Pruning Network Coding Traffic By Network Coding

A New Class of Max-Flow Algorithms

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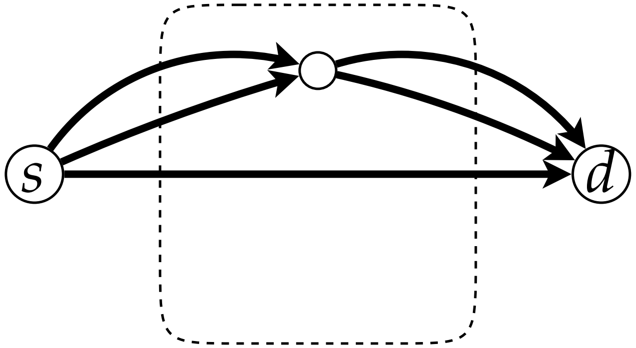

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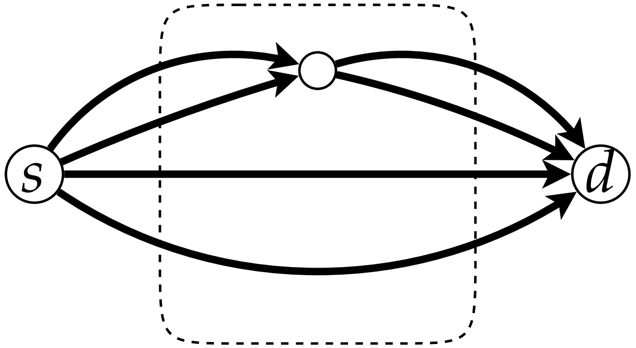

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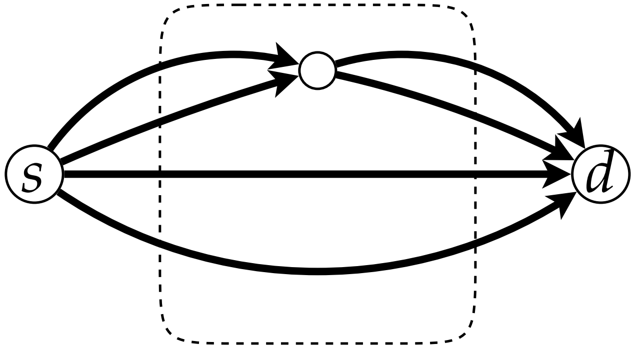

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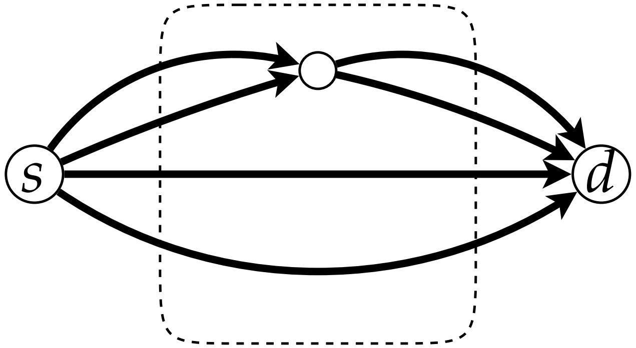

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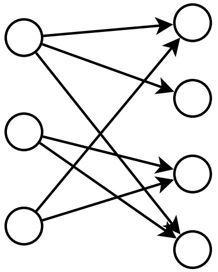

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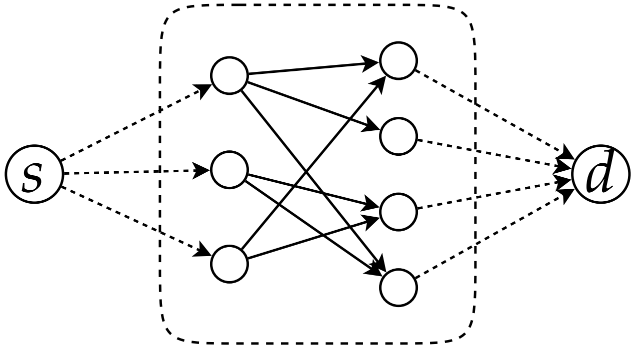

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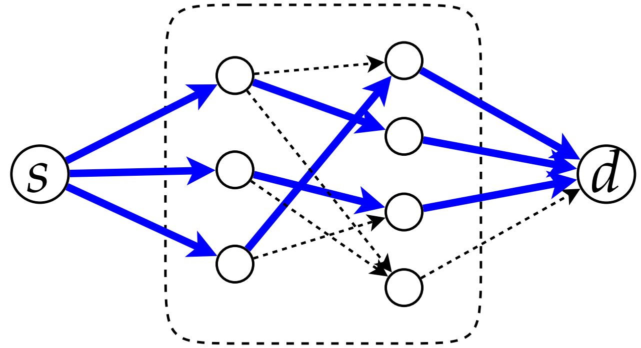

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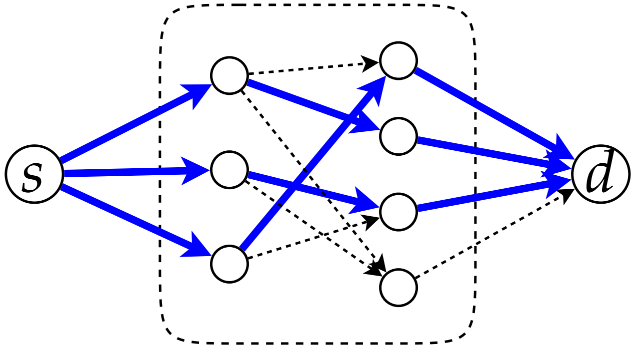

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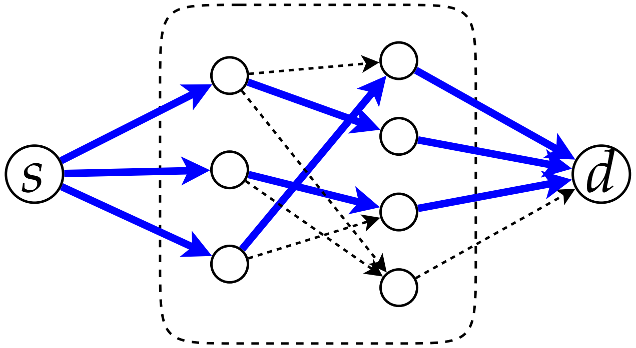

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  - A multicast rate r is supportable iff $r \leq \mathrm{MFV}_i$ for all source-destination pairs (s,d_i). [Ahlswede *et al.* 00], [Li *et al.* 03]

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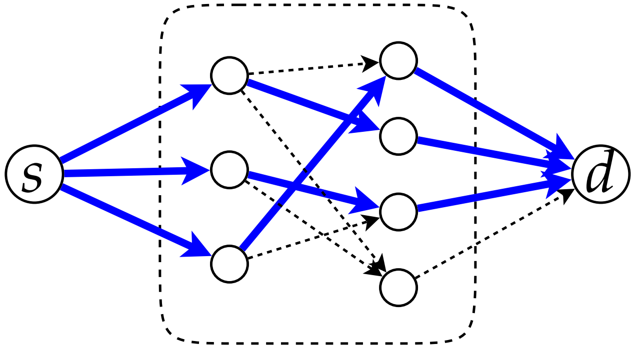

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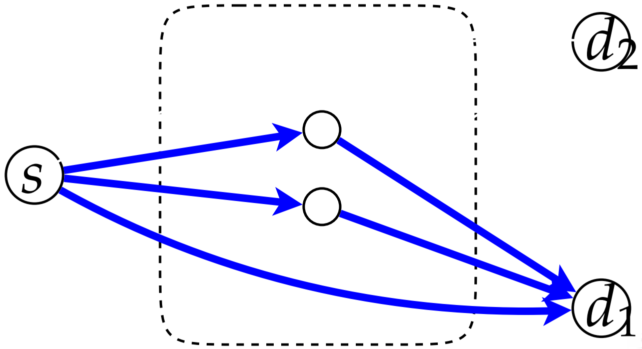

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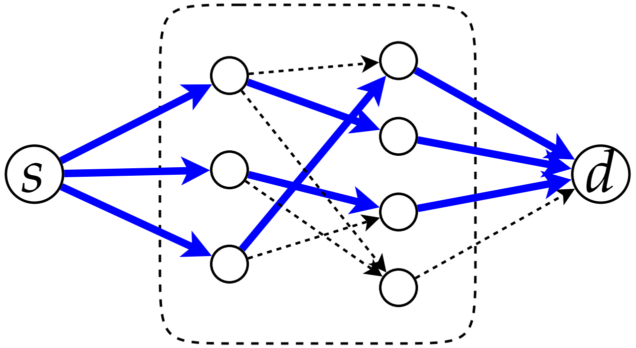

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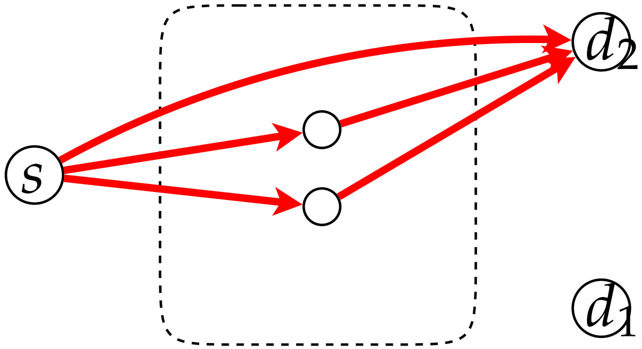

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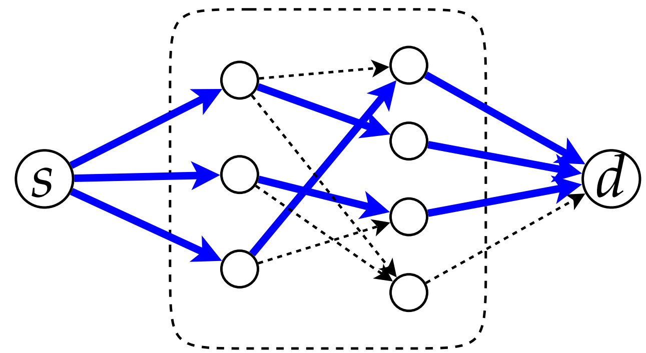

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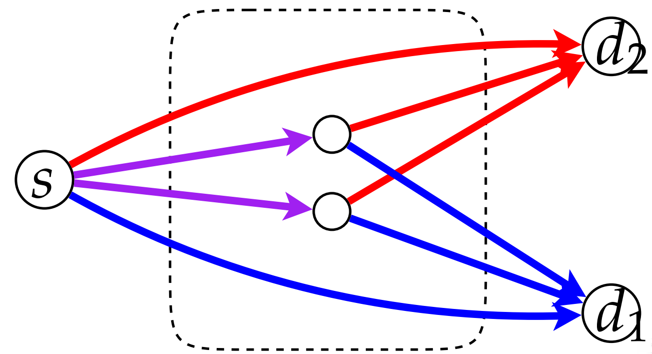

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- Linear-programming (LP) based max-flow algorithms
  - $$\max_{f_e \geq 0} \sum_{e \in \mathrm{Out}(s)} f_e$$
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  - Fractional rate vs. packet-by-packet coding operations.
    - Time-averaging? Practical generation size (# of to-be-mixed packets) is 30–100.

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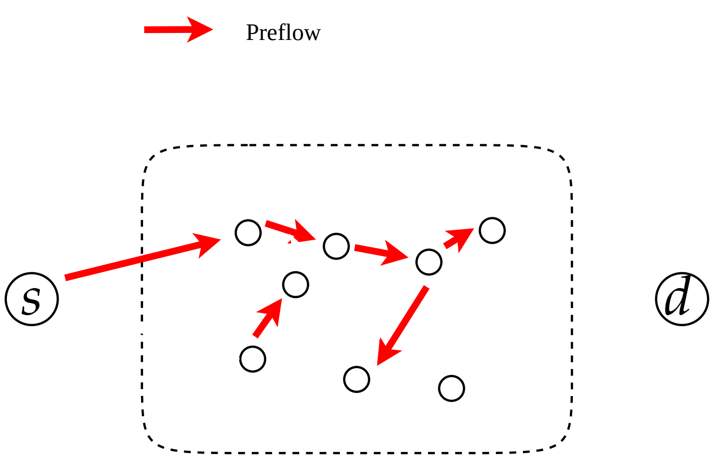

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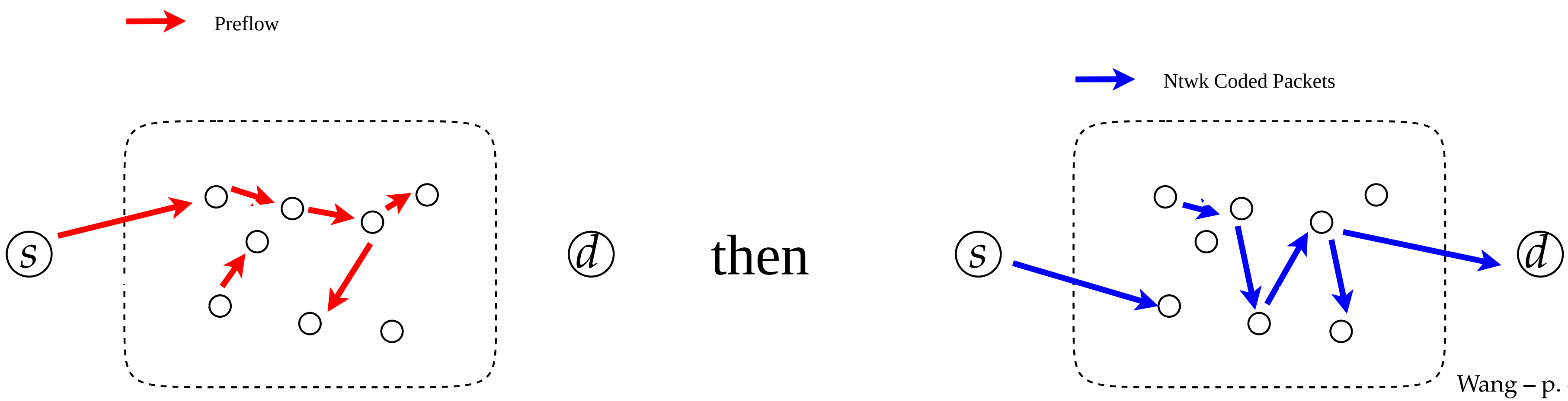

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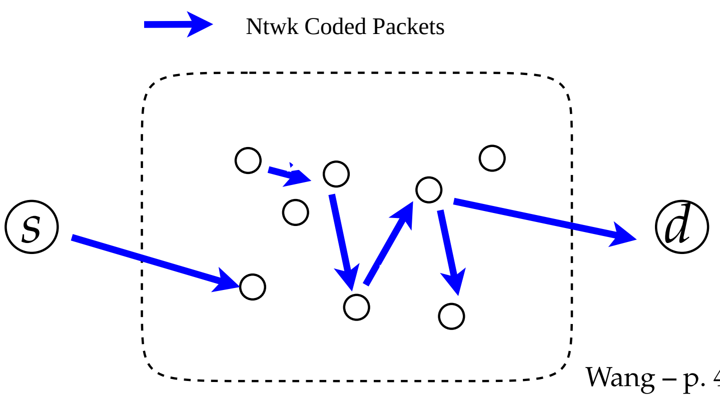

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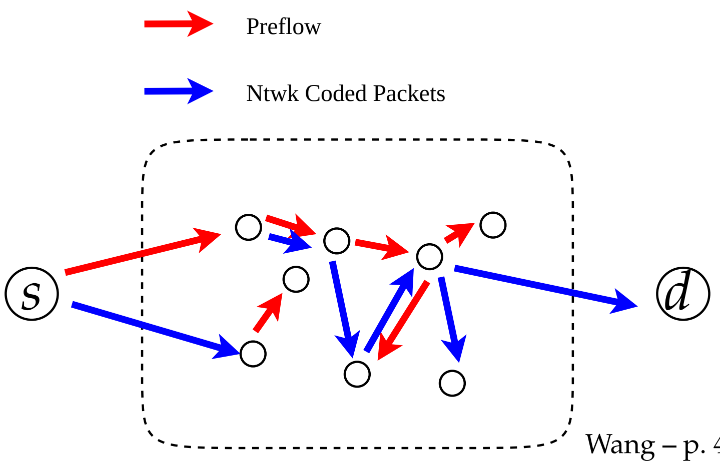

Delay Minimality of NC

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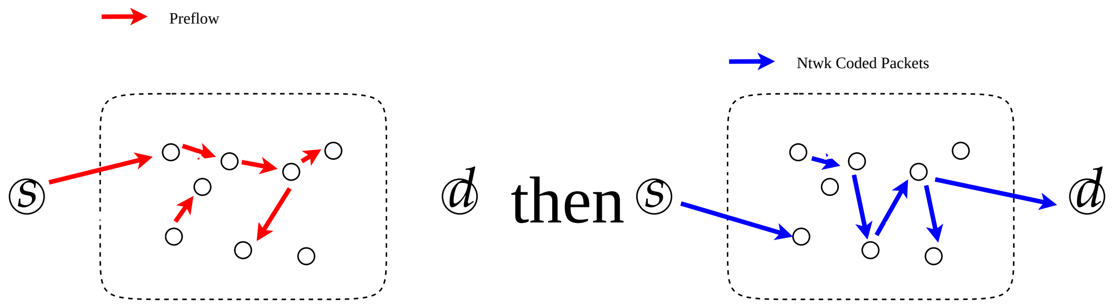

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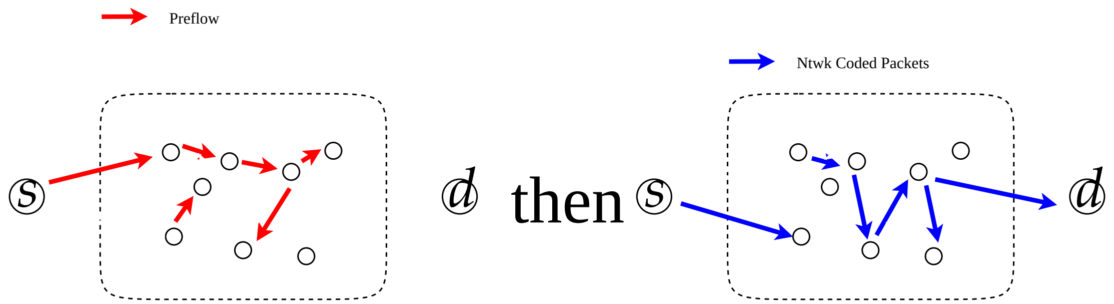

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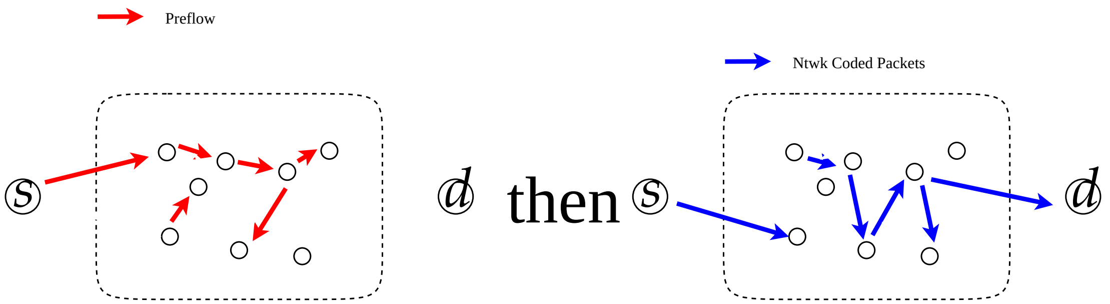

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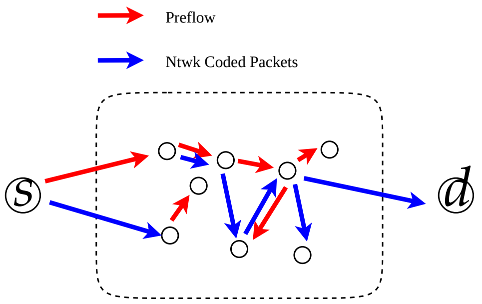

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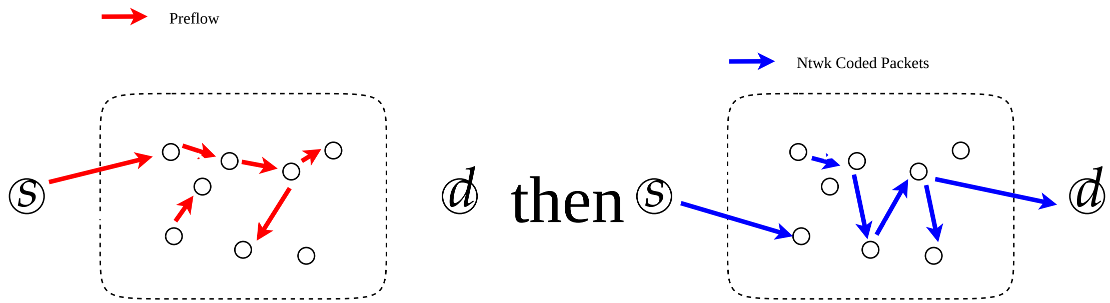

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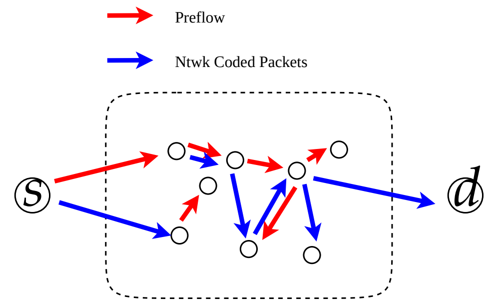

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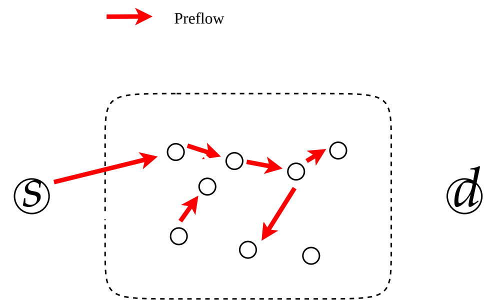

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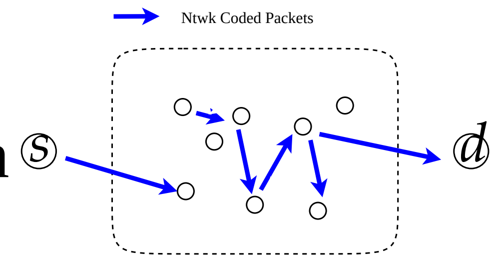

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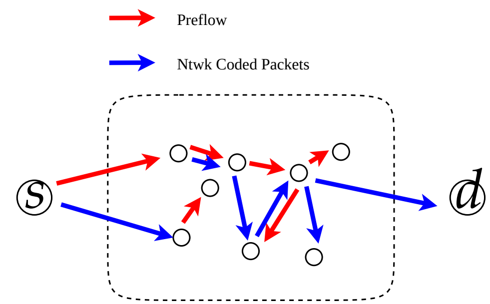

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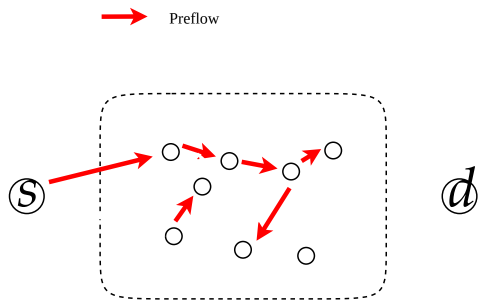

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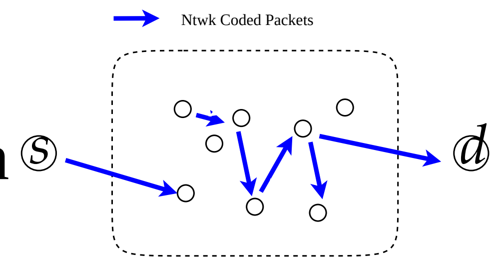

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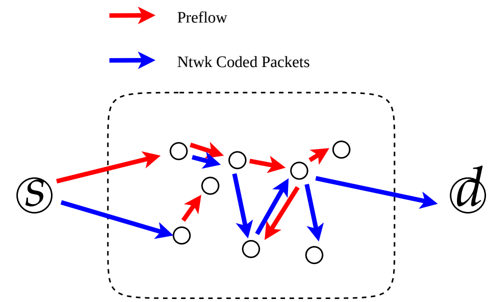

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  - One simply performs random mixing + broadcasting.
  - Network coding is delay-optimal.
  - Coding eliminates the need to decide which edge to send.
  - Significant control and communication overhead.

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- ...ing

The key question: How to find distributedly the redundant edges?

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- Arbitrary GF(q), ex: $q = 2^1, 2^8, 2^{16}$ or $q = 3$.

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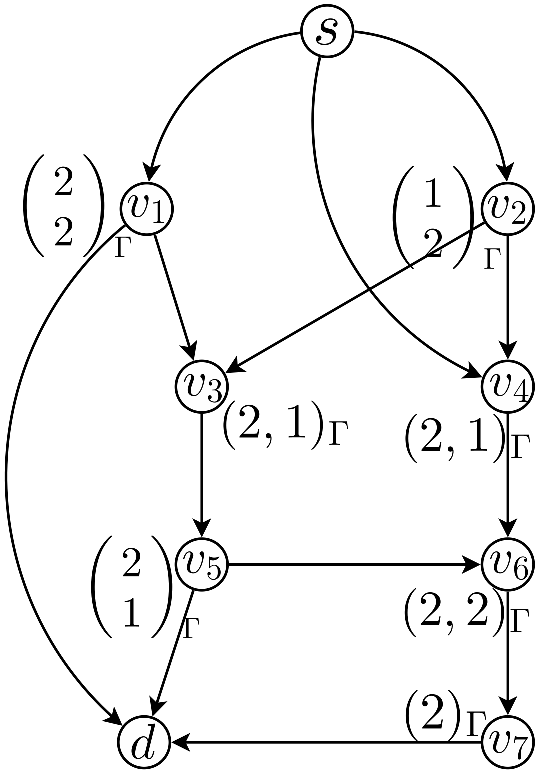

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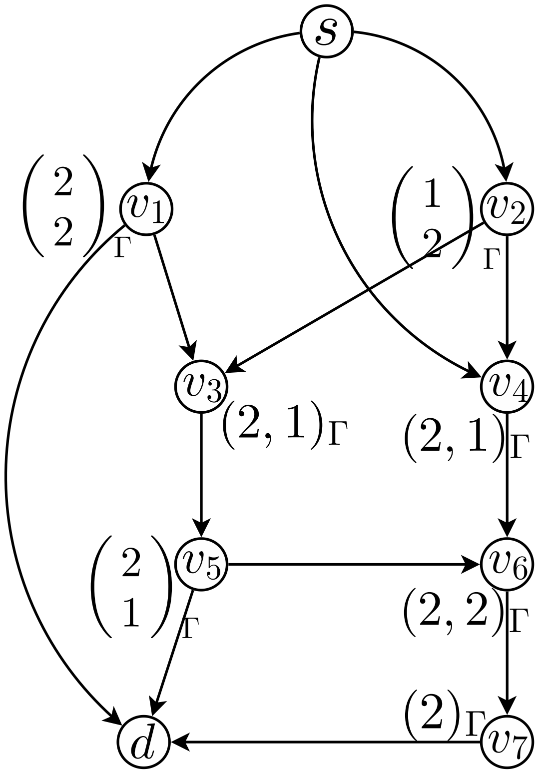

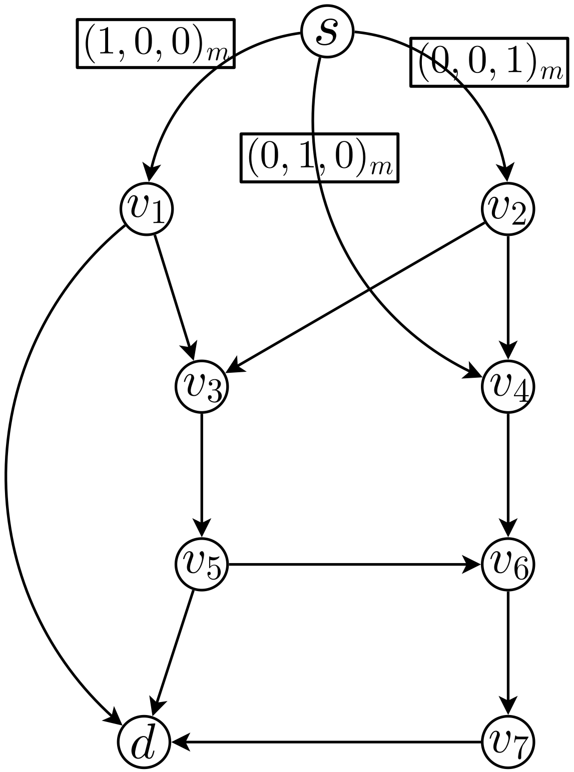

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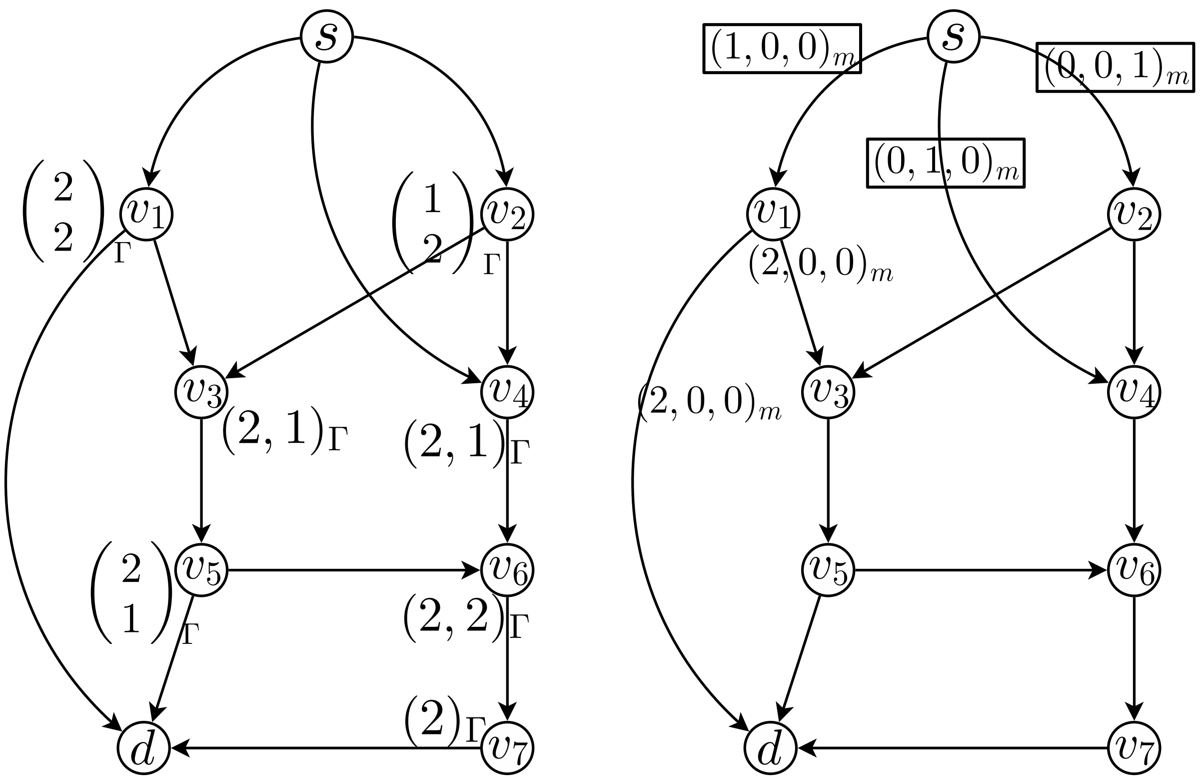

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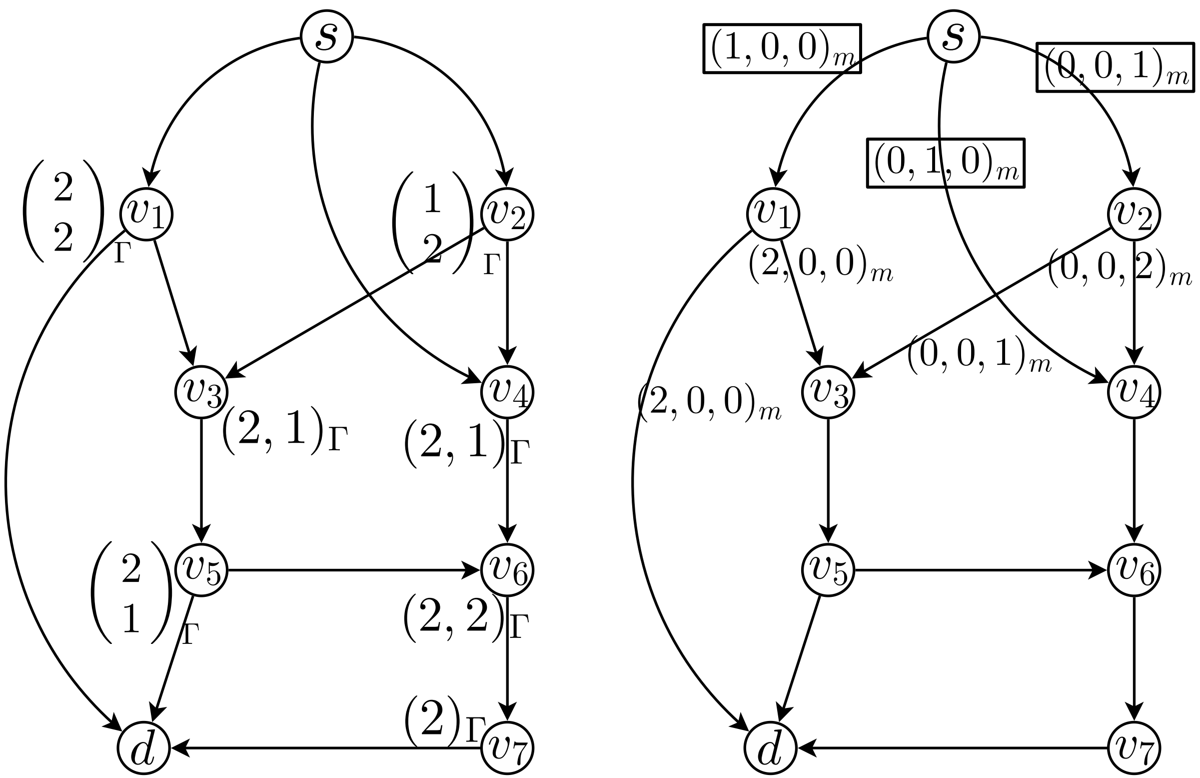
Step 2: Compute the coding vectors m_e

Network coding on GF(3)



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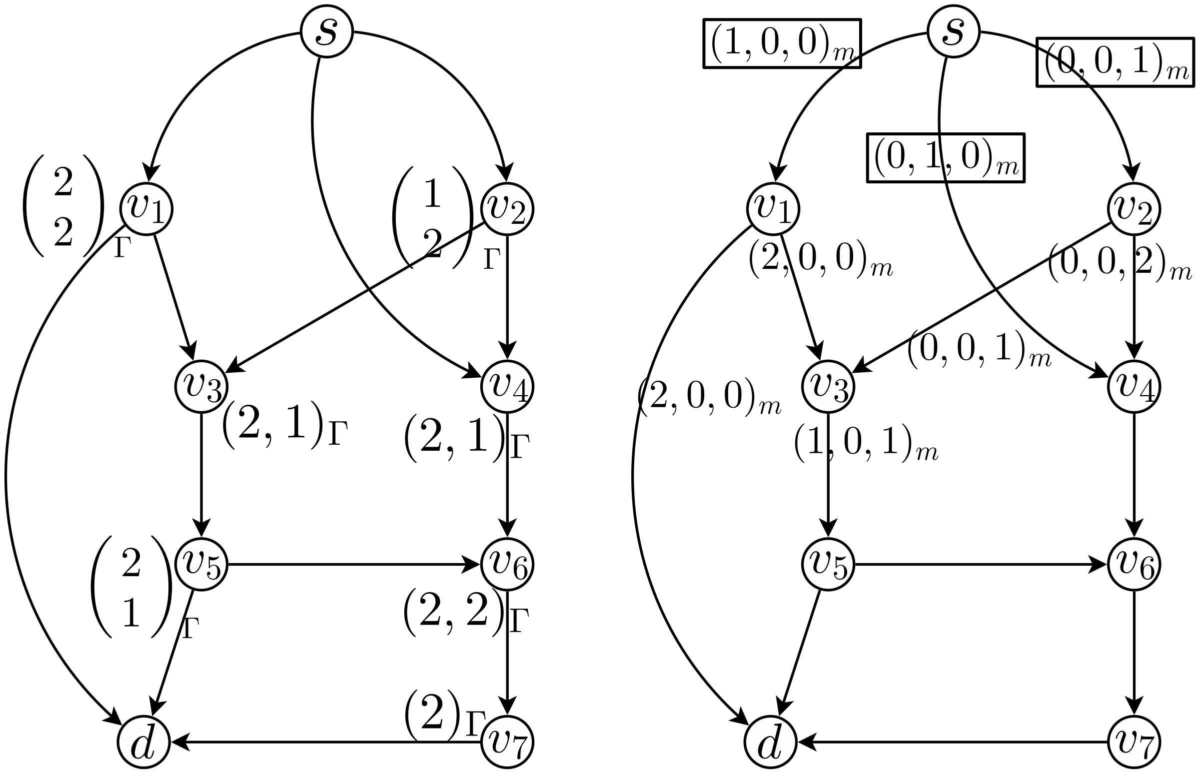

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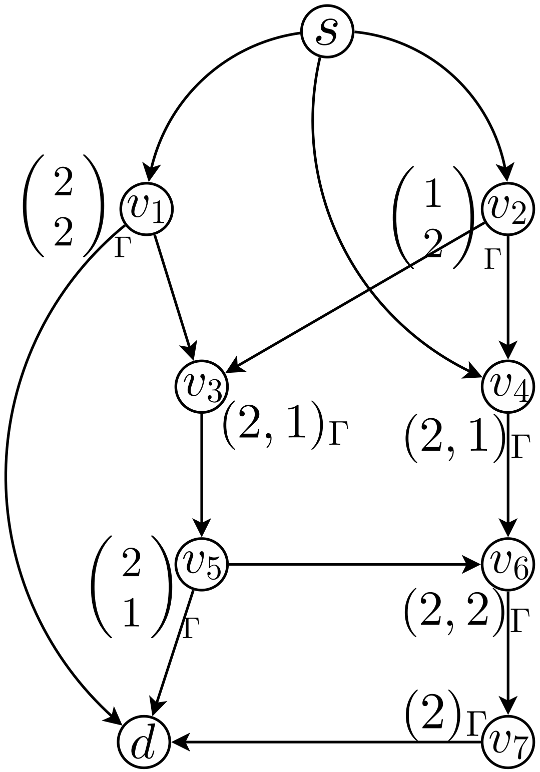

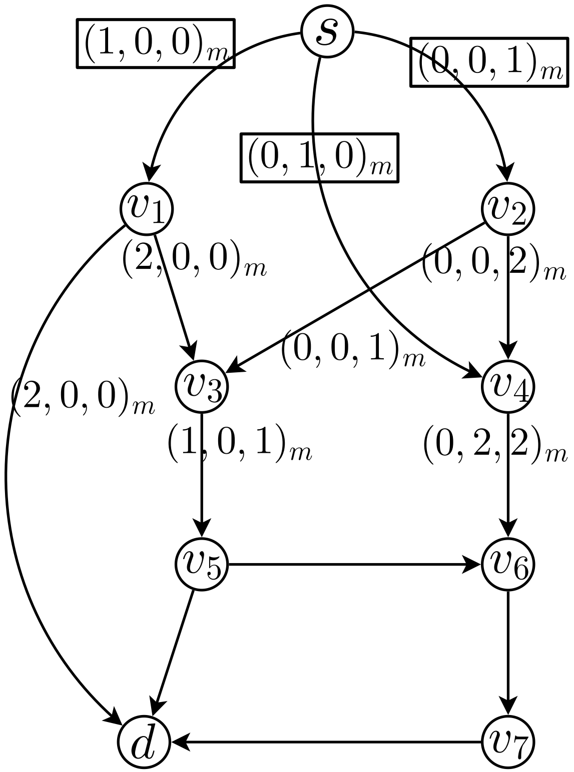

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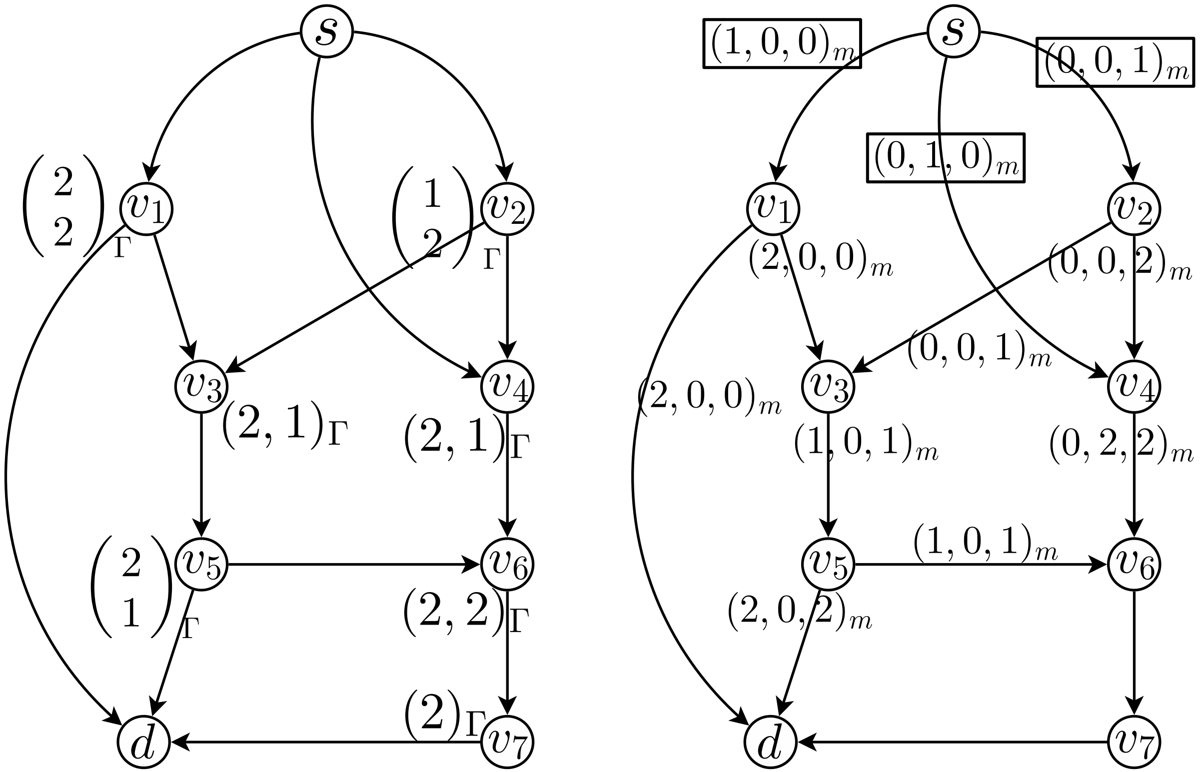

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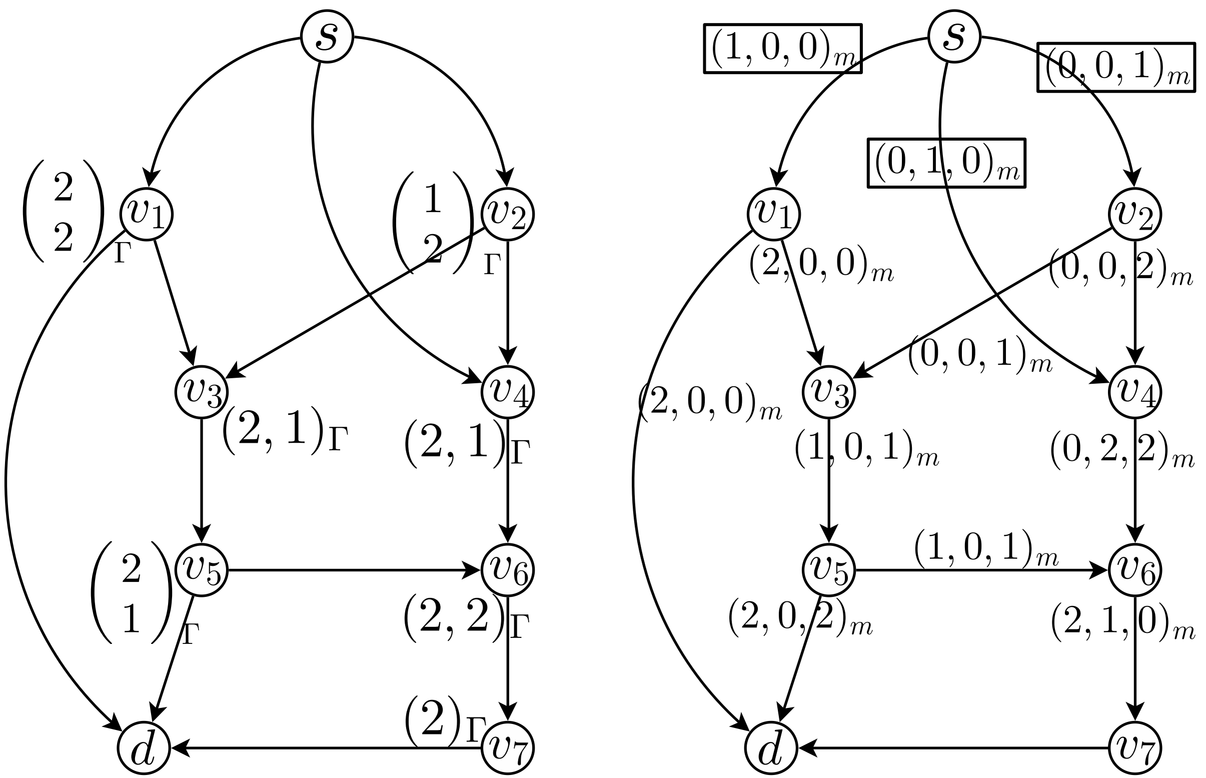

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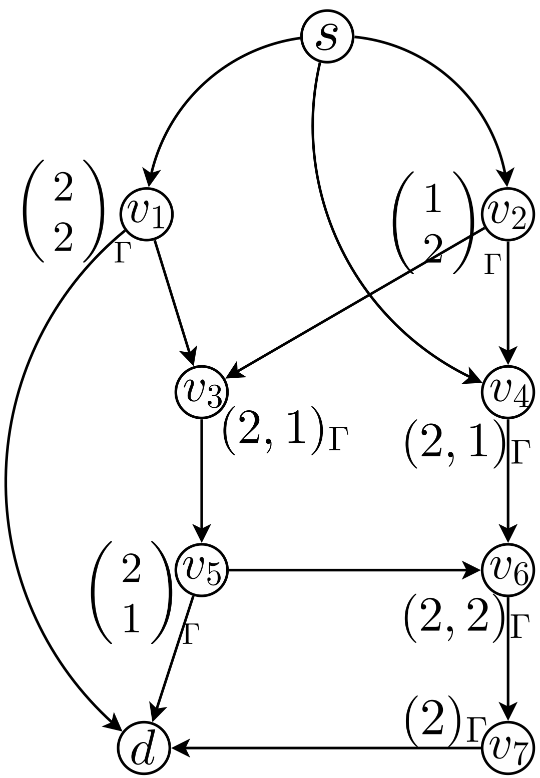

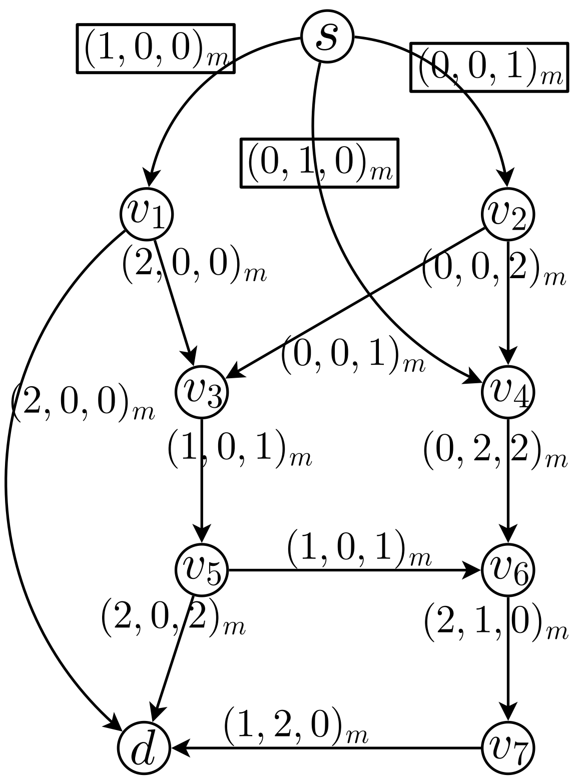

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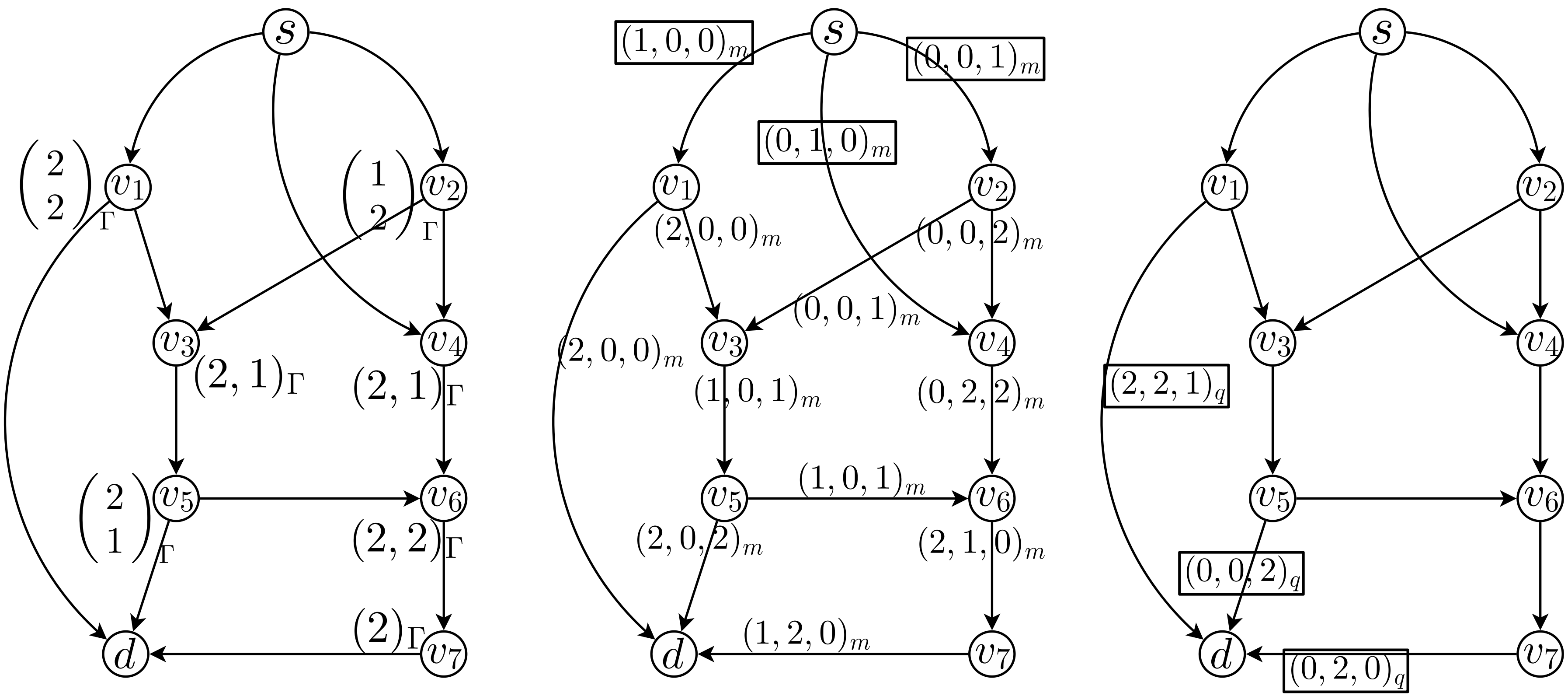
Network coding on GF(3)

Step 3: Compute the coded feedback q_e



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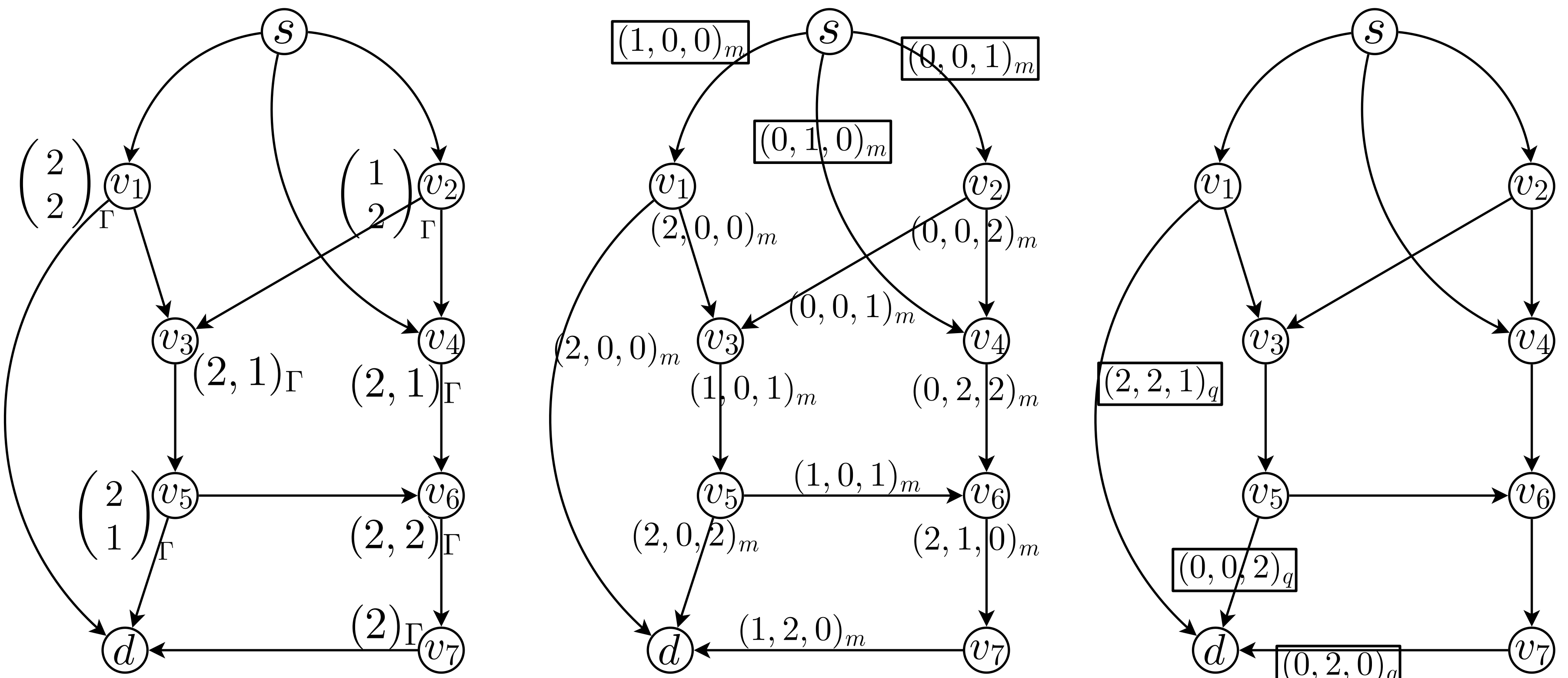
Network coding on GF(3)

♡ **Orthogonal Coded Feedback**



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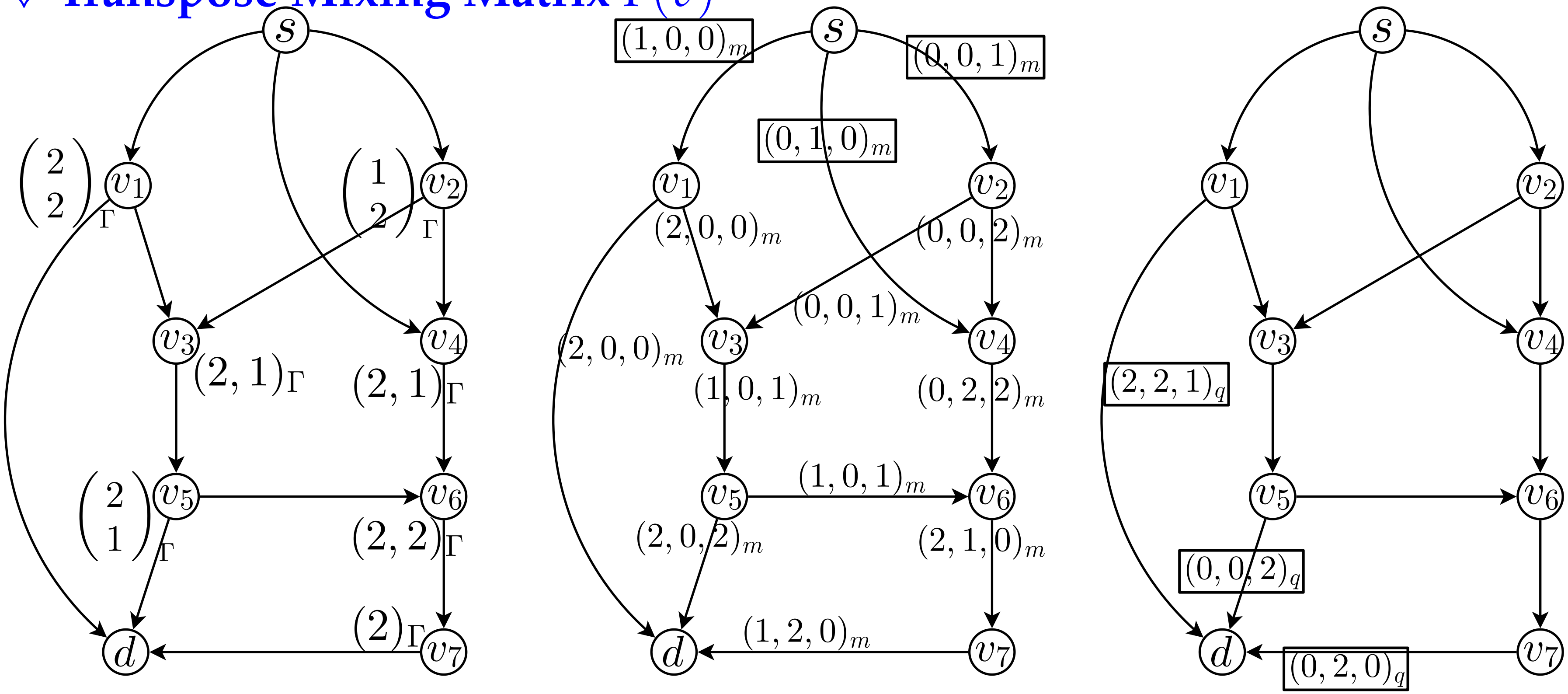
♡ **Transpose Mixing Matrix $\Gamma(v)^T$**



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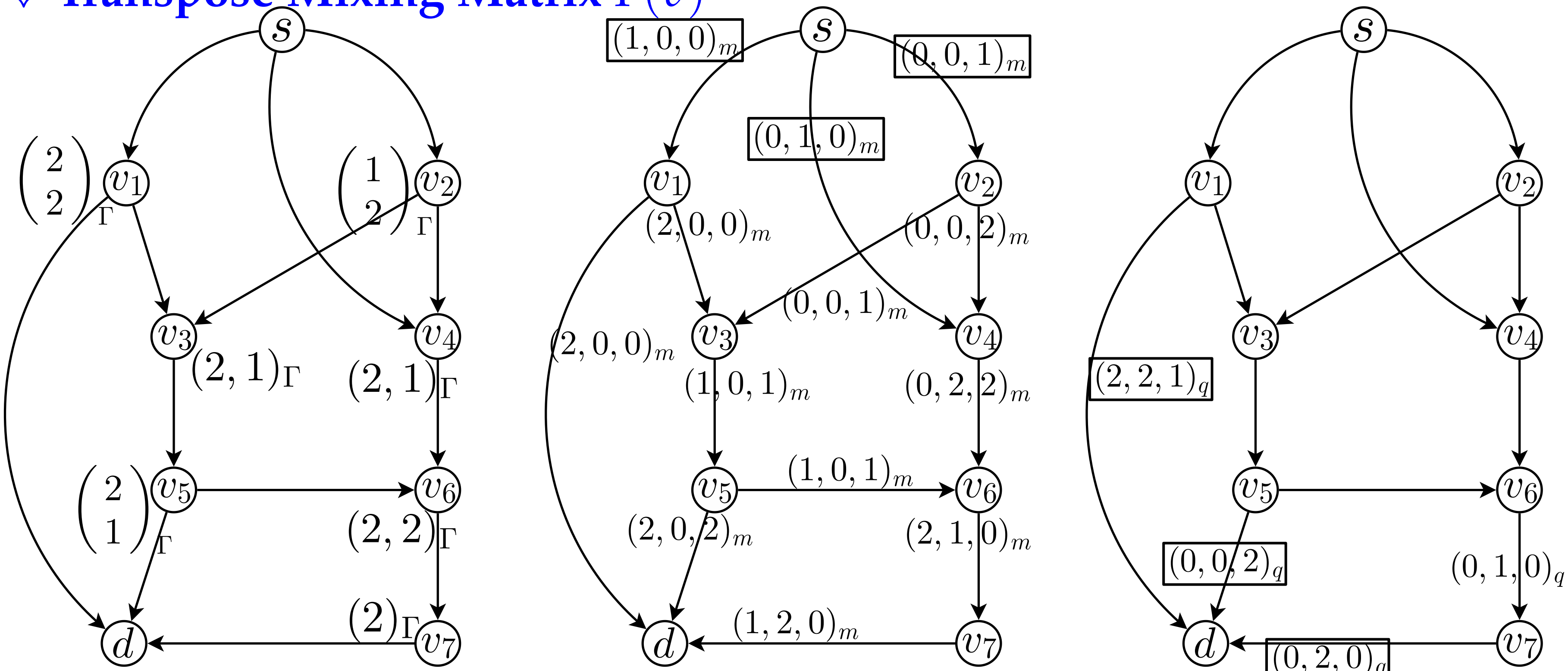

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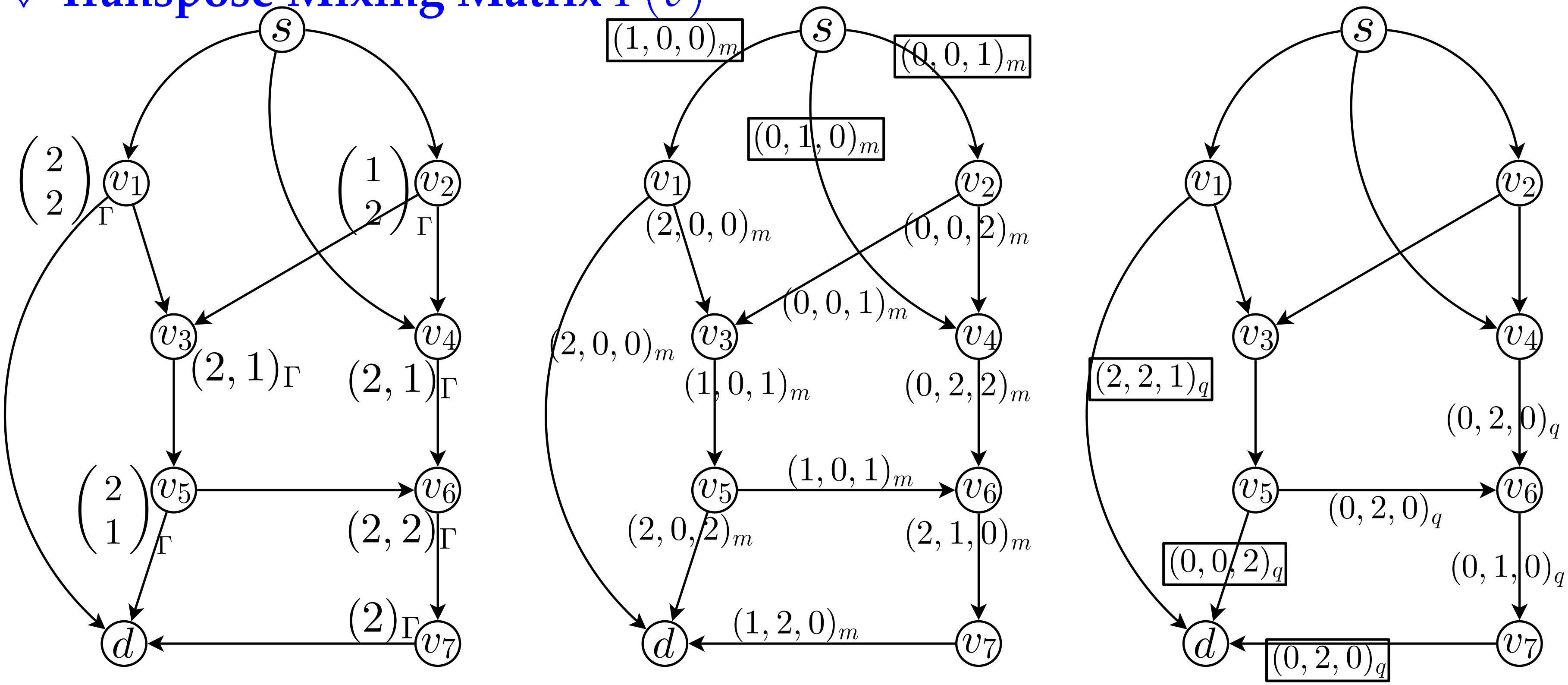

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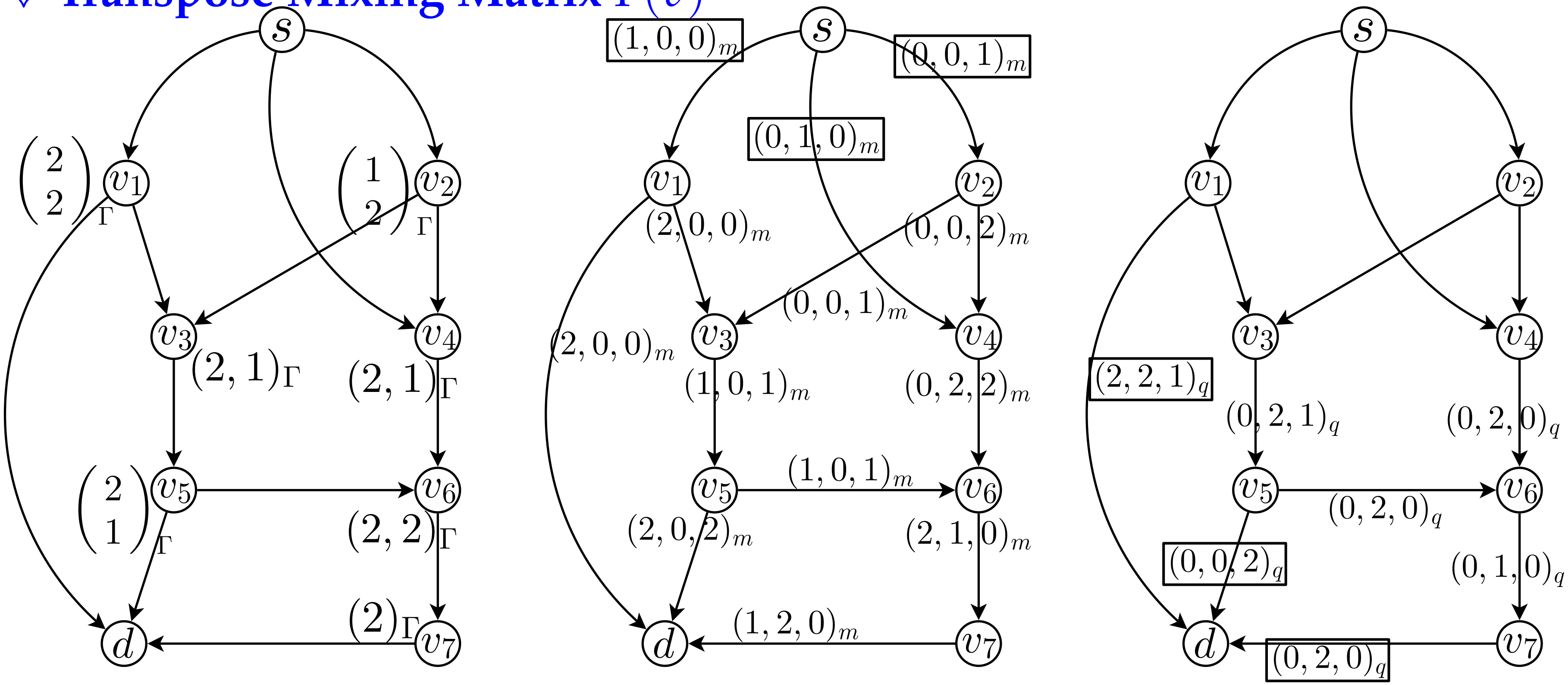

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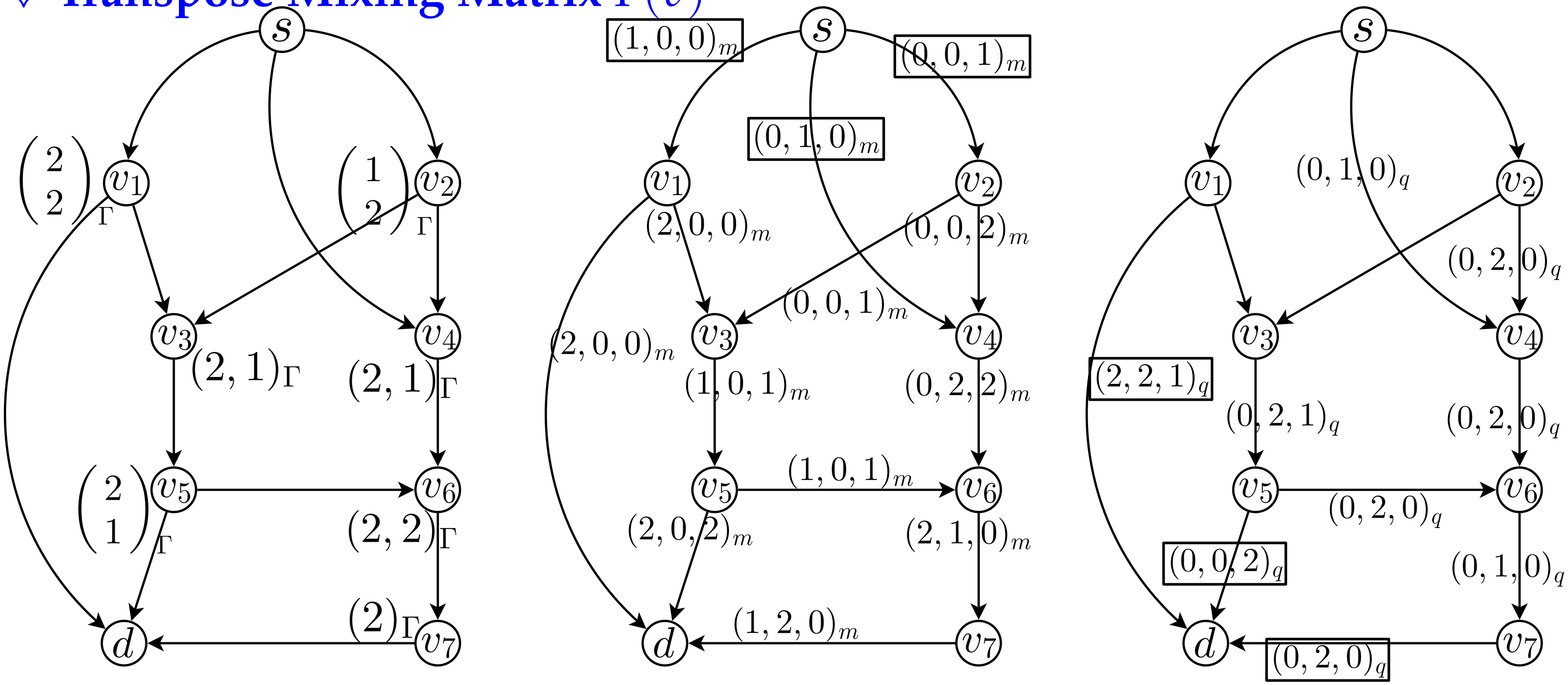

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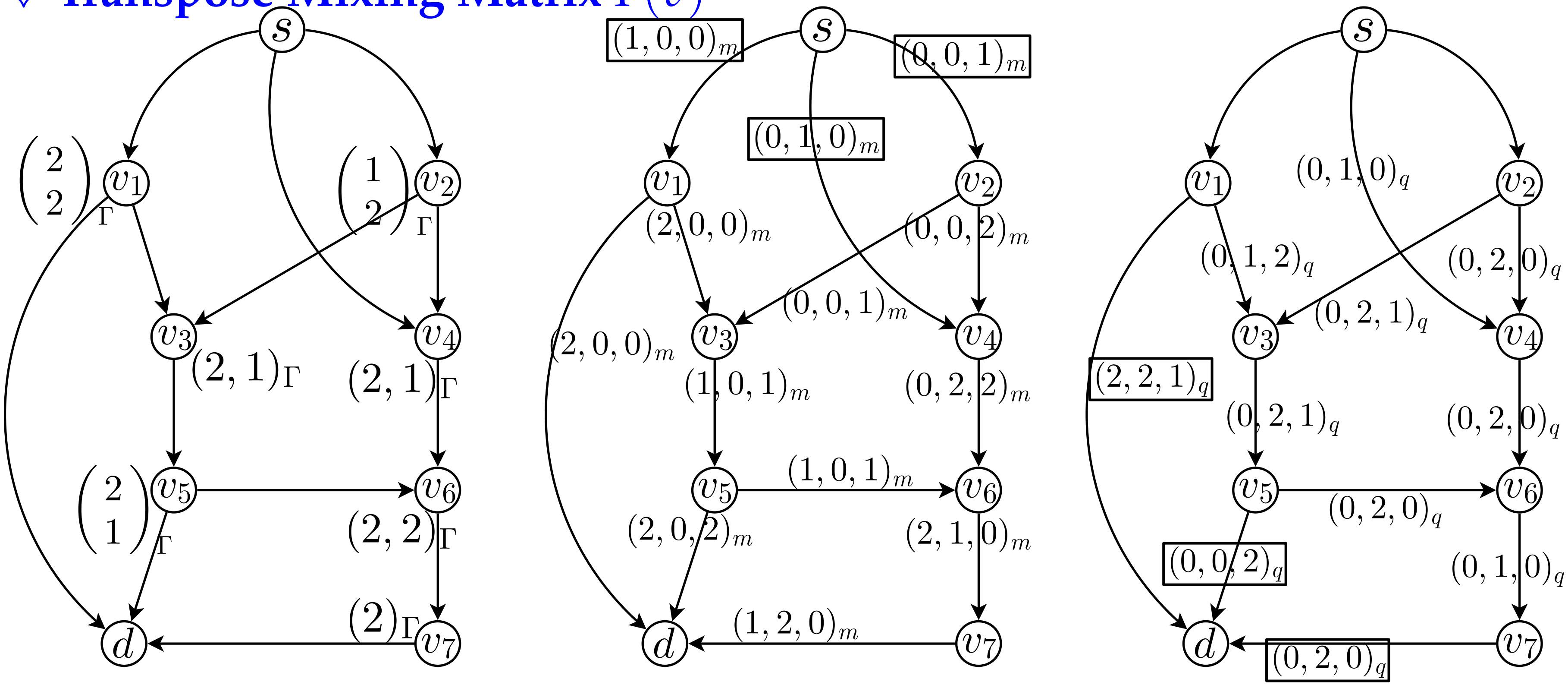

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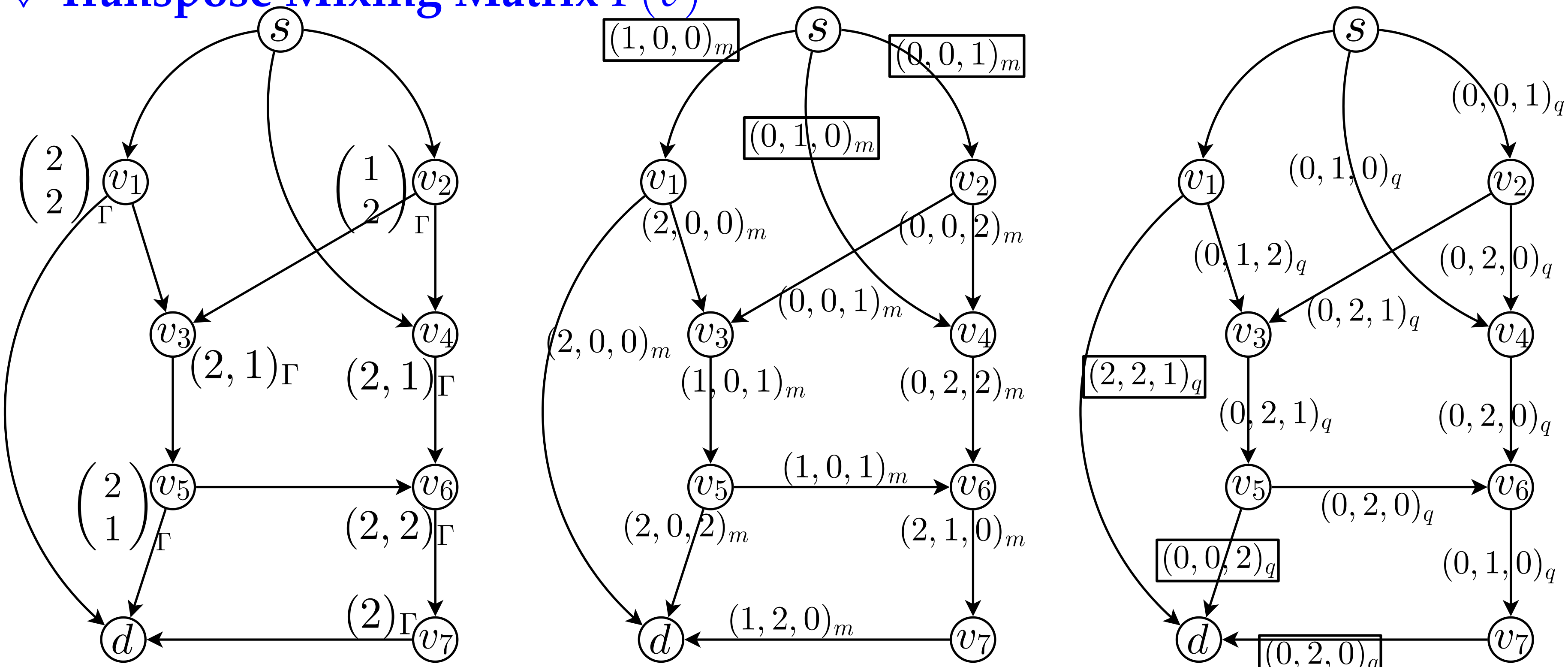

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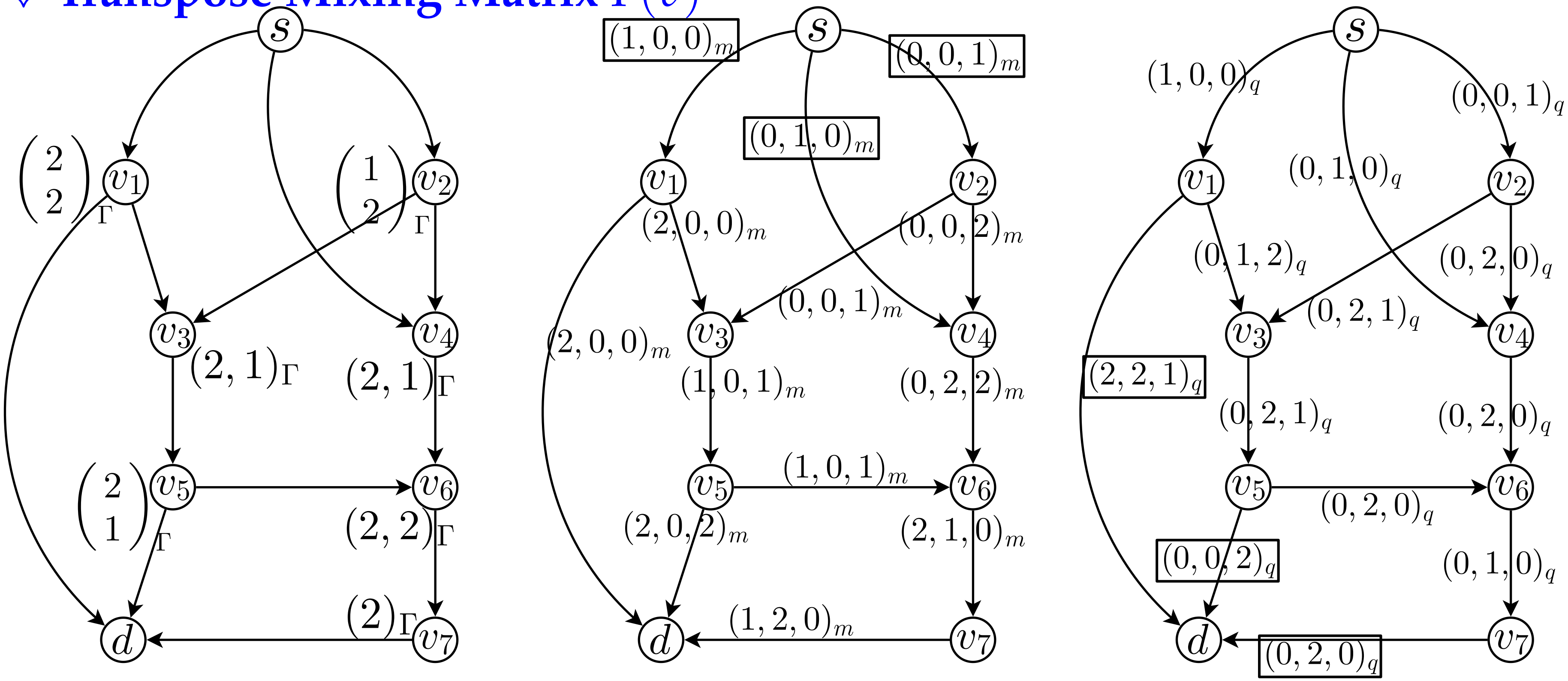

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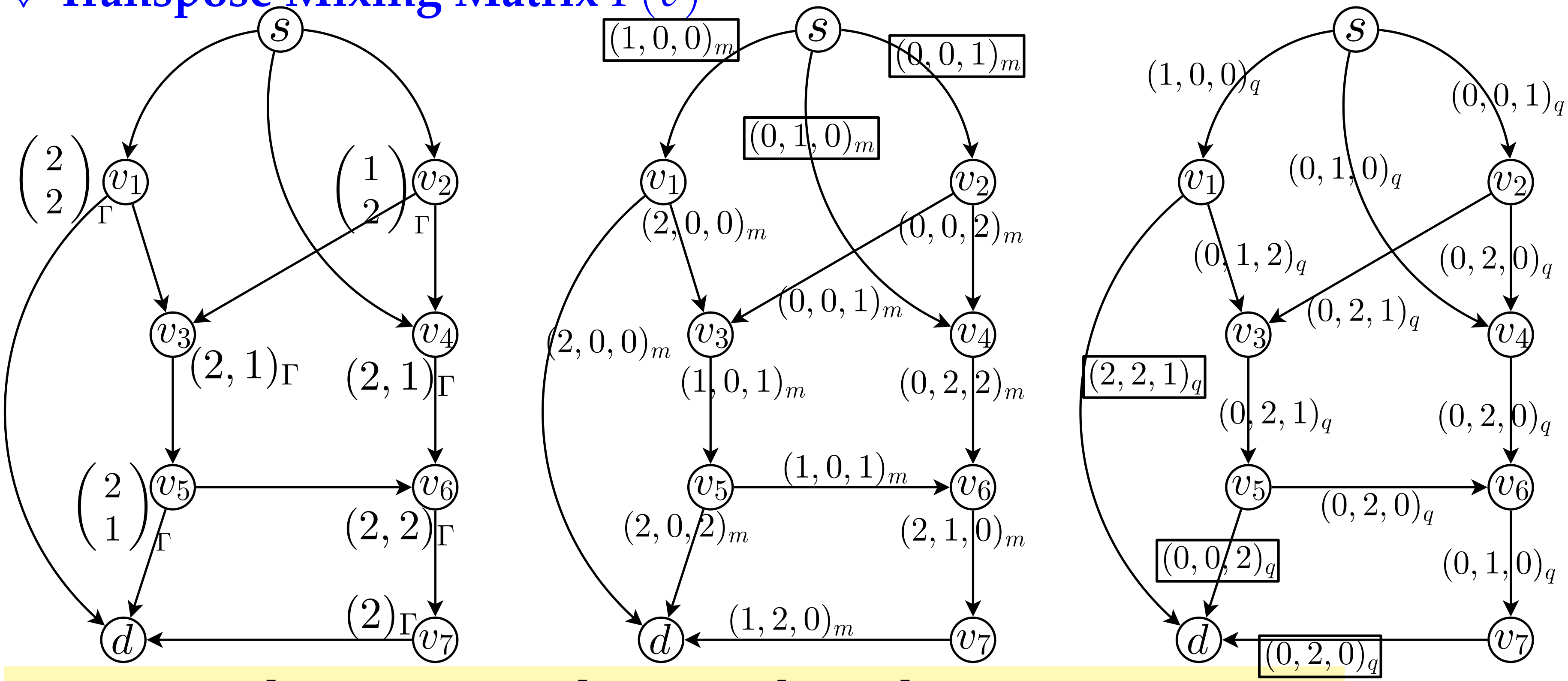
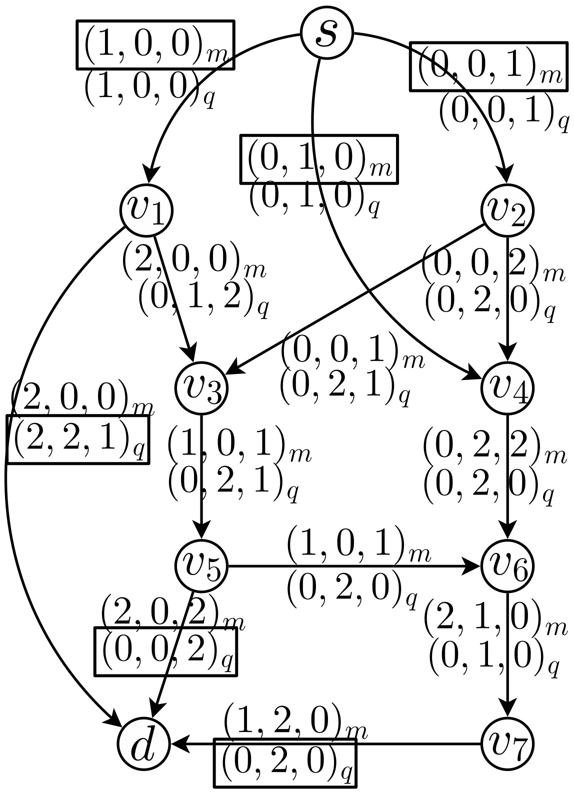


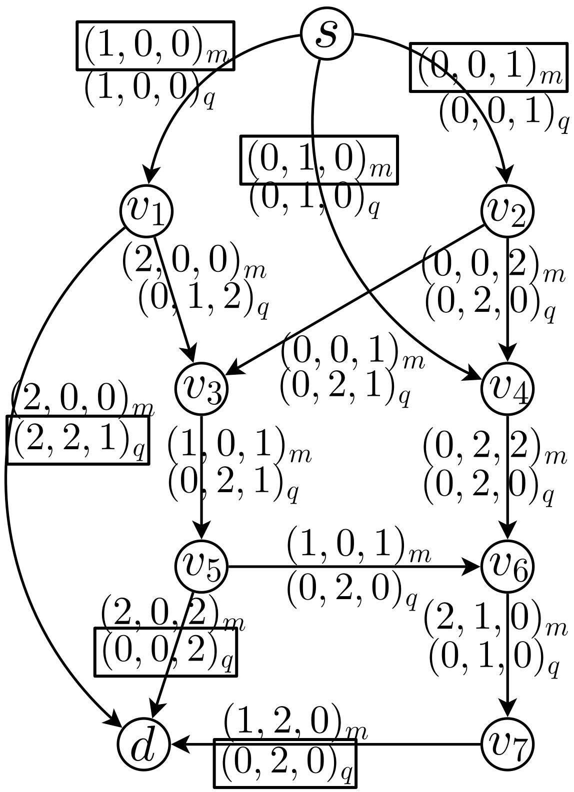
Steps 1 and 2 are Normal Network Coding. Step 3 is new.

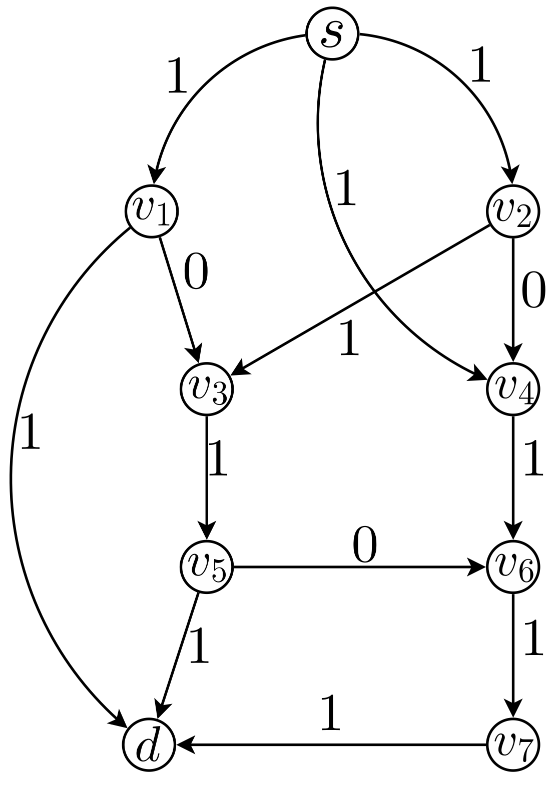
Cont'd


s
$(1,0,0)_m$
$(1,0,0)_q$
$(0,0,1)_m$
$(0,0,1)_q$
$(0,1,0)_m$
$(0,1,0)_q$
v_1
v_2
$(2,0,0)_m$
$(0,1,2)_q$
$(0,0,2)_m$
$(0,2,0)_q$
$(0,0,1)_m$
$(0,2,1)_q$
v_3
v_4
$(2,0,0)_m$
$(2,2,1)_q$
$(1,0,1)_m$
$(0,2,1)_q$
$(0,2,2)_m$
$(0,2,0)_q$
v_5
v_6
$(1,0,1)_m$
$(0,2,0)_q$
$(2,0,2)_m$
$(0,0,2)_q$
$(2,1,0)_m$
$(0,1,0)_q$
d
v_7
$(1,2,0)_m$
$(0,2,0)_q$

Cont'd

Step 4: Compare the inner products





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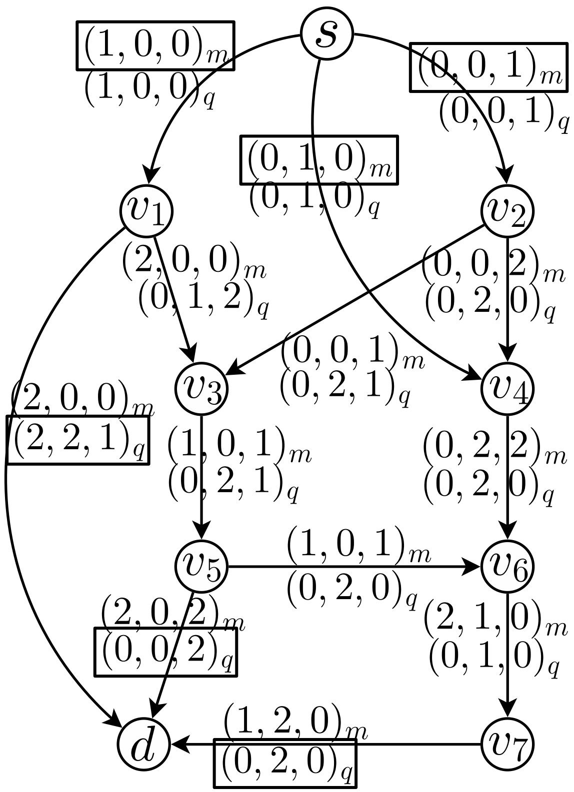
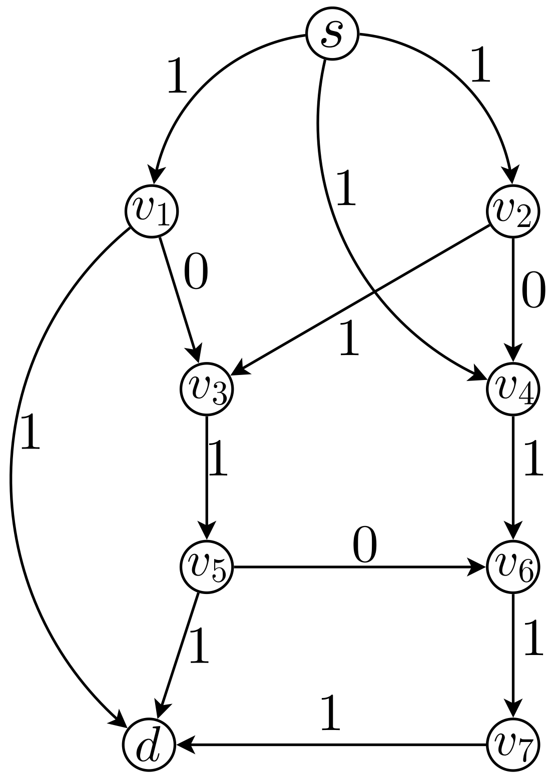
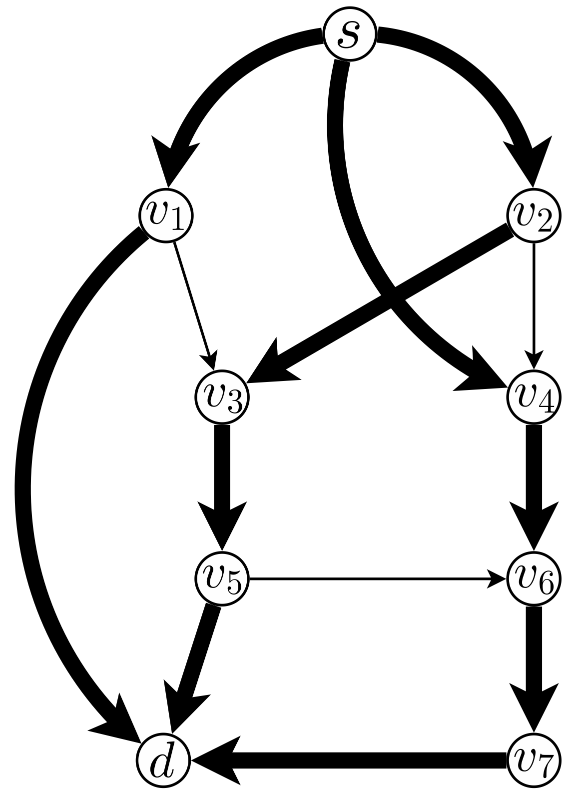
Step 4: Compare the inner products

Comparison to the true max flow found offline







Cont'd

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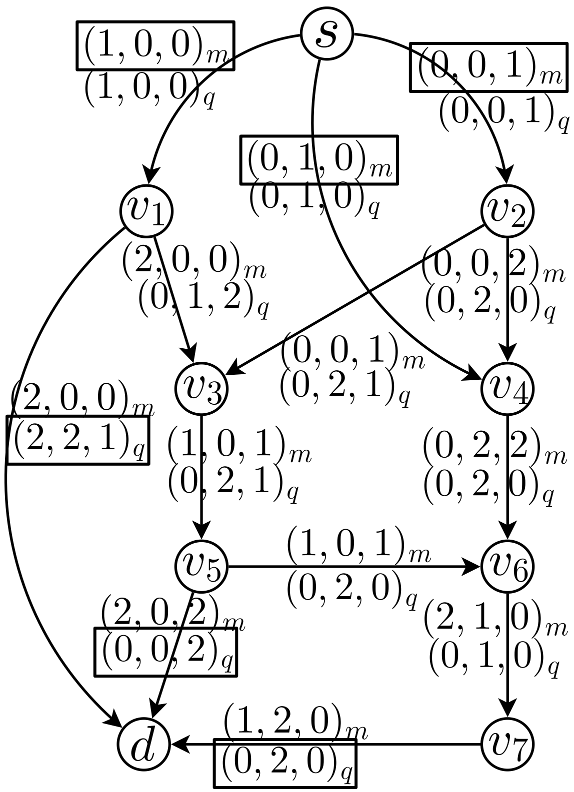

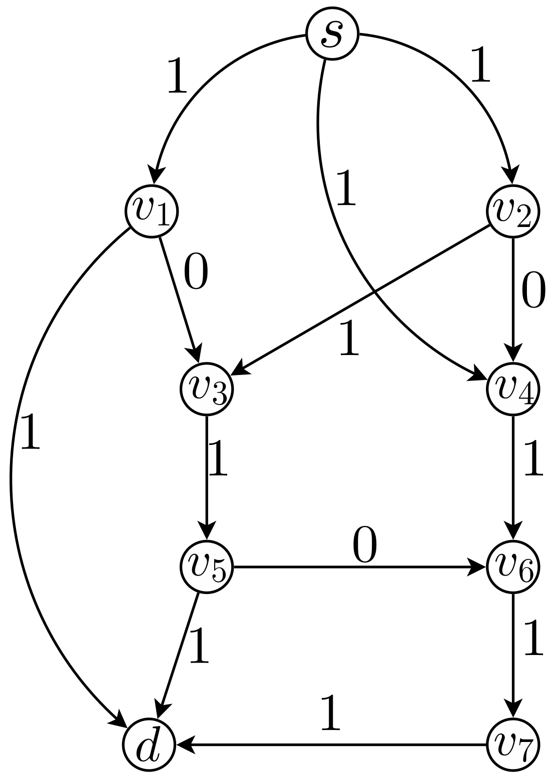

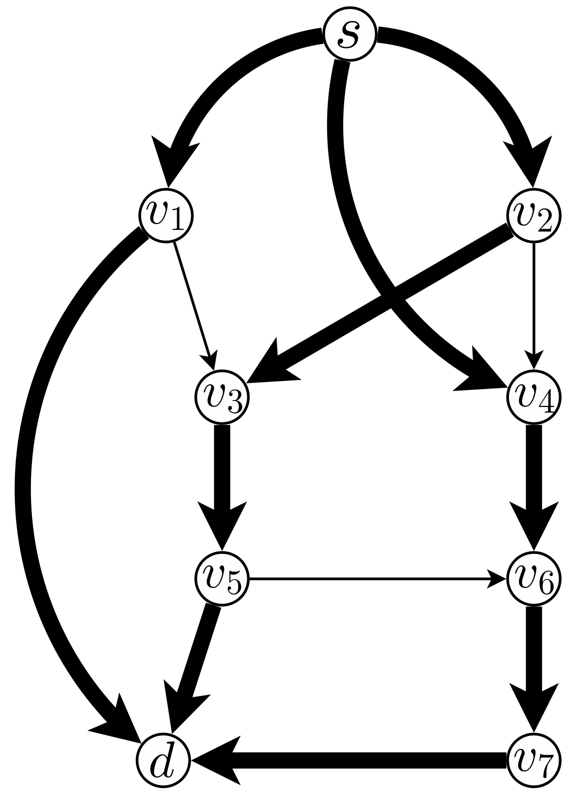

Voila!

Cont'd

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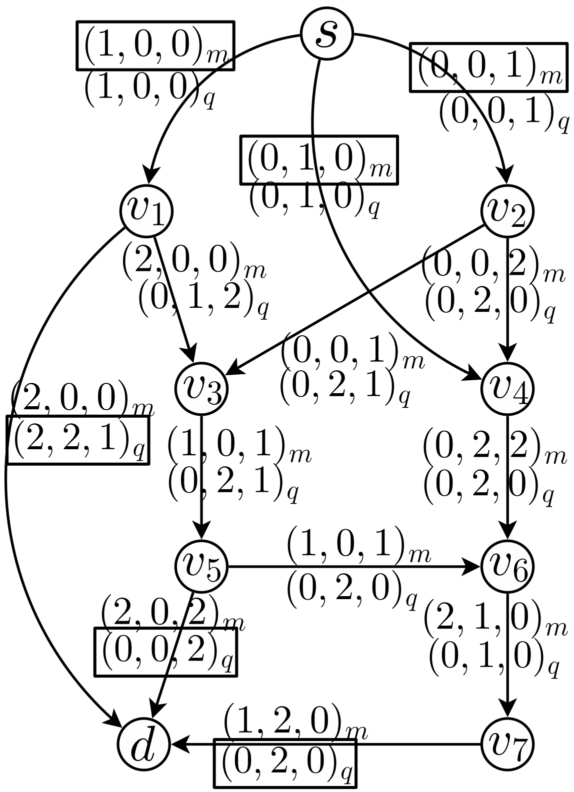
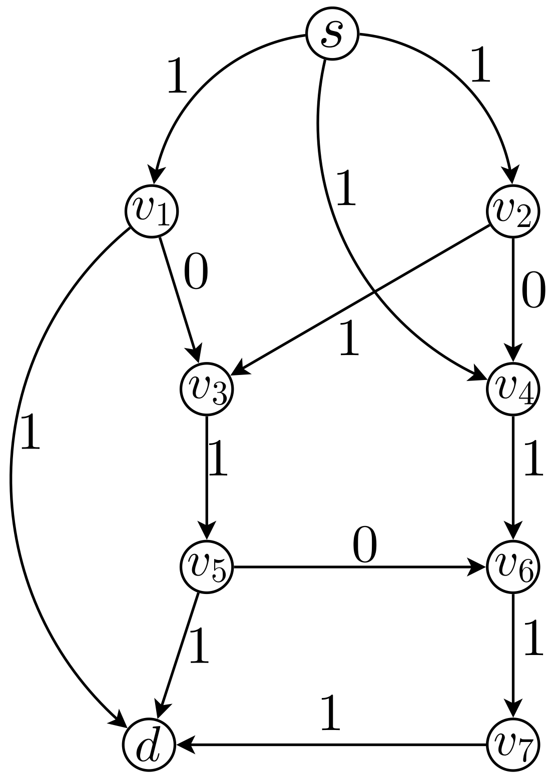
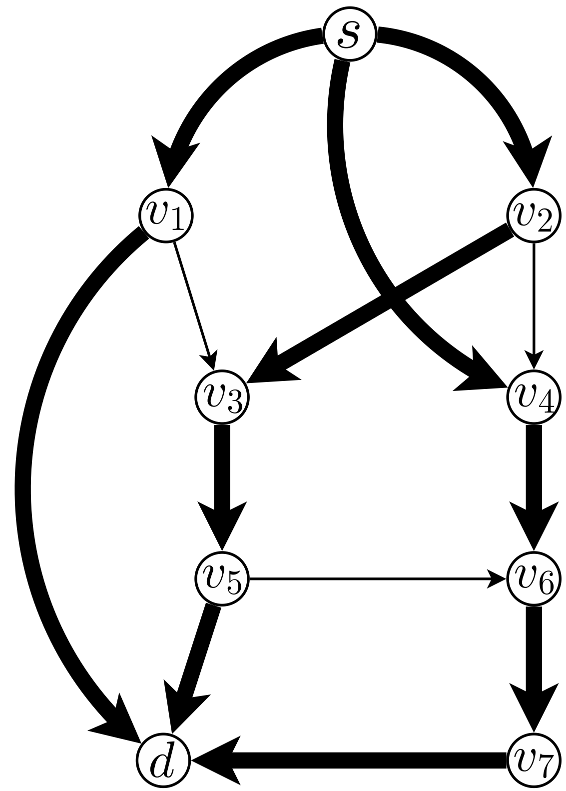
Comparison to the true max flow found offline







Voila!

Not so fast! For more complicated networks, some unexpected scenario may arise. We need a provably correct algorithm.

The Detailed Description

High-level description:

Choose $\Gamma(v)$
loop
  Compute Forward Messages m_e
  Compute Coded Feedback q_e
  Find redundant edge set $E_R(v)$
  if $E_R(v) \neq \emptyset$ **then**
    Remove $E_R(v)$.
  else
    return the remaining graph G
  end if
end loop

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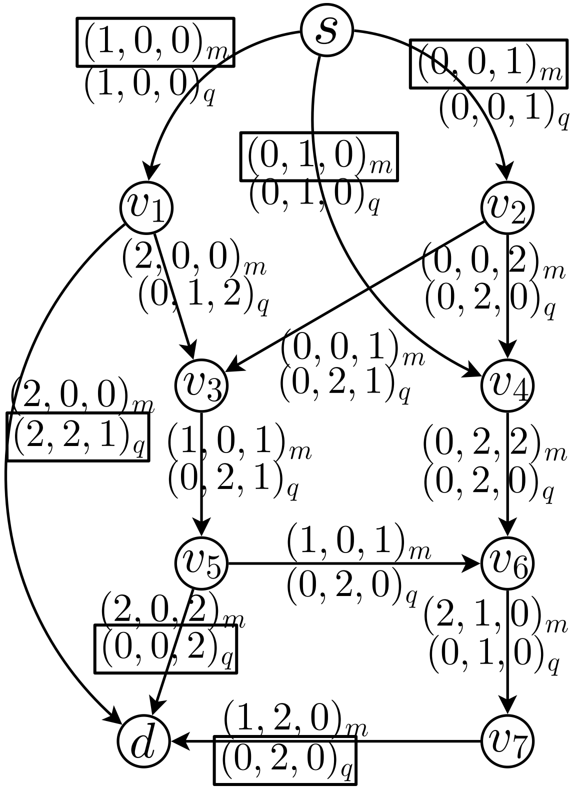
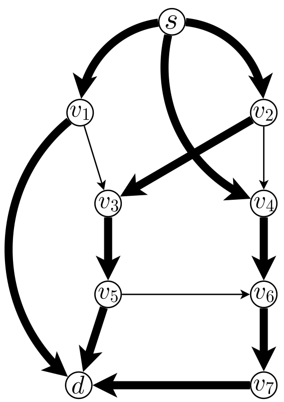
end loop

Find redundant edge set $E_R(v)$:

while searching v from downstream d back to upstream s **do**
  Generate the inner product matrix Φ between $[q_e : e \in \mathrm{Out}(v)]$ and $[m_e : e \in \mathrm{In}(v)]$.
  Choose an $E_v \subseteq \mathrm{In}(v)$ such that the corresponding submatrix of Φ being of a full rank square matrix.
  if $E_v \neq \mathrm{In}(v)$ **then**
    return $E_R(v) \leftarrow \mathrm{In}(v) \backslash E_v$.
  end if
end while

An Illustrative Example





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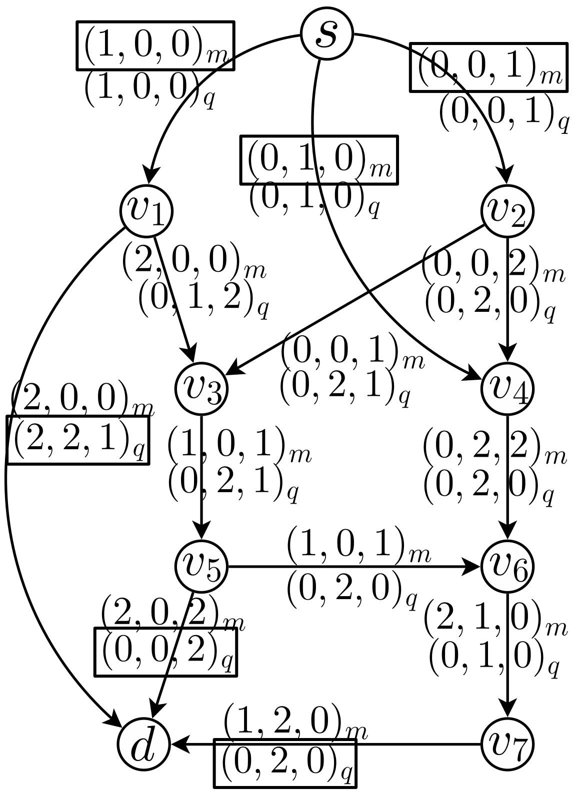
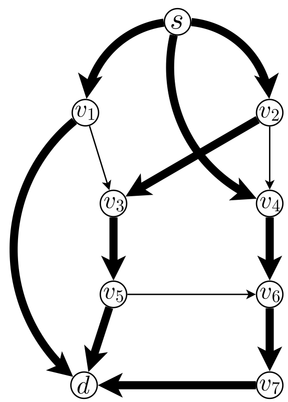


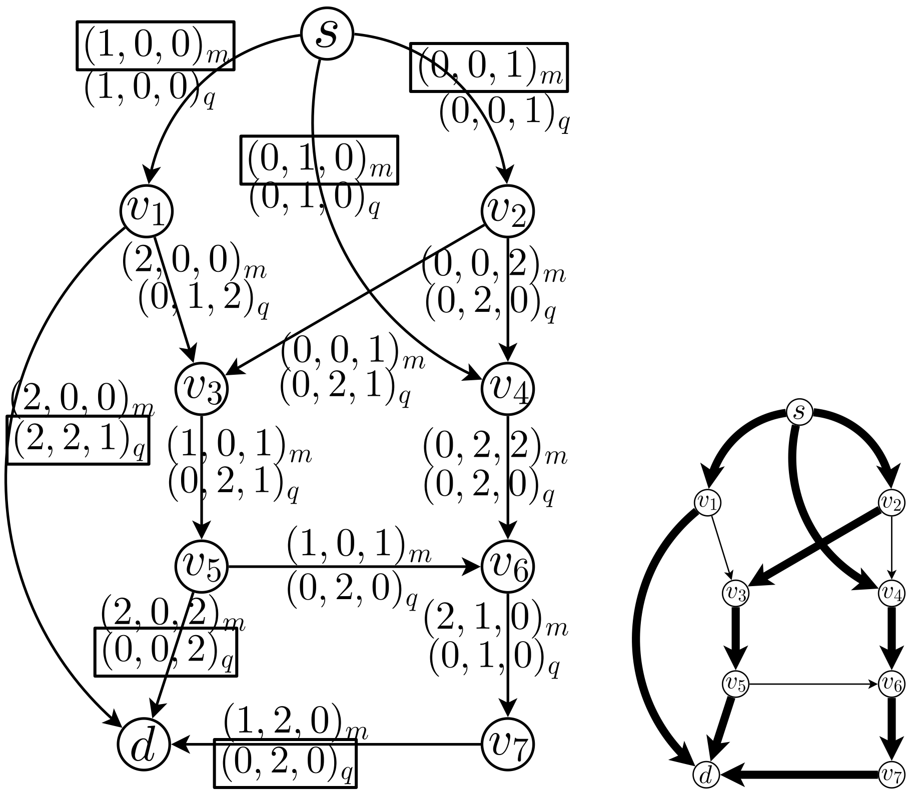


Search v_7:

Out(v_7): $q_{v_7,d} = (0, 2, 0)$
In(v_7): $m_{v_6,v_7} = (2, 1, 0)$

Φ: 2 $\Rightarrow$ Full rank

An Illustrative Example

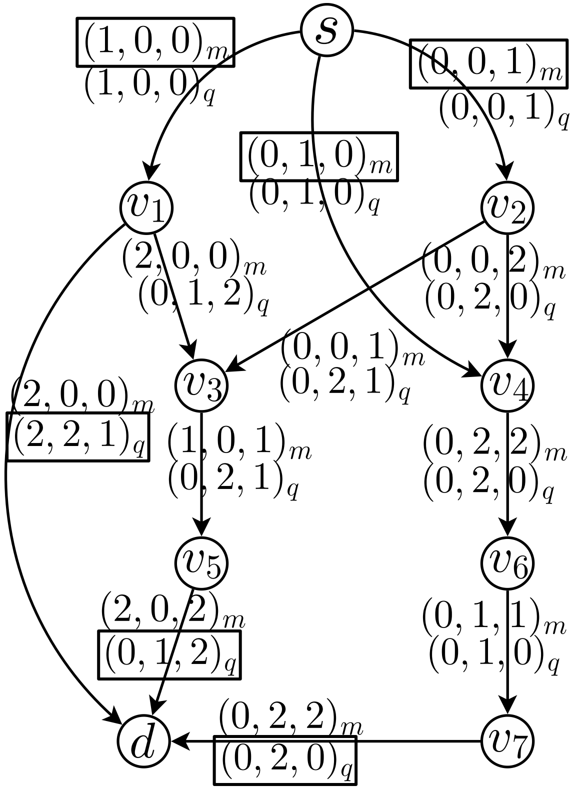


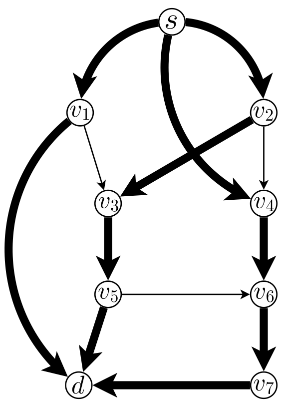
Search v_6:

Out(v_6): $q_{v_6,v_7} = (0,1,0)$
In(v_6): $m_{\cdot,v_6} = (1,0,1), (0,2,2)$

Φ: $[0, 2] \Rightarrow E_v = \{(v_4, v_6)\}$
$E_R(v) = \{(v_5, v_6)\}$

An Illustrative Example





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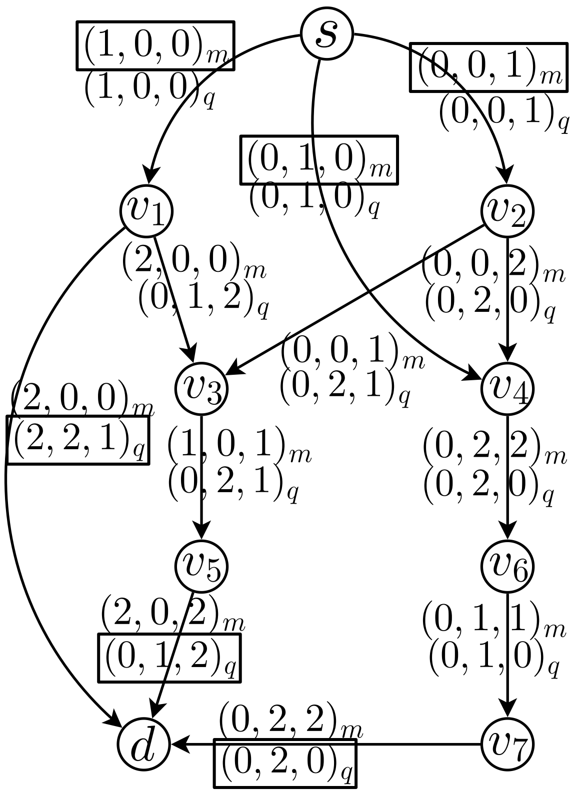
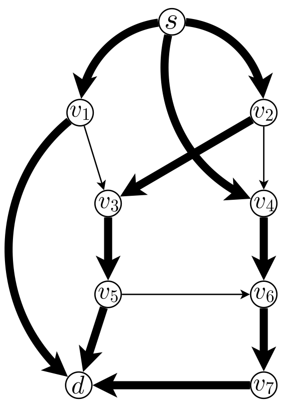


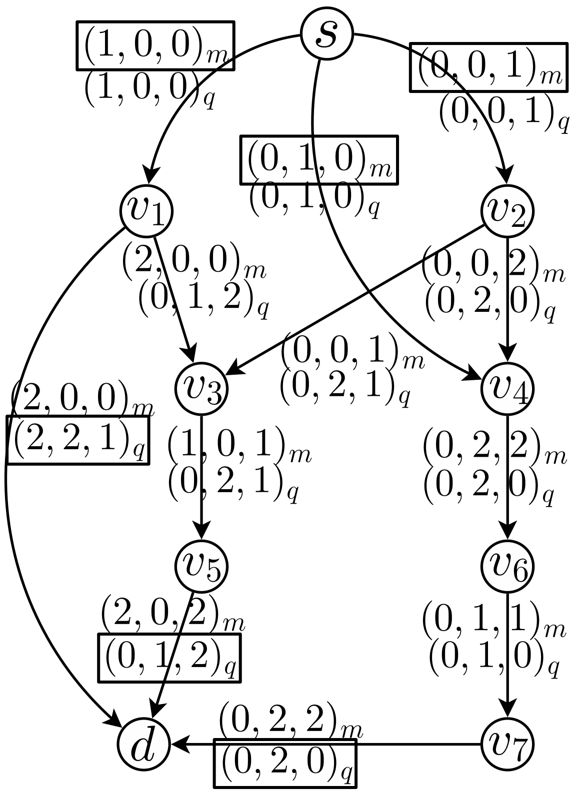


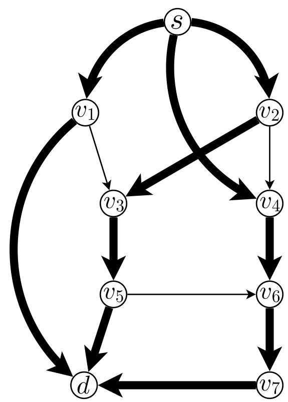
Search v_5:

Out(v_5): $q_{v_5,d} = (0,1,2)$
In(v_5): $m_{v_3,v_5} = (1,0,1)$

Φ: 2 $\Rightarrow$ Full rank

An Illustrative Example



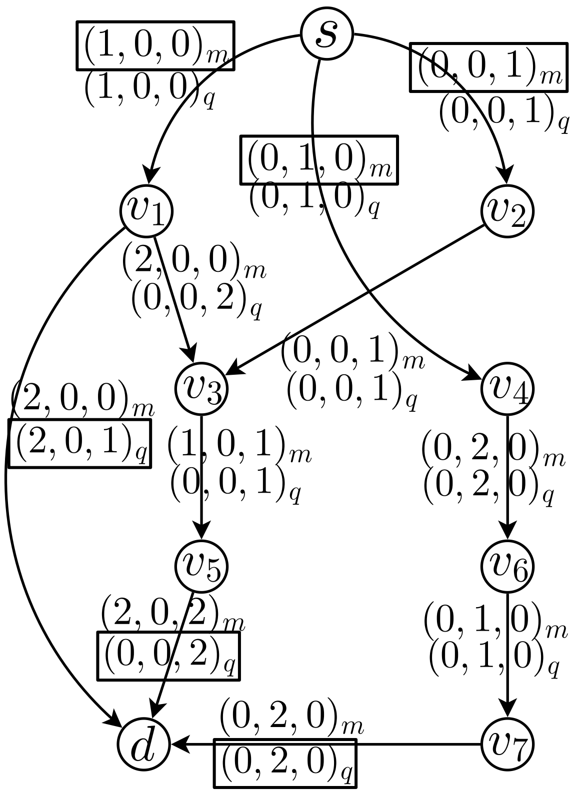


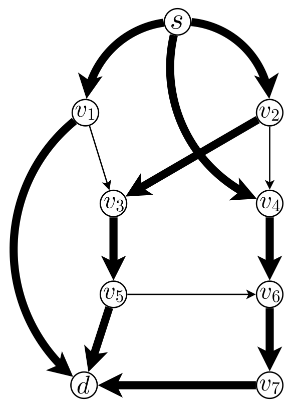
Search v_4:

Out(v_4): $q_{v_4,v_6} = (0,2,0)$
In(v_4): $m_{\cdot,v_4} = (0,1,0), (0,0,2)$

Φ: $[\,2, 0] \Rightarrow E_v = \{(s, v_4)\}$
$E_R(v) = \{(v_2, v_4)\}$

An Illustrative Example





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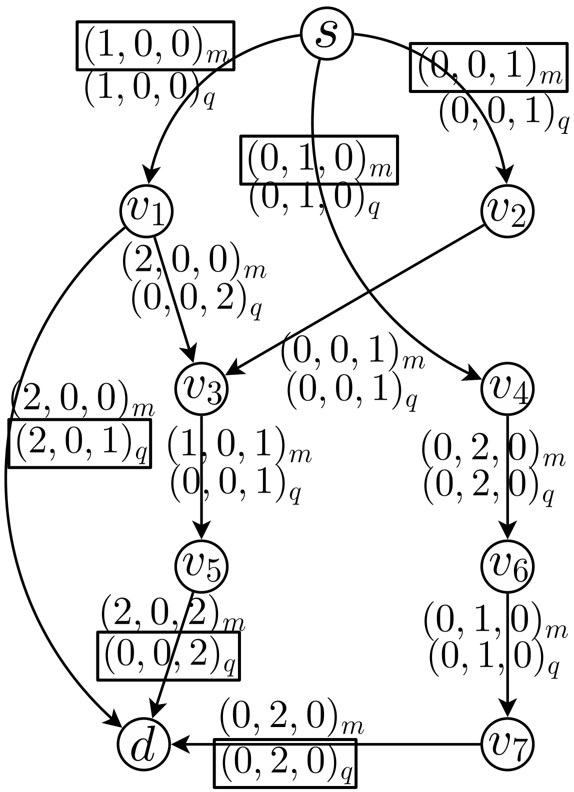
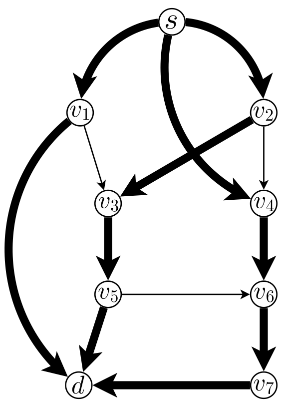


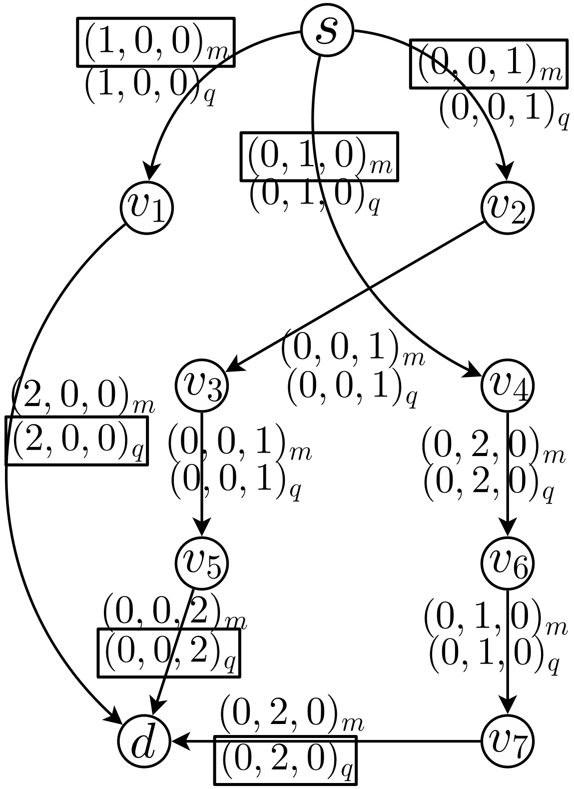


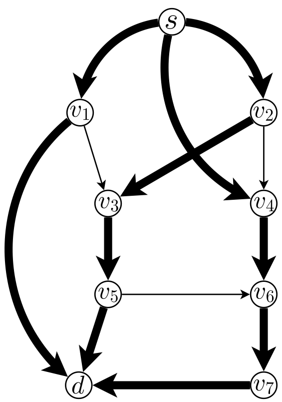
Search v_3:

Out(v_3): $q_{v_3,v_5} = (0,0,1)$
In(v_3): $m_{\cdot,v_3} = (2,0,0), (0,0,1)$

Φ: $[0, 1] \Rightarrow E_v = \{(v_2, v_3)\}$
$E_R(v) = \{(v_1, v_3)\}$

An Illustrative Example





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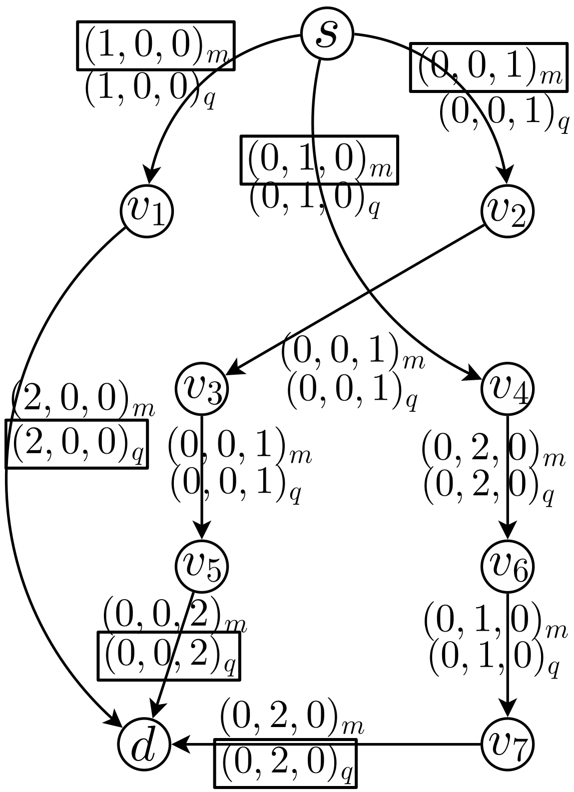
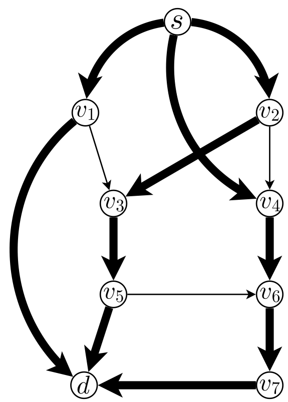


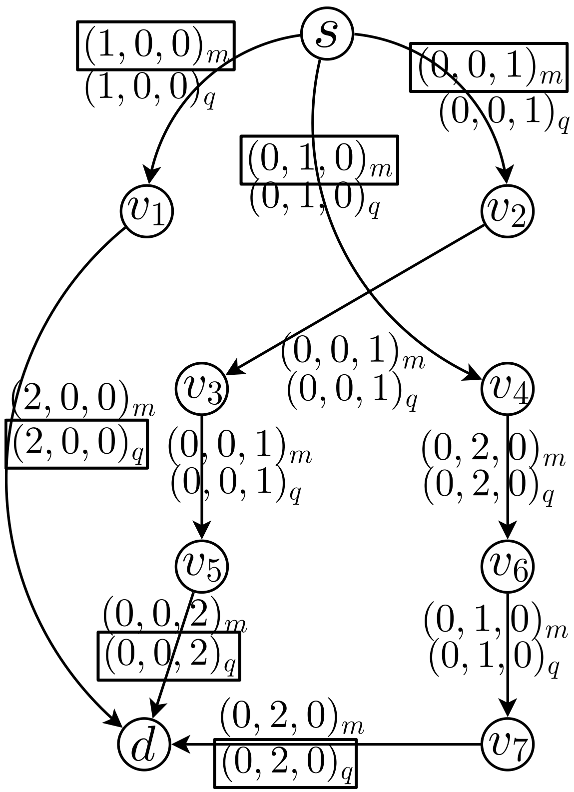


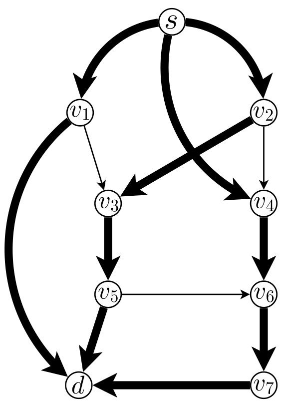
Search v_2:

Out(v_2): $q_{v_2,v_3} = (0,0,1)$
In(v_2): $m_{s,v_2} = (0,0,1)$

Φ: 1 $\Rightarrow$ Full rank

An Illustrative Example





Search v_1:

Out(v_1): $q_{v_1,d} = (2,0,0)$
In(v_1): $m_{s,v_1} = (1,0,0)$

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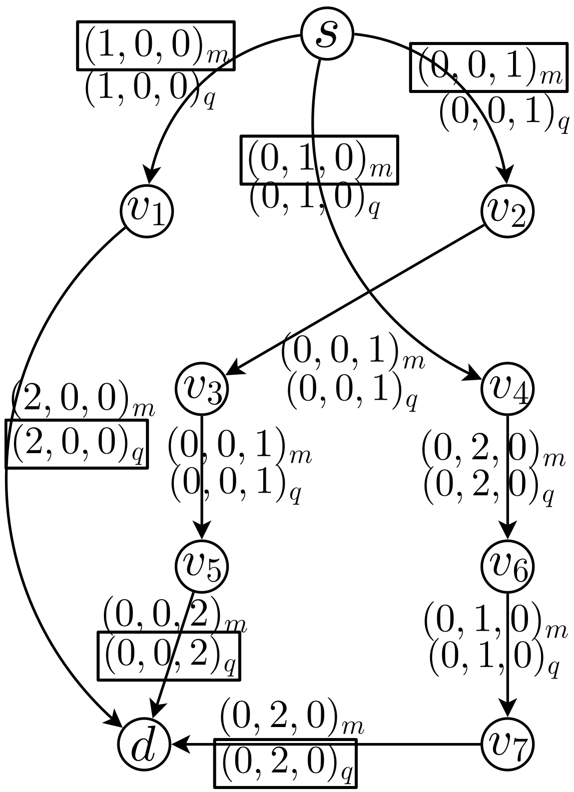

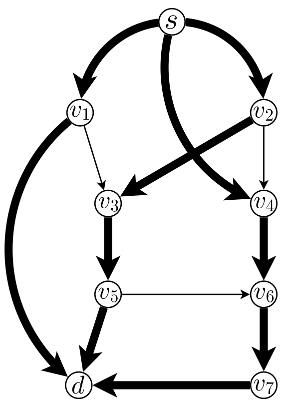

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The feedback q_e tells node v what is critical for d. Choose m_e that indeed carries the critical info.

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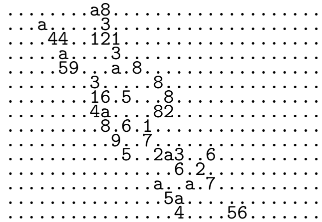
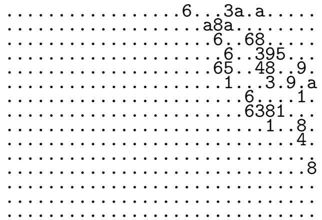
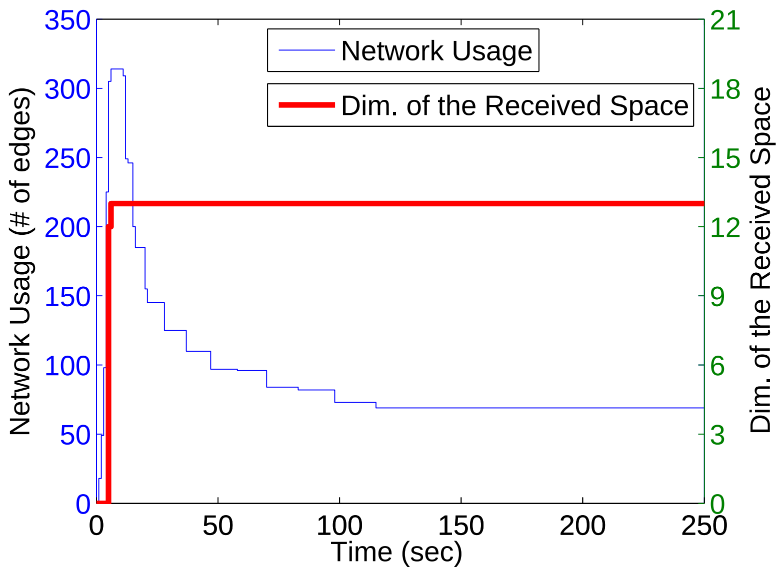
Simulation Results

A 30-node network with





The coding-theoretic approach



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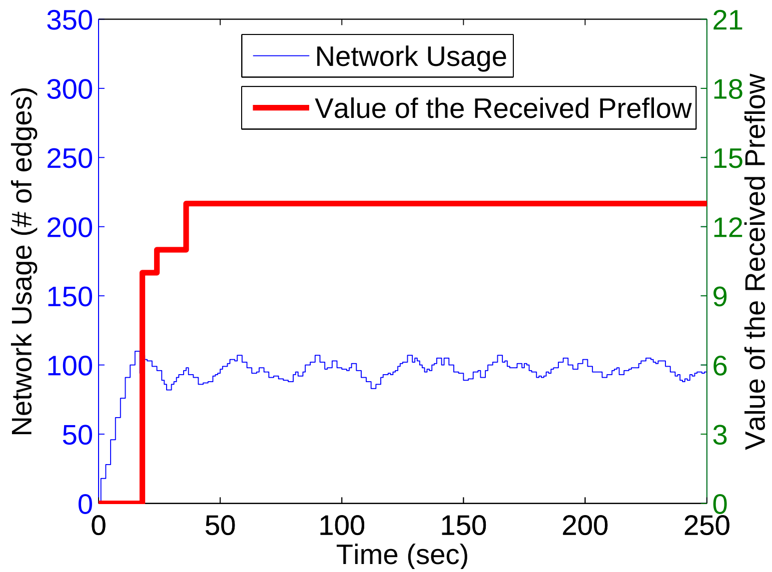

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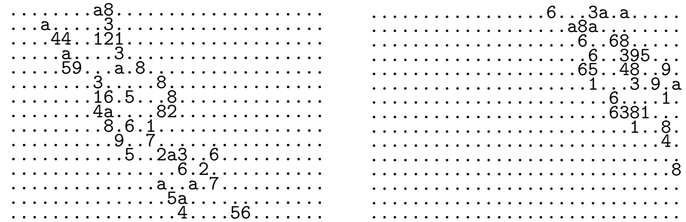

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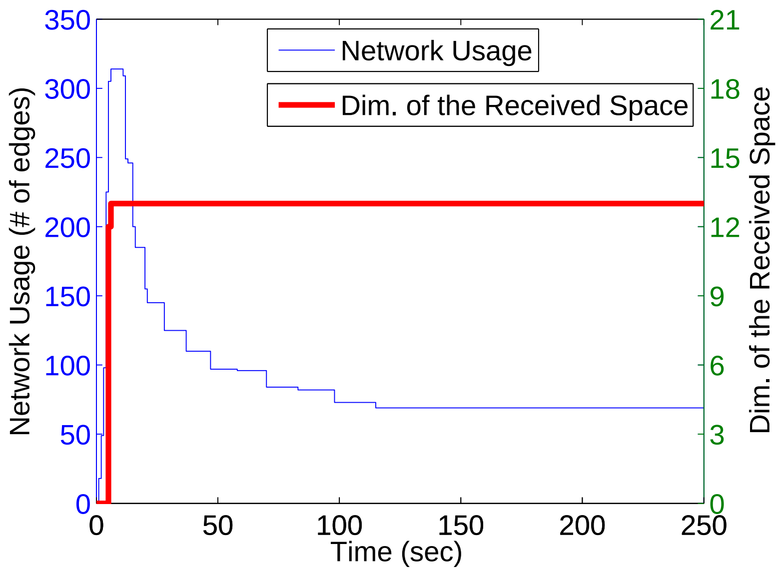

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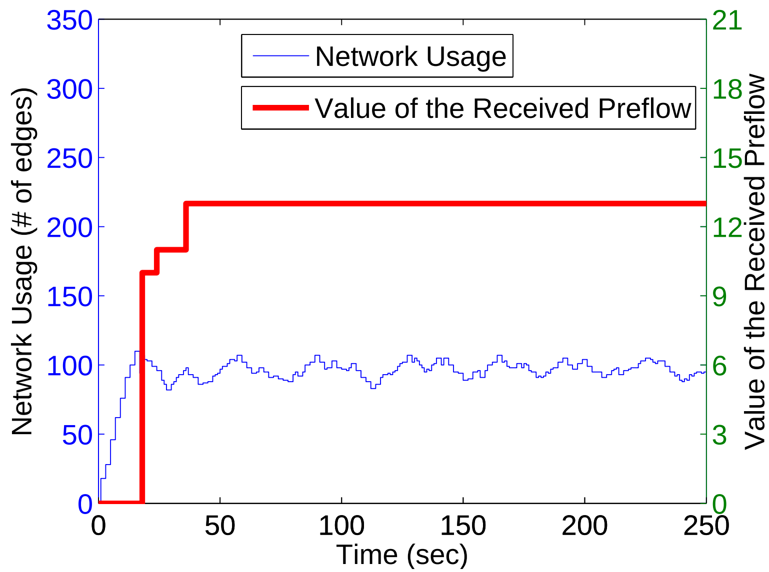

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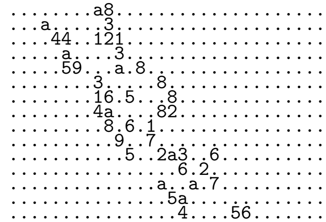
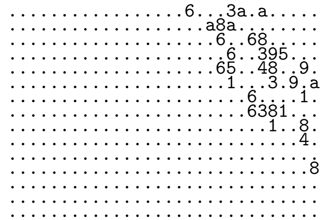
A 30-node network with  

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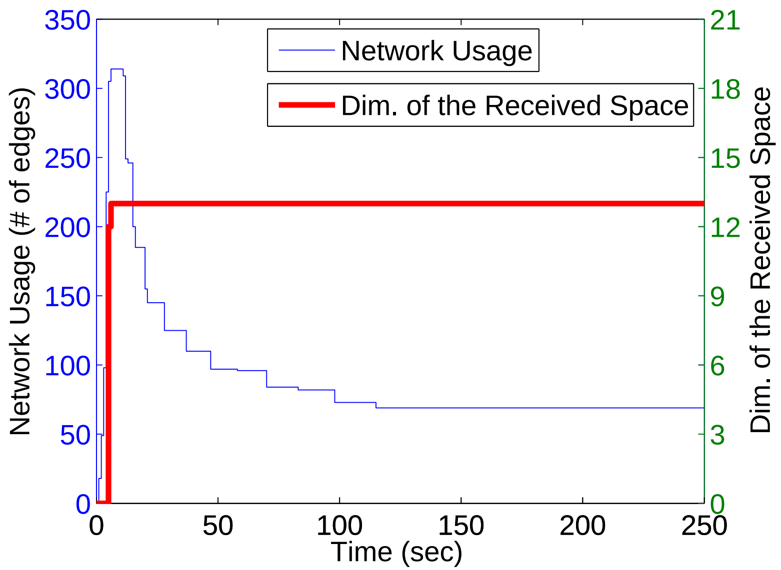

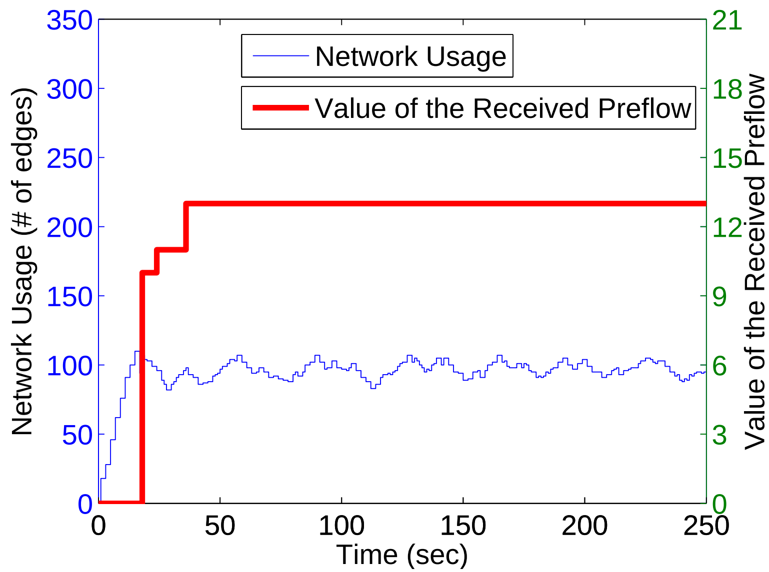

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- No extra hardware requirement. Only linear operations.

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