu + \epsilon)$.

■

From the expression of $G(\epsilon, \gamma)$, the upper bound $C_U(\epsilon)$ is dependent on the distribution of the service time S_i. Uniform BSTC has the smallest $C_U(\epsilon)$ among all BSTC.

In the next two sections, we will provide two lower bounds $C_{L,1}$ and $C_{L,2}$ on the capacity for BSTC and compare them with C_U.

III. Two Lower Bounds on the Capacity of BSTC

A. The First Lower Bound

In this section, we will provide a sub-optimal lower bound $C_{L,1}$ on the capacity of BSTC. This lower bound is obtained by using a coding scheme in which the inter-arrival times $A_1, A_2, \cdots$ are *i.i.d.* geometric random variables. We require $A_i \geq 1/\mu + \epsilon$ for all i in order to avoid queuing. Further, the possible values of A_i for each i are spaced 4ϵ apart to allow error-free decoding. More precisely, $A_1, A_2, \cdots$ are *i.i.d.* random variables with probability density function:

$$P[A_i = 1/\mu + \epsilon + k(4\epsilon)] = p_1(1 - p_1)^k, \quad k = 0, 1, \cdots$$

This encoding scheme does not require the prior knowledge of service time distribution, and yield a lower bound $C_{L,1}$ on BSTC. We now state our first lower bound Lemma without proof. The proof for a tighter lower bound $C_{L,2}$ will provided in Section III-B.

Lemma 1: Consider a BSTC where the service times $S_1, S_2, \cdots$, are *i.i.d.* random variables with service rate μ and $P[S_i \in (1/\mu - \epsilon, 1/\mu + \epsilon)] = 1$. A lower bound $C_{L,1}(\epsilon)$ on the capacity of the timing channel is:

$$C_{L,1}(\epsilon) = \mu \sup_{0<\gamma<(1+\epsilon\mu)^{-1}} \gamma[H(p_1)/p_1] \text{ bits/sec}$$

where

$$p_1 = \frac{4\epsilon\mu}{1/\gamma - 1 + 3\epsilon\mu}$$

and $H(p) = -p\log_2(p) - (1-p)\log_2(1-p)$.

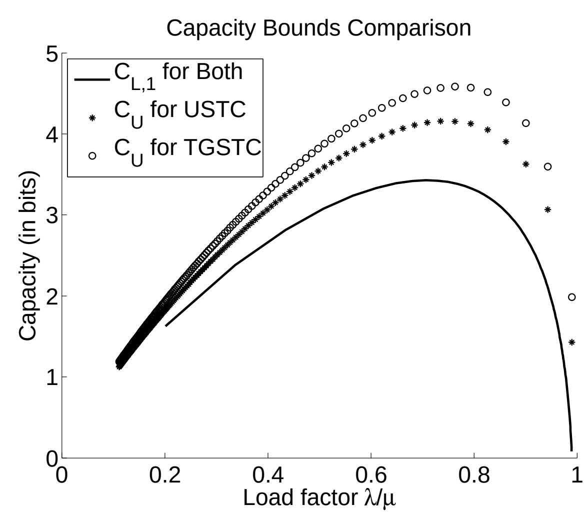

Fig. 1. Capacity Bounds C_U and $C_{L,1}$ (in bits per average service time) for Uniform BSTC and Gaussian BSTC when $\mu\epsilon = 0.01$.

Figures 1 shows $C_{L,1}(\epsilon)$ as a function of load factor $\gamma = \lambda/\mu$ when $\epsilon\mu = 0.01$, along with the upper bounds $C_U(\epsilon)$ for *Uniform BSTC* and *Gaussian BSTC* as functions of load factor $\gamma = \lambda/\mu$ when $\epsilon\mu = 0.01$.

As expected, C_U for the Uniform BSTC is smaller than that of the Gaussian BSTC. Moreover, $\Delta C_1 = C_U - C_L$, the differences between the upper and lower bounds, are smaller than 1.5μ bits/sec for both Uniform and Gaussian BSTC.

In general, we can show that for Uniform and Gaussian BSTC, $\Delta C_1(\epsilon) = C_U(\epsilon) - C_{L,1}(\epsilon)$ is bounded by a constant K, and K is independent of ϵ. Further, ΔC_1 for the Uniform BSTC is the smallest among all BSTC with service rate μ and support interval $(1/\mu - \epsilon, 1/\mu + \epsilon)$.

Proposition 2: Let $\Delta C_1(\epsilon) = C_U(\epsilon) - C_{L,1}(\epsilon)$. For BSTC with service rate μ and support interval $(1/\mu - \epsilon, 1/\mu + \epsilon)$,
(a) $\Delta C_1(\epsilon) < \mu(\log_2(e) - h(S_n) + \log_2(2\epsilon))$ bits/sec .
(b) $\Delta C_1(\epsilon)$ for the Uniform BSTC with service rate μ and support interval $(1/\mu - \epsilon, 1/\mu + \epsilon)$ is the smallest among all BSTC with the same service rate and support interval.
[Proof] See Appendix.

We obtain $\Delta C_1(\epsilon) < \log_2(e)\mu$ bits/sec $\approx 1.447\mu$ bits/sec for Uniform BSTC, and $\Delta C_1(\epsilon) \leq 2.004\mu$ bits/sec for Gaussian BSTC using Proposition 2.

Table I shows the values of $C_{L,1}(\epsilon)$ for BSTC when $\epsilon\mu = 10^{-1}, 10^{-2},$ and 10^{-3}, along with the values of $C_U(\epsilon)$ and $\Delta C_1(\epsilon)$ for the Uniform BSTC and the Gaussian BSTC. As we have already pointed out, $C_{L,1}(\epsilon)$ is independent of the service time distribution. We can see that $\Delta C_1 < 2\mu$ bits/sec for all ϵ in this Table as expected.

TABLE I
THE UPPER AND LOWER BOUNDS ON TIMING CHANNEL CAPACITY FOR UNIFORM BSTC AND GAUSSIAN BSTC (IN BITS PER SERVICE TIME).

	ALL	Uniform	Uniform	Gaussian	Gaussian
$\epsilon\mu$	$C_{L,1}$	C_U	ΔC_1	C_U	ΔC_1
10^{-1}	1.4500	2.0314	0.5814	2.3927	0.9428
10^{-2}	3.4287	4.1582	0.7295	4.5833	1.1547
10^{-3}	5.9081	6.7469	0.8388	7.2127	1.3045

Even though $C_{L,1}(\epsilon)$ is sub-optimal by using a naive coding scheme with big (4ϵ) spacing, it is tight in the sense that for Uniform BSTC, $\Delta C_1(\epsilon) < 1.447$ bits/sec for all ϵ.

Moreover, this coding scheme is derived without knowing the service distribution in prior. When the service time distribution is Uniform, ΔC_1 is the smallest among all BSTC with same service rate and support interval. This result corresponds to a min-max optimality similar to that of ESTC in seminal paper [1].

Next, we will derive a optimal lower bound $C_{L,2}$ for BSTC which is also independent of the service time distribution.

B. The Second Lower Bound

When the receiver uses more computational power to recover the actual time, we only need to space the possible values of the inter-arrival times $A_1, A_2 \cdots$ to be 2ϵ apart. This will yield a better lower bound $C_{L,2}$. Further, we will show that $C_{L,2}(\epsilon)$ is asymptotically optimal in the sense that without prior knowledge of the service time distribution, $C_{L,2}(\epsilon)$ is the best one could do when ϵ is small.

Lemma 2: Consider a BSTC where the service times $S_1, S_2, \cdots$, are *i.i.d.* random variables with service rate μ and $P[S_i \in (1/\mu - \epsilon, 1/\mu + \epsilon)] = 1$. A lower bound $C_{L,2}(\epsilon)$ on the capacity of the timing channel is:

$$C_{L,2}(\epsilon) = \mu \sup_{0<\gamma<(1+\epsilon\mu)^{-1}} \gamma[H(p_2)/p_2] \text{ bits/sec}$$

where

$$p_2 = \frac{2\epsilon\mu}{1/\gamma - 1 + \epsilon\mu}$$

and $H(p) = -p\log_2(p) - (1-p)\log_2(1-p)$.

Proof. When the receiver uses more computational power to recover the time, we could use a smaller spacing, 2ϵ, between two possible inter-arrival times. Consider the following example. Choose $a_1 = 0$, then the first bit depart before time $t = 1/\mu + \epsilon$. Then, the second bit can be sent at one of the two possible times $a_2^{(1)} = 1/\mu + \epsilon$, or $a_2^{(2)} = 1/\mu + \epsilon + 2\epsilon$. without confusing the receiver. Because the two possible departing time $d_2^{(1)}$ and $d_2^{(2)}$ are contained within two non-overlapping intervals, with $d_2^{(1)} \in (1/\mu, 1/\mu + 2\epsilon)$ and $d_2^{(2)} \in (1/\mu + 2\epsilon, 1/\mu + 4\epsilon)$. This allows the receiver to decode the message error free.

In this coding scheme, we choose the inter-arrival times $A_1, A_2, \cdots$ to be *i.i.d.* geometric random variables with 2ϵ spacing, i.e.

$$P[A_i = 1/\mu + \epsilon + k(2\epsilon)] = p_2(1-p_2)^k, \quad k = 0, 1, \cdots$$

This coding scheme guarantee no queuing if the queue is initially empty, because $A_i \geq 1/\mu + \epsilon > S_j$. Moreover, it allows error-free decoding as we have discussed.

Let λ be the departure rate and $\gamma = \lambda/\mu$. We have
$1/\lambda = E[D_i] = E[A_i] = 1/\mu + \epsilon + 2\epsilon(1/p_2 - 1)$. Thus,

$$p_2 = \frac{2\epsilon\mu}{1/\gamma - 1 + \epsilon\mu}$$

Since $I(A_i; D_i) = h(A_i) = H(p_2)/p_2$, we have

$$C \geq \frac{I(A_i; D_i)}{E[D_i]} = \mu[\gamma H(p_2)/p_2] \text{ for all } \gamma, 0 < \gamma < (1+\epsilon\mu)^{-1}$$

Therefore,

$$C_{L,2}(\epsilon) = \mu \sup_{0<\gamma<(1+\epsilon\mu)^{-1}} \gamma[H(p_2)/p_2] \text{ bits/sec}$$

■

Figures 2 shows $C_{L,2}(\epsilon)$ as a functions of load factor $\gamma = \lambda/\mu$ for $\mu\epsilon = 0.01$, along with $C_U(\epsilon)$ for *Uniform* BSTC and *Gaussian* BSTC as functions of load factor $\gamma = \lambda/\mu$.

As we can see in Figure 2, C_U for Uniform BSTC is smaller that of Gaussian BSTC, just as expected. Further, for the *Uniform* BSTC, $C_{L,2}(\epsilon)$ is extremely close to $C_U(\epsilon)$.

In the next Proposition, we will show that $C_{L,2}$ is optimal in the sense that for Uniform BSTC, $C_{L,2}(\epsilon) \to C_U(\epsilon)$ as $\epsilon \to 0$.

Proposition 3: Denote $\Delta C_2(\epsilon) = C_U(\epsilon) - C_{L,2}(\epsilon)$.
(a) ΔC_2 for the Uniform BSTC is the smallest among all BSTC with same service rate and support interval.
(b)$\Delta C_2(\epsilon) \to 0$ as $\epsilon \to 0$ for Uniform BSTC.

Proof See Appendix.

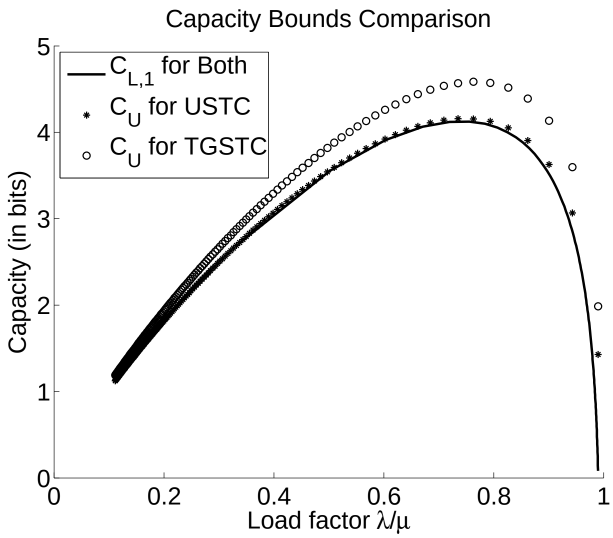

Fig. 2. Gaussian BSTC Capacity Bounds C_U and C_L for $\mu = 1$ and $\epsilon = 0.01$ (in bits per average service time).

Table II shows the values of $C_{L,2}$ of BSTC, which is independent of the service time distribution, for various values of $\epsilon\mu$. The values of C_U and ΔC_2 for Uniform BSTC and Gaussian BSTC are also shown in this Table. We observe that when $\epsilon\mu = 10^{-3}$, $\Delta C_2 = 0.0086\mu$ bits/sec for Uniform BSTC, which agrees with Proposition 3(b). Moreover, we can infer that when $\epsilon\mu = 10^{-3}$, the true capacity C for Uniform BSTC is 6.74μ bits/sec, and our second coding scheme almost achieves this capacity.

TABLE II

THE UPPER AND LOWER BOUNDS ON TIMING CHANNEL CAPACITY FOR UNIFORM BSTC AND GAUSSIAN BSTC (IN BITS PER SERVICE TIME).

$\epsilon\mu$	ALL $C_{L,2}$	Uniform C_U	Uniform ΔC_2	Gaussian C_U	Gaussian ΔC_2
10^{-1}	1.9106	2.0314	0.1198	2.3927	0.4812
10^{-2}	4.1240	4.1582	0.0342	4.5833	0.4593
10^{-3}	6.7384	6.7469	0.0086	7.2127	0.4743

C. Discussion

Importance of Proposition x, it corresponds to a min-max optimality with no sufficient prior information.

Although we developed the bounds on the BSTC with symmetric support interval around the mean service time, these results can be easily generalized to general BSTC with asymmetric support intervals.

IV. CONCLUSION

We have studied capacities for the timing channels with bounded service times. In particular, we have obtained an upper bound on the capacity of these channels using a feedback capacity, and two lower bounds by using geometrically distributed inter-arrival times. Both lower bounds are independent of the service time distributions. We further compared the difference between the upper bound and lower bound and found that our lower bound for the uniform service timing channel is aymptotically tight. When the receiver does not spend any computing power to recover the time, we have shown that $C_U - C_L \leq \mu \log_2(e)$ bit/sec; when the receiver spends computing time to recover the time, we have shown that $C_U - C_L \to 0$ as $\epsilon \to 0$. Our future research direction is to develop practical efficient coding schemes for BSTC.

REFERENCES

[1] N. Anantharam and S. Verdu, "Bits through queues," *IEEE Trans. Inform. Theory,* vol. 42, pp. 4 – 18, Jan. 1996

[2] R. Sundaresan and S. Verdu "Robust decoding for timing channels," *IEEE Trans. Inform. Theory,* vol. 46, pp. 405 – 419, Mar. 2000

[3] R. Sundaresan and S. Verdu "Sequential decoding for the exponential server timing channel," *IEEE Trans. Inform. Theory,* vol. 46, pp. 405 – 419, Mar. 2000

[4] R. Sundaresan and S. Verdu "Capacity of Queues Via Point-Process Channels," *IEEE Trans. Inform. Theory,* vol. 52, pp. 2697 – 2709, June 2006

[5] J. Giles and B. Hajek "An Information-theoretic and game-theoretic study of timing channels," *IEEE Trans. on Inform. Theory,* VOL. 48, pp. 2455 – 2477, Mar. 2002

[6] X. Liu and R. Srikant "The timing capacity of single-server queues with multiple flows," *DIMACS Series in Discrete Mathematics and Theoretical Computer Science,* 2004

[7] A. B. Wagner and V. Anantharam "Zero-reate reliability of the exponetial-server timing channel," *IEEE Trans. on Inform. Theory,* VOL 52, No 2, pp. 447 – 465, Feb. 2005

[8] P. Mimcilovic "Mismatch Decoding of a Compound Timing Channel," *Forty-Fourth Annual Allerton Conference on Communication, Control, and Computing,* Sep. 2006

[9] A. Bedekar and M. Azizoglu "The information-theoretic capacity of discrete-time queues," *IEEE Trans. Inform. Theory,* vol. 44, pp. 446 – 461, Mar. 1998

[10] J. A. Thomas "On the Shannon capacity of discrete time queues," *IEEE Int. Symp. Inform. Theory*, pp. 333, July 1997

[11] S. H. Sellke, N. B. Shroff, S. Bagchi, and C. C. Wang "Timing Channel Capacity for Uniform and Gaussian Servers," *Forty-Fourth Annual Allerton Conference on Communication, Control, and Computing*, Sep. 2006

[12] T. Cover and J. Thomas *Elements of Information Theory, New York: Wiley,* 1991

APPENDIX

[Proof of Proposition 2]

(1) To show $\Delta C_1(\epsilon) \leq \mu[\log_2(e) - h(S_n) + \log_2(2\epsilon)]$ bits/sec.
By *Proposition 1*, $C_U(\epsilon) = \mu \sup_{0<\gamma<1} G(\epsilon, \gamma)$ bits/sec, where
$G(\epsilon, \gamma) = \gamma[\log_2(\epsilon\mu + \frac{1}{\gamma} - 1) + \log_2(e) - \log_2(\mu) - h(S_i)]$.

By *Lemma 1*,

$$C_{L,1}(\epsilon) = \mu \sup_{0<\gamma<(1+\epsilon\mu)^{-1}} \gamma[H(p_1)/p_1] \text{ bits/sec}$$

where

$$p_1 = \frac{4\epsilon\mu}{1/\gamma - 1 + 3\epsilon\mu}$$

and $H(p) = -p\log_2(p) - (1-p)\log_2(1-p)$. Thus,

$$\begin{aligned} \Delta C_1 &= C_U(\epsilon) - C_{L,1}(\epsilon) \\ &< \mu \sup_{0<\gamma<(1+\epsilon\mu)^{-1}} [G(\epsilon,\gamma) - \gamma H(p_1)/p_1] \quad (1) \end{aligned}$$

First, express the first term of $G(\epsilon, \gamma)$, $\log_2(\epsilon\mu + 1/\gamma - 1)$, in terms of p_1.

$$p_1 = \frac{4\epsilon\mu}{1/\gamma - 1 + 3\epsilon\mu} \qquad \Rightarrow \epsilon\mu = \frac{(1/\gamma-1)p_1}{4-3p_1}$$

Thus,

$$\epsilon\mu + 1/\gamma - 1 = (1/\gamma - 1)(\frac{4-2p_1}{4-3p_1}) = (\epsilon\mu)(\frac{4-2p_1}{p_1})$$

Therefore, $\log_2(\epsilon\mu + 1/\gamma - 1) = \log_2[(\epsilon\mu)(\frac{4-2p_1}{p_1})]$, so that

$$\begin{aligned}
& G(\epsilon,\gamma) \\
= \; & \gamma[\log_2(\epsilon\mu + \frac{1}{\gamma} - 1) + \log_2(e) - \log_2(\mu) - h(S_i)] \\
= \; & \gamma\{\log_2[(\epsilon\mu)(\frac{4-2p_1}{p_1})] + \log_2(e) - \log_2(\mu) - h(S_i)\} \\
= \; & \gamma\{\log_2[(\epsilon)(\frac{4-2p_1}{p_1})] + \log_2(e) - h(S_i)\}
\end{aligned}$$

Thus,

$$\begin{aligned}
& G(\epsilon,\gamma) - \gamma H(p_1)/p_1 \\
= \; & \gamma\{\log_2[(\epsilon)(\frac{4-2p_1}{p_1})] + \log_2(e) - h(S_i)\} - \gamma H(p_1)/p_1 \\
= \; & \gamma[\log_2(2\epsilon(2-p_1)/p_1) + \log_2(e) - h(S_i) \\
& +(\log_2(p_1) + (1-p_1)/p_1 \log_2(1-p_1))] \\
= \; & \gamma[\log_2(e) - h(S_n) + \log_2(2\epsilon)] \\
& +\gamma[\log_2(2-p_1) + (\frac{1-p_1}{p_1})\log_2(1-p_1))]
\end{aligned}$$

Since $\log_2(2-p) + \frac{1-p}{p}\log_2(1-p) < 0$ and $\gamma > 0$,
We have

$$\begin{aligned}
G(\epsilon,\gamma) - \gamma H(p_1)/p_1 &< \gamma[\log_2(e) - h(S_n) + \log_2(2\epsilon)] \\
&< \log_2(e) - h(S_n) + \log_2(2\epsilon)
\end{aligned}$$

By equation (1),
$\Delta C_1(\epsilon) \le \mu[\log_2(e) - h(S_n) + \log_2(2\epsilon)]$ bits/sec.

(2) By *Proposition 1* part (b), $C_U(\epsilon)$ for the Uniform BSTC with service rate μ and support interval $(1/\mu - \epsilon, 1/\mu + \epsilon)$ is the smallest among all BSTC with the same service rate and support interval, and by Lemma 1, $C_{L,1}(\epsilon)$ is independent of the service distribution. Therefore, ΔC_1 for the Uniform BSTC is the smallest among all BSTC with service rate μ and support interval $(1/\mu - \epsilon, 1/\mu + \epsilon)$

■

[Proof of Proposition 3]

(a) Same argument as in the proof of Proposition (2)(b).

(b) To show $\Delta C_2(\epsilon) \to 0$ as $\epsilon \to 0$ for Uniform BSTC.

Similar to the proof of Proposition 2,

$$\begin{aligned}
\Delta C_2(\epsilon) &= C_U(\epsilon) - C_{L,2}(\epsilon) \\
&= \mu \sup_{0<\gamma<1} G(\epsilon,\gamma) - \sup_{0<\gamma<(1+\epsilon\mu)^{-1}} [\gamma H(p_2)/p_2]
\end{aligned}$$

First, express the first term of $G(\epsilon,\gamma)$, $\log_2(\epsilon\mu + 1/\gamma - 1)$, in terms of p_2. Since

$$p_2 = \frac{2\epsilon\mu}{1/\gamma - 1 + \epsilon\mu} \qquad \Rightarrow \epsilon\mu = \frac{(1/\gamma-1)p_2}{2-p_2}$$

Thus, $\epsilon\mu + 1/\gamma - 1 = (1/\gamma - 1)(\frac{2}{2-p_2}) = \frac{2\epsilon\mu}{p_2}$

Therefore,

$$\begin{aligned}
& G(\epsilon,\gamma) = \gamma[\log_2(\epsilon\mu + \frac{1}{\gamma} - 1) + \log_2(e) - \log_2(\mu) - h(S_i)] \\
= \; & \gamma\{\log_2[(\epsilon\mu)(\frac{2}{p_2})] + \log_2(e) - \log_2(\mu) - h(S_i)\} \\
= \; & \gamma\{\log_2[(2\epsilon)(\frac{1}{p_2})] + \log_2(e) - h(S_i)\}
\end{aligned}$$

Thus,

$$\begin{aligned}
& G(\epsilon,\gamma) - \gamma H(p_2)/p_2 \\
= \; & \gamma\{\log_2[(2\epsilon)(\frac{1}{p_2})] + \log_2(e) - h(S_i)\} - \gamma H(p_2)/p_2 \\
= \; & \gamma[\log_2(2\epsilon/p_2) + \log_2(e) - h(S_i) \\
& +(\log_2(p_2) + (1-p_2)/p_2 \log_2(1-p_2))] \\
= \; & \gamma[\log_2(e) + \log_2(2\epsilon) - h(S_i)] + \gamma[(\frac{1-p_2}{p_2})\log_2(1-p_2))] \\
= \; & \gamma[\log_2(e) + (\frac{1-p_2}{p_2})\log_2(1-p_2))]
\end{aligned}$$

The last equality is from $h(S_i) = \log_2(2\epsilon)$ for Uniform BSTC with support interval $(1/\mu - \epsilon, 1/\mu + \epsilon)$.

Let $\gamma^* = \gamma^*(\epsilon)$ be the value when $G(\epsilon,\gamma)$ achieve its maximum, i.e. $C_U(\epsilon) = G(\epsilon,\gamma^*)$. The corresponding value of p_2^* sastifies

$$(1/p_2^*) \cdot \ln(1/p_2^*) = (1/2\epsilon - 1)/2$$

Therefore, $p_2^* \to 0$ as $\epsilon \to 0$. Thus,

$$\log_2(e) + (\frac{1-p_2^*}{p_2^*})\log_2(1-p_2^*) \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Thus for Uniform BSTC,

$$(G(\epsilon,\gamma^*) - \gamma^* H(p_2^*)/p_2^*) \longrightarrow 0 \quad \text{as} \quad \epsilon \to 0.$$

Therefore, $\Delta C_2(\epsilon) \to 0$ as $\epsilon \to 0$ for Uniform BSTC.

■