Capacity of 1-to-K Broadcast Packet Erasure Channels with Channel Output Feedback — A Packet Evolution Approach

Chih-Chun Wang, Purdue University

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Joint work with Y. Charlie Hu (Purdue), Ness B. Shroff (The OSU), Dimitrios Koutsonikolas, Abdallah Khreishah.

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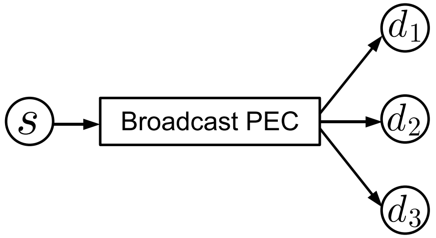
Two Ingredients

- Packet Erasure Channels (PECs):
  - Input: $X \in \mathsf{GF}(2^b)$ for large b.
  - A packet X either arrives perfectly (with the help of CRC), or is considered as erasure and discarded. (No hybrid ARQ).
  - Memoryless, time-invariant.



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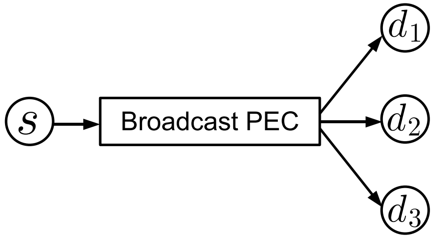

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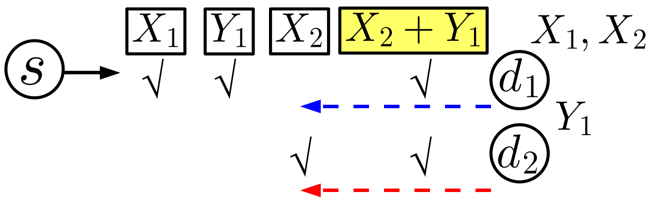

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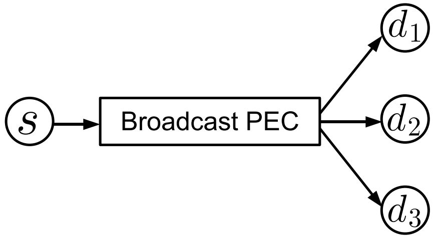

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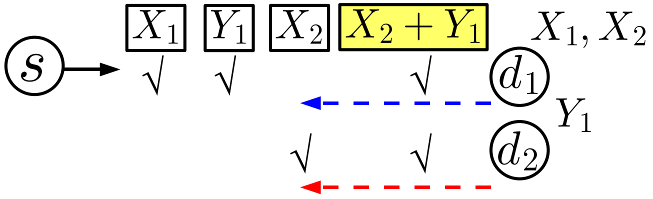
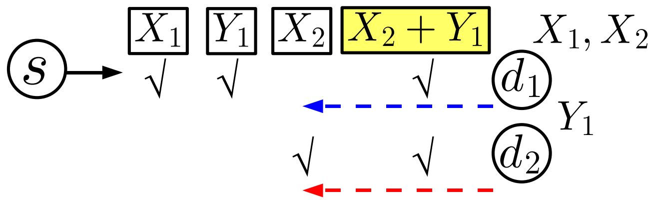


  - Create its own SI through spatial diversity.
  - Empirically, 10–20% throughput improvement.

- Our goal: Finding the Shannon capacity of PECs with channel output feedback (COF) for arbitrary number $M \geq 3$ of sessions.

Main Results & Contents

- The benefits of ER follows from the channel output feedback (COF).



# of sessions	ER-like Protocols (Broadcast PECs w. **COF**)	Gaussian broadcast channels w. **COF**
M=2	Full capacity region [Georgiadis et al. 09]	Outer and inner bounds [Ozarow 84]
M=3	Full capacity region	?
General M	(1) Capacity for fair systems; (2) Outer and inner bounds that meet numerically.	?

Capacity-achieving schemes by code alignment.

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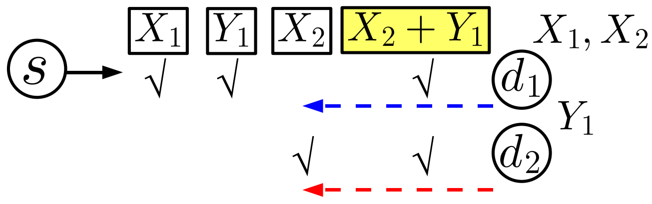

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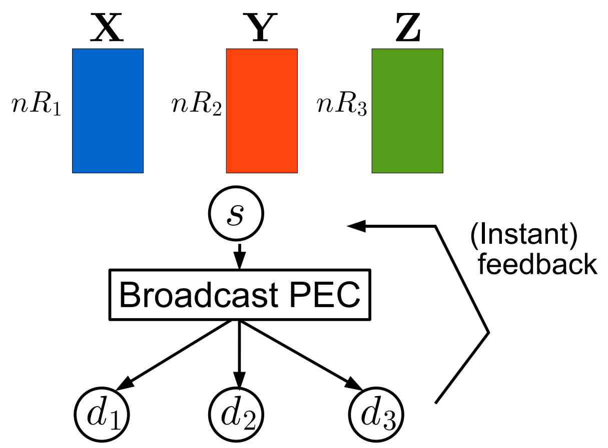
Capacity-achieving schemes by code alignment.

- The problem setting.
- Existing results for $M = 2$ [Georgiadis *et al.* 09].
- New concepts of code alignment and packet evolution.
- Main theorems and numerical evaluation.

1-Hop Cellular (AP) Networks

- 1-hop access point networks. M dest.
- M can be large, say ≈ 20.
- Each session has nR_i packets.
- The source s uses the channel n times.



- Our goal is to maximize the achievable rate vector $(R_1, \cdots, R_M)$.

Formal Definition of Feasibility

A network code is defined by the following functions:

info. channel output feedback

$$Y(t) = f_t(\{X_{k,l} : k \in [M], l \in [nR_k]\}, \{Z_k(\tau) : k \in [M], \tau \in [t-1]\}),$$

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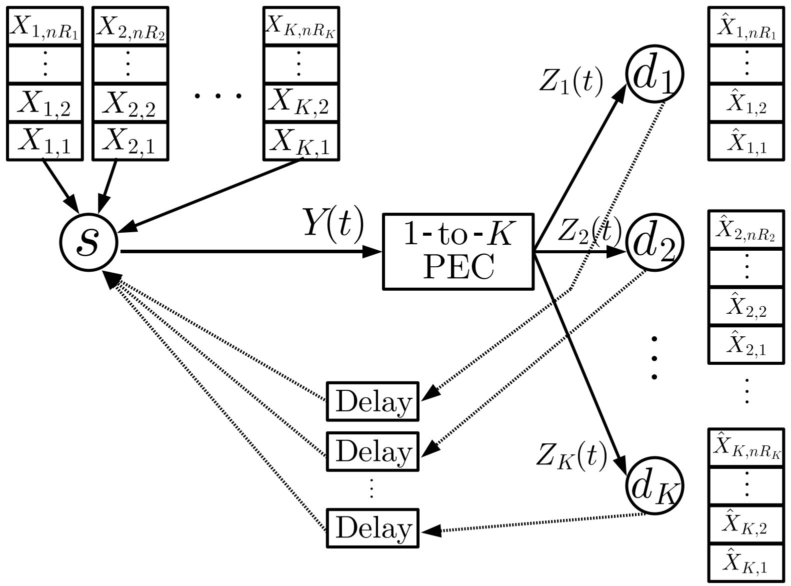

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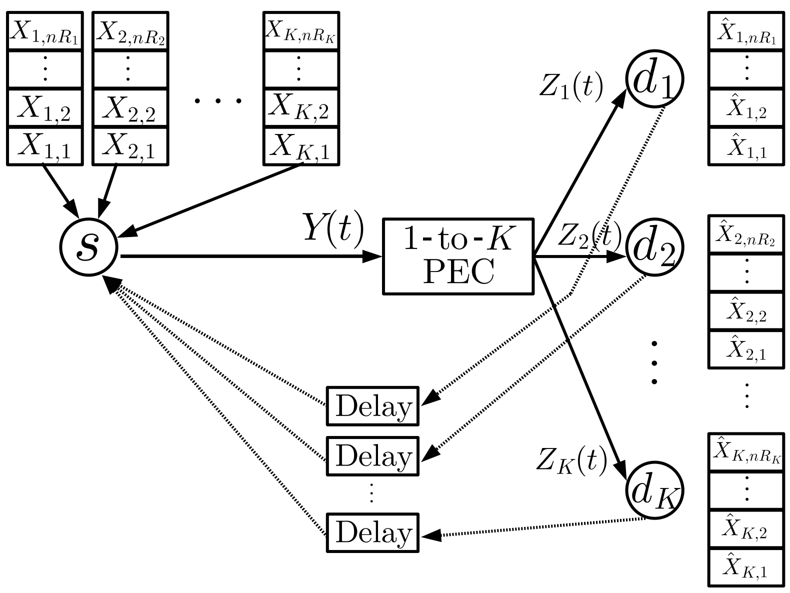

Definition 1 $(R_1, \cdots, R_M)$ *is achievable if* $\forall \epsilon > 0$, *there exist a sufficiently large* n, *a sufficiently large finite field* $\mathrm{GF}(2^b)$, *and a corresponding network code, such that for independently and uniformly distributed* $\mathbf{X}_k$, $k \in [M]$:

$$\max_{k \in [M]} \mathsf{P}\left(\hat{\mathbf{X}}_k \neq \mathbf{X}_k\right) < \epsilon.$$

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  (For 2-hop relay networks $M \leq 6$).
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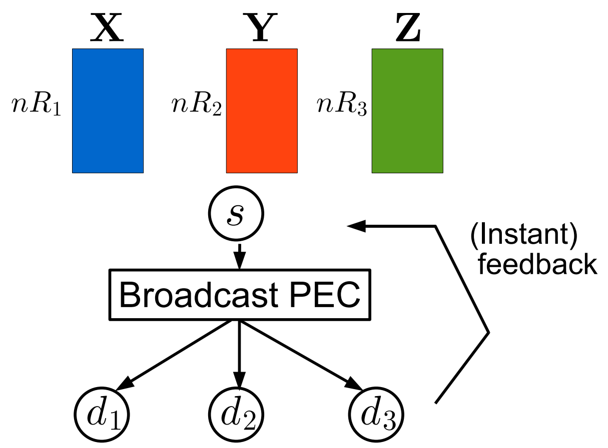

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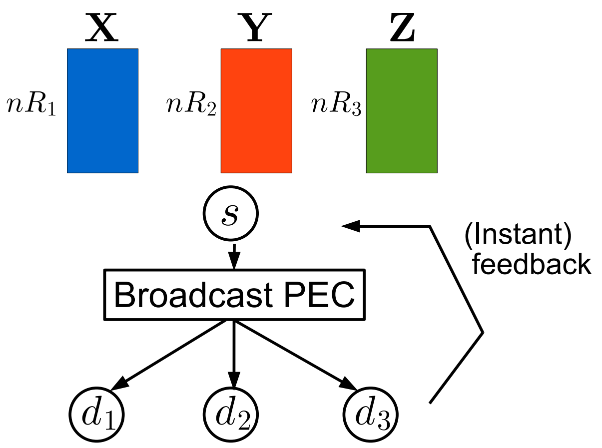

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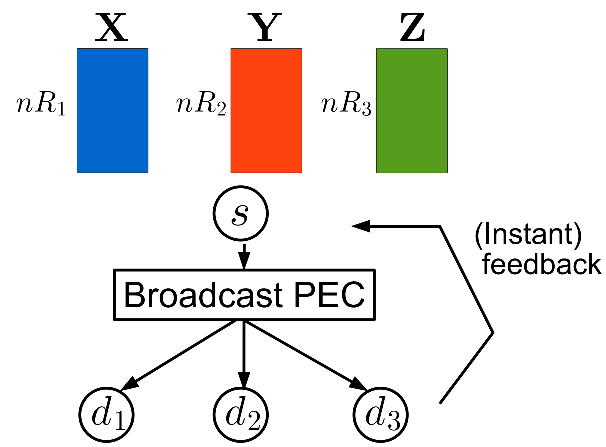
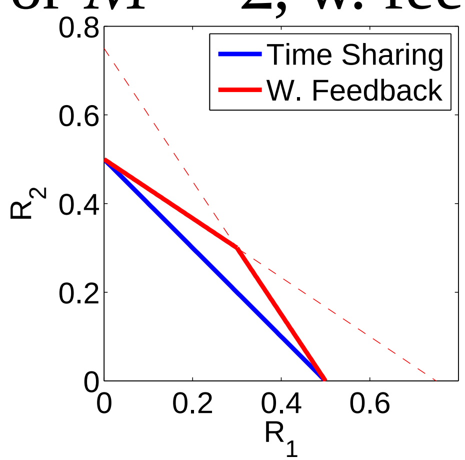
- For $M = 2$, w. feedback, the capacity is [Georgiadis *et al.* 09].

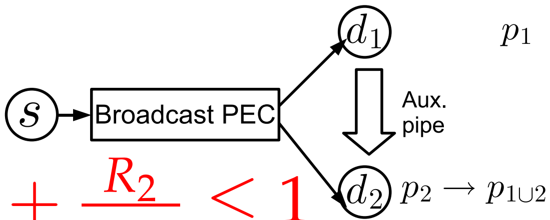




$$\begin{cases} \frac{R_1}{p_{1\cup 2}} + \frac{R_2}{p_2} \leq 1 \\ \frac{R_1}{p_1} + \frac{R_2}{p_{1\cup 2}} \leq 1 \end{cases}$$

Georgiadis' Proof

- Outer bound [Ozarow *et al.* 84]: Introduce auxiliary pipes to convert it into *physically degraded channels*, for which feedback does not increase the capacity [El Gamal 78].



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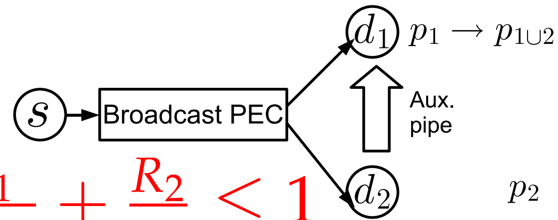

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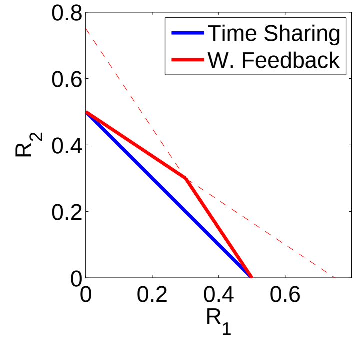

The cap. of the original CH with feedback
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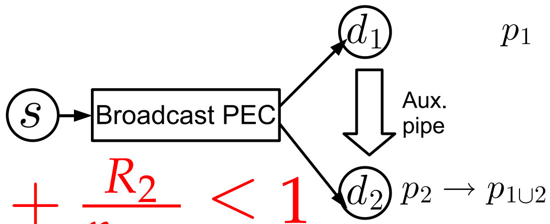

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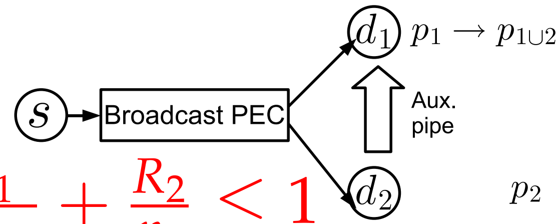

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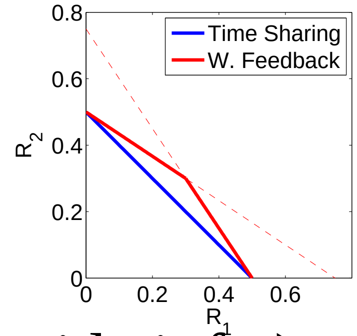

- Inner bound: A 2-phase approach. (Creating its own side info.)

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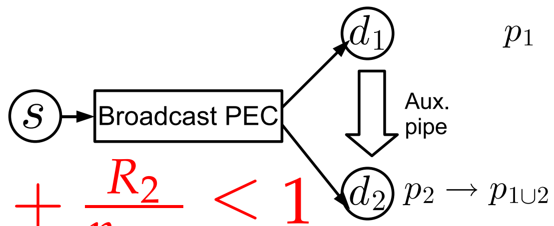

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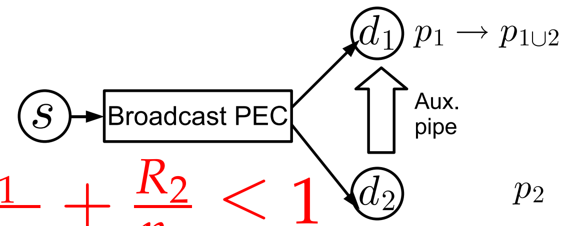

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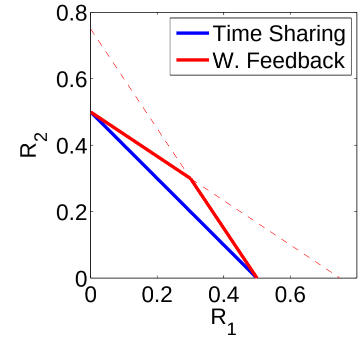

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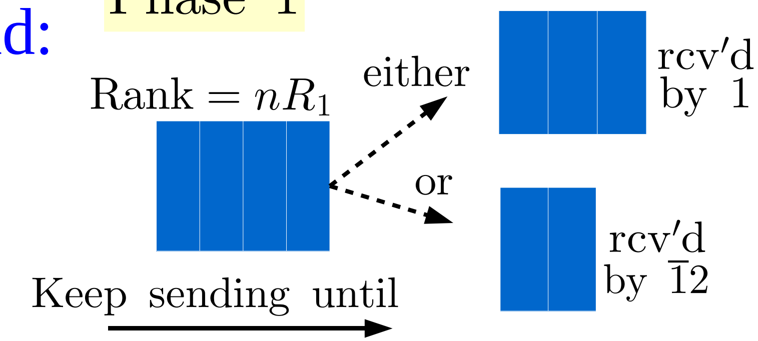

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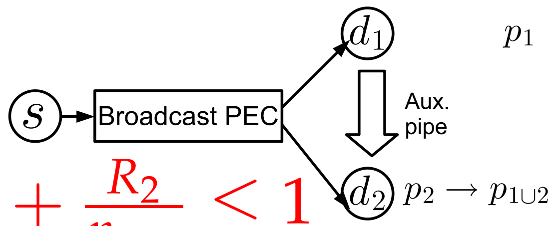

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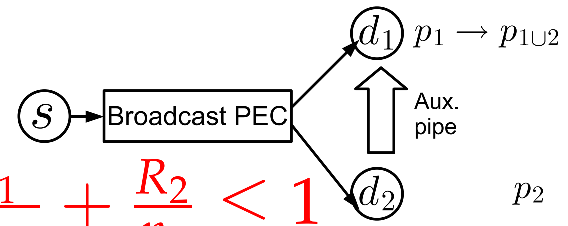

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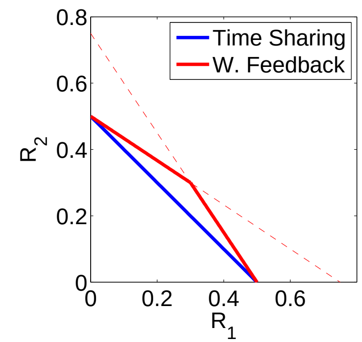

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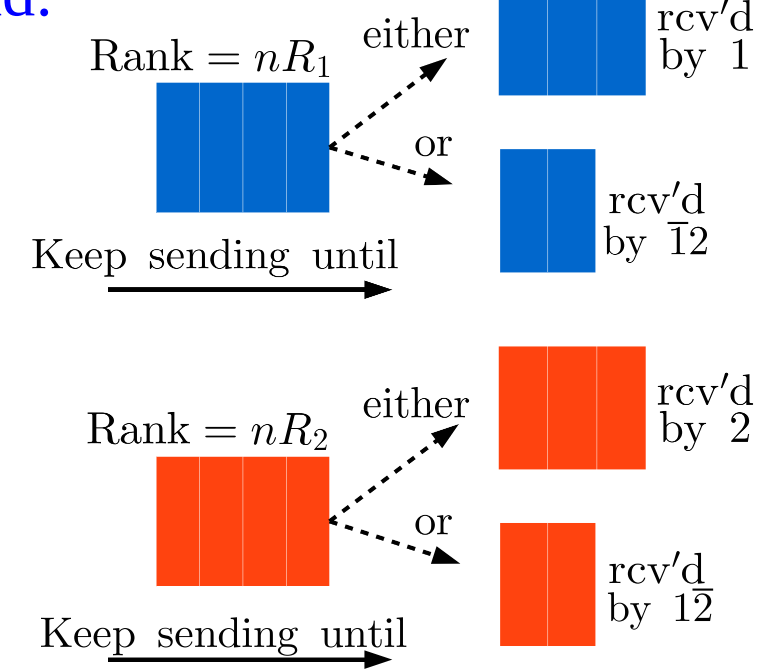

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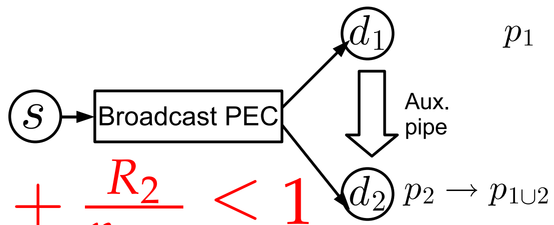

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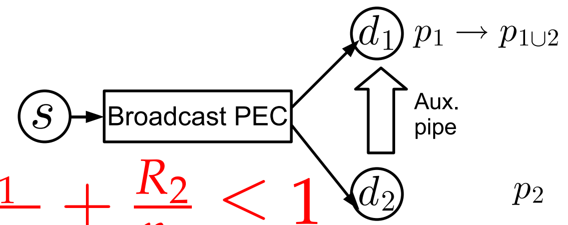

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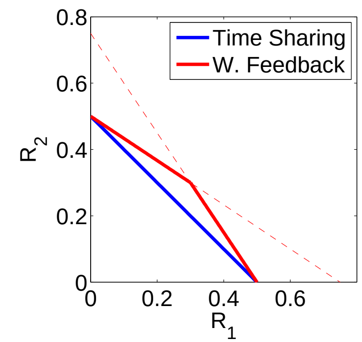

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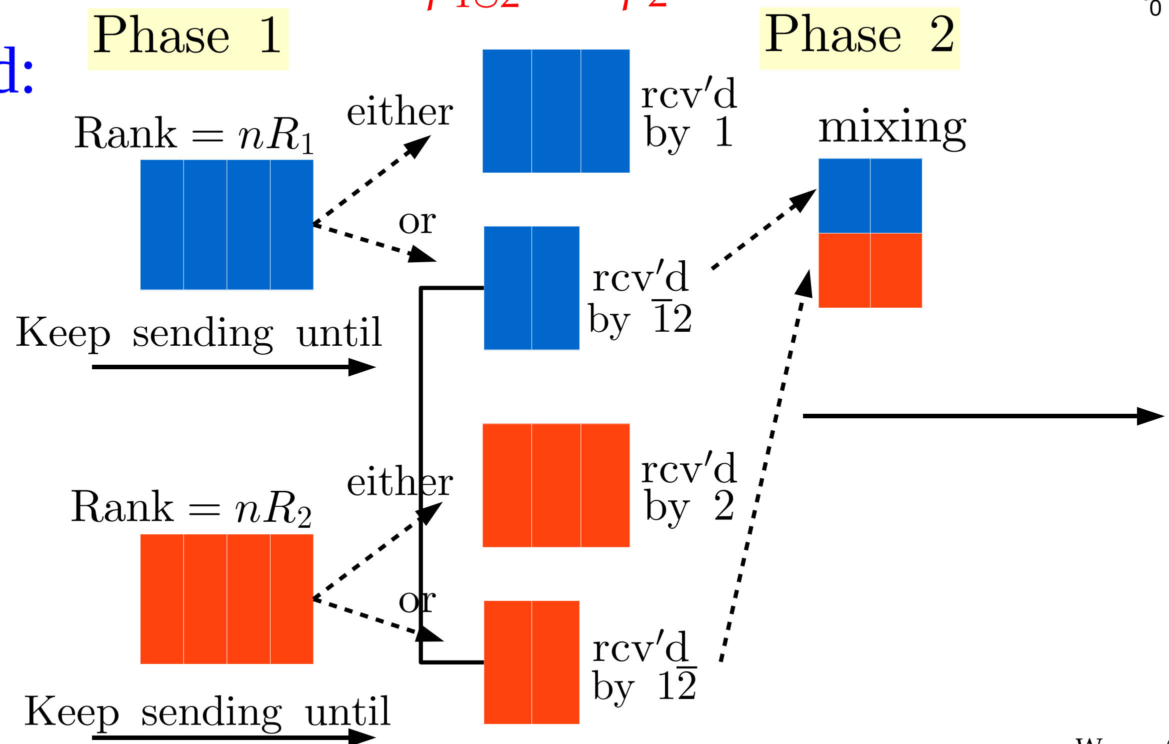

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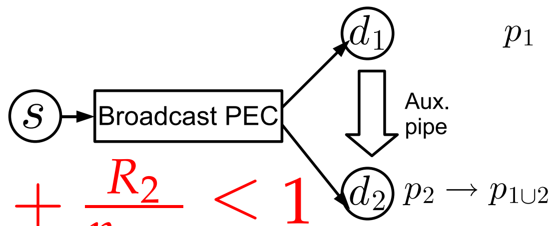

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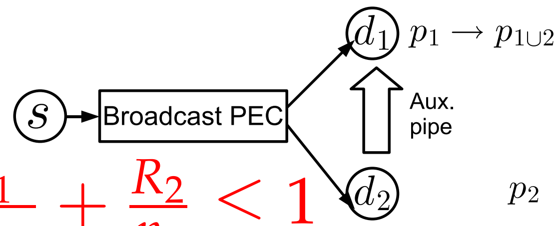

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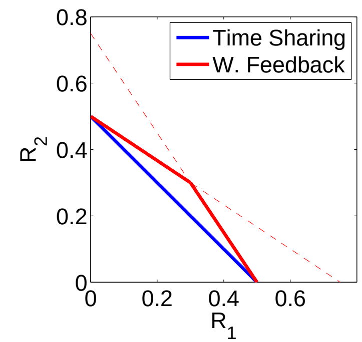

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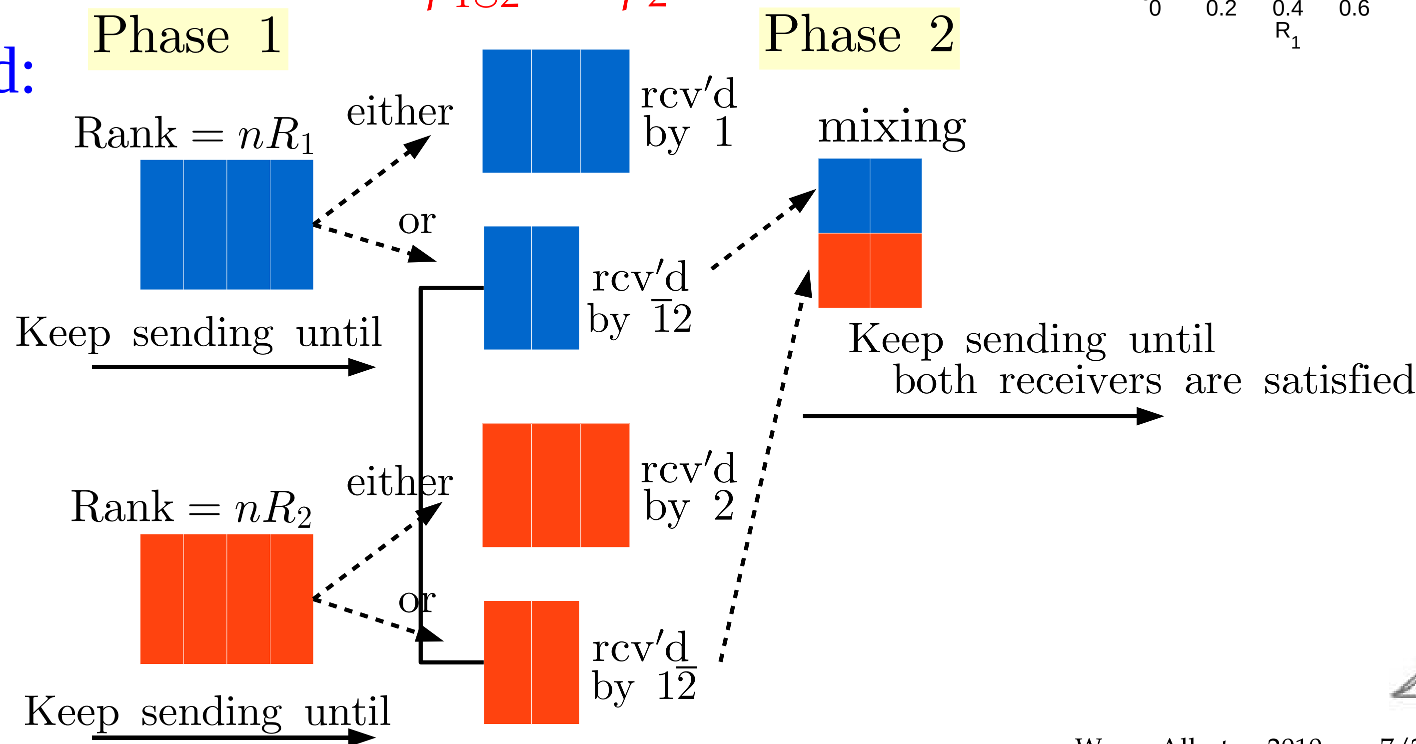

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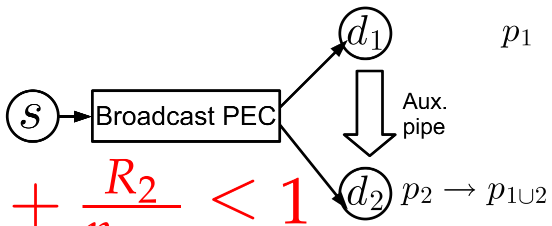

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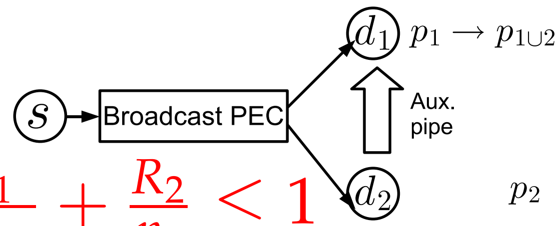

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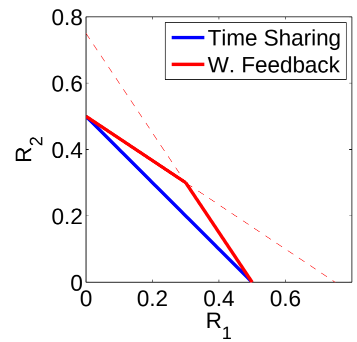

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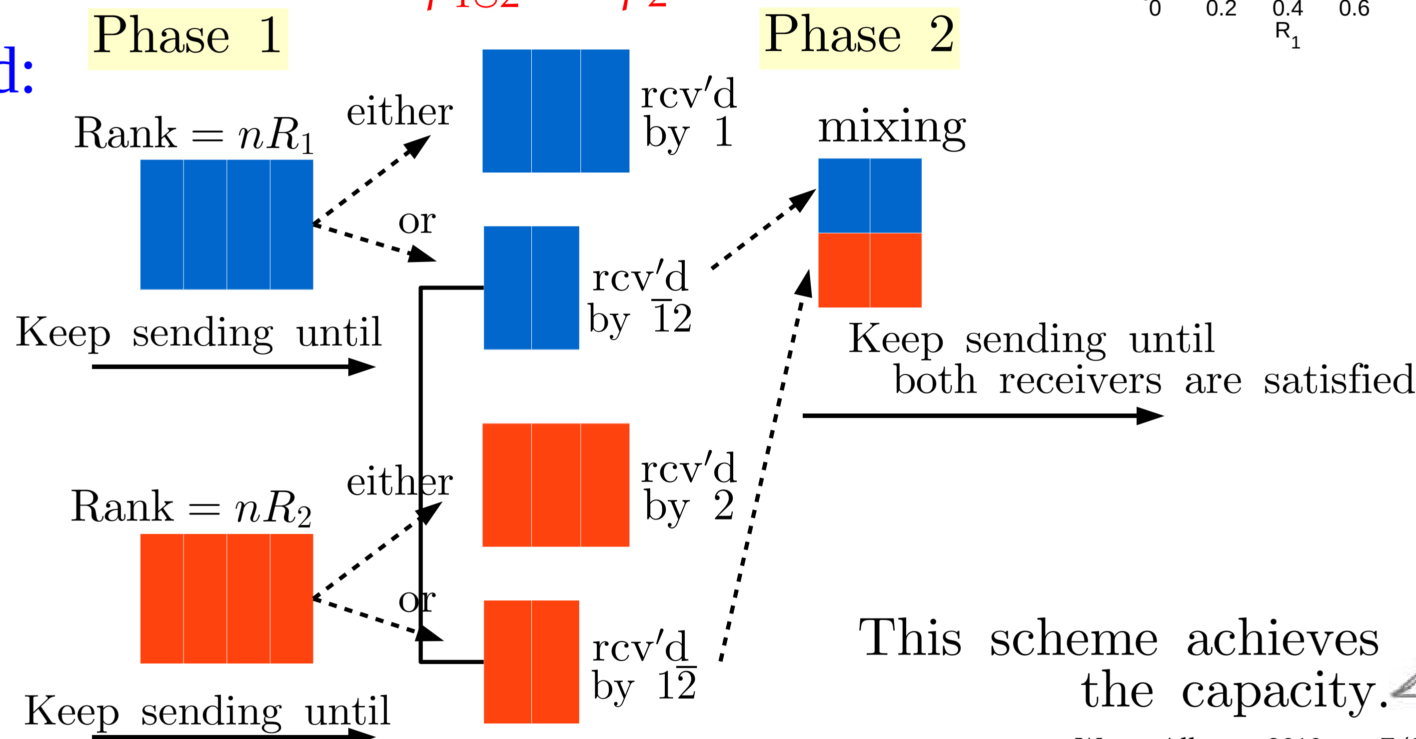
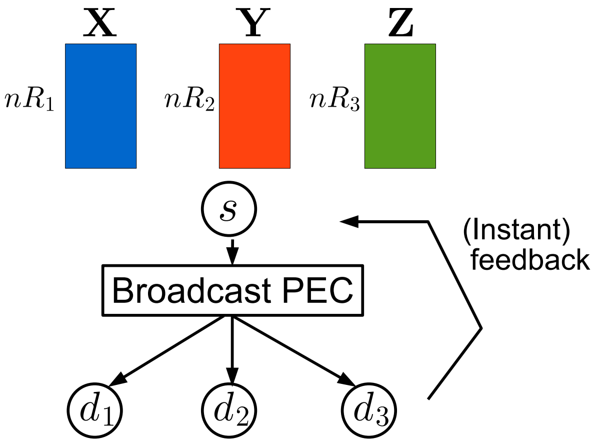


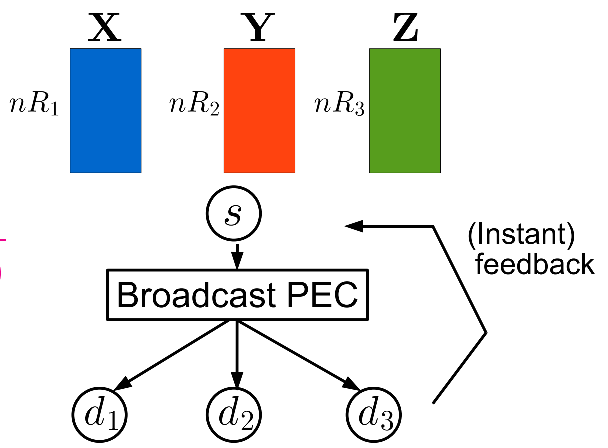
This scheme achieves the capacity.

What if $M \geq 3$?



What if $M \geq 3$?

- The CH. parameters become more involved.
  - $M = 2$: p_{12}, p_{12^c}, p_{1^c2}, $p_{1^c2^c}$.
  - $M \geq 3$: the success probability $p_{S\overline{([M]\backslash S)}}$ that a packet is received *by and only by* $d_i \in S$. We have 2^M such parameters.



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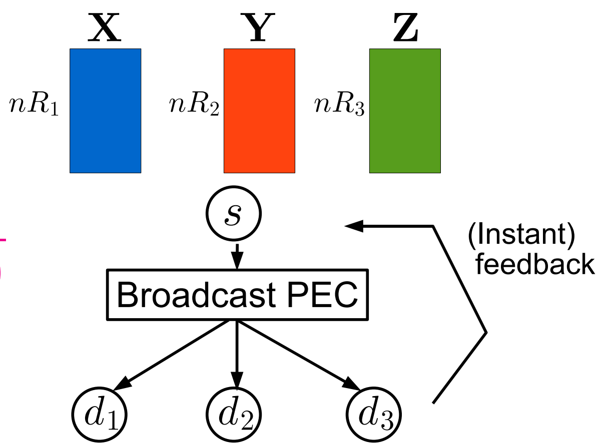

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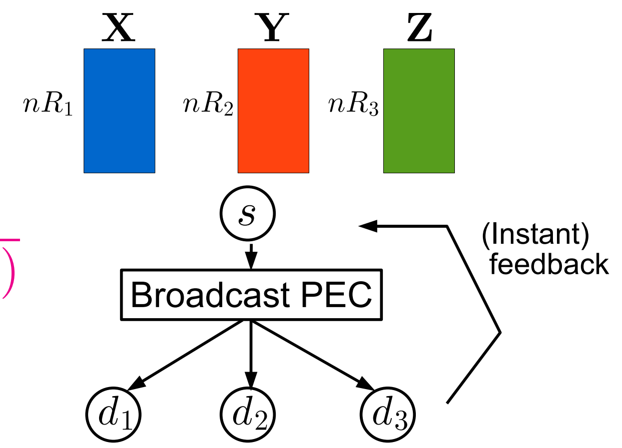
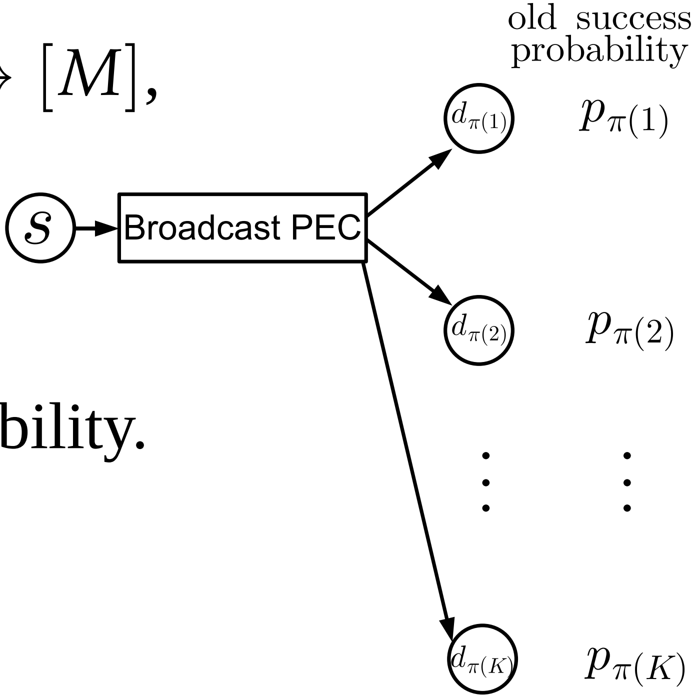


- Can we also quantify the Shannon capacity for $M \geq 3$?
  - Generalization of the outer bound is straightforward.
  - Generalization of the inner bound is more difficult.

Simple Cap. Outer Bound

- For any permutation $\pi : [M] \mapsto [M]$,



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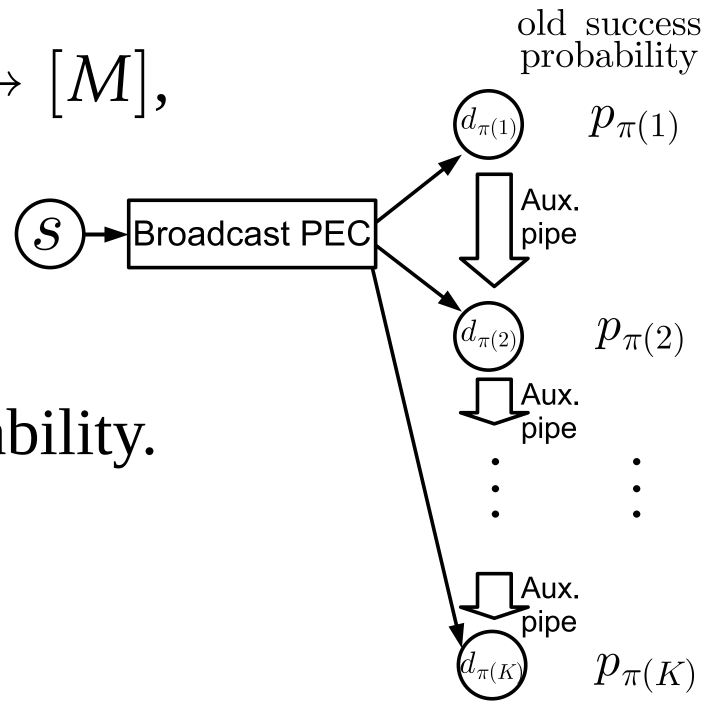

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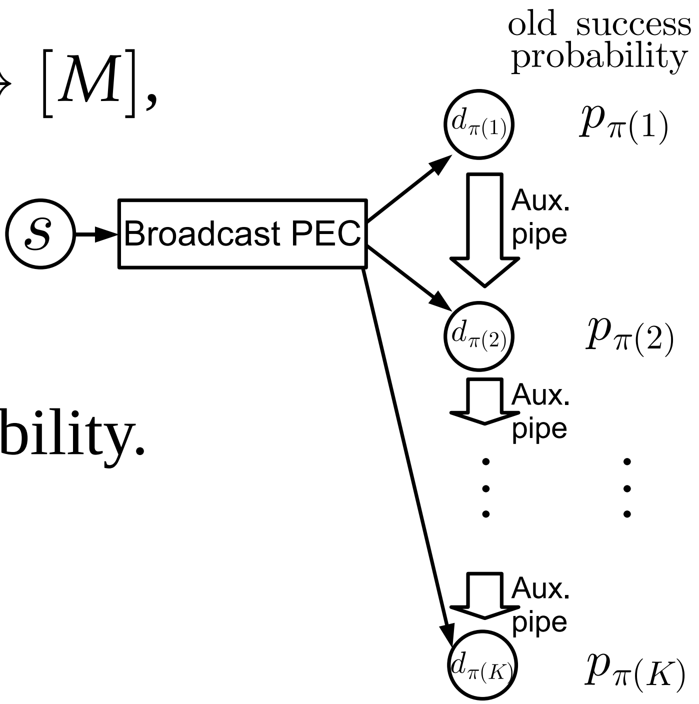

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- $p_{\cup S}$: Prob. at least one $d_i \in S$ is successful.
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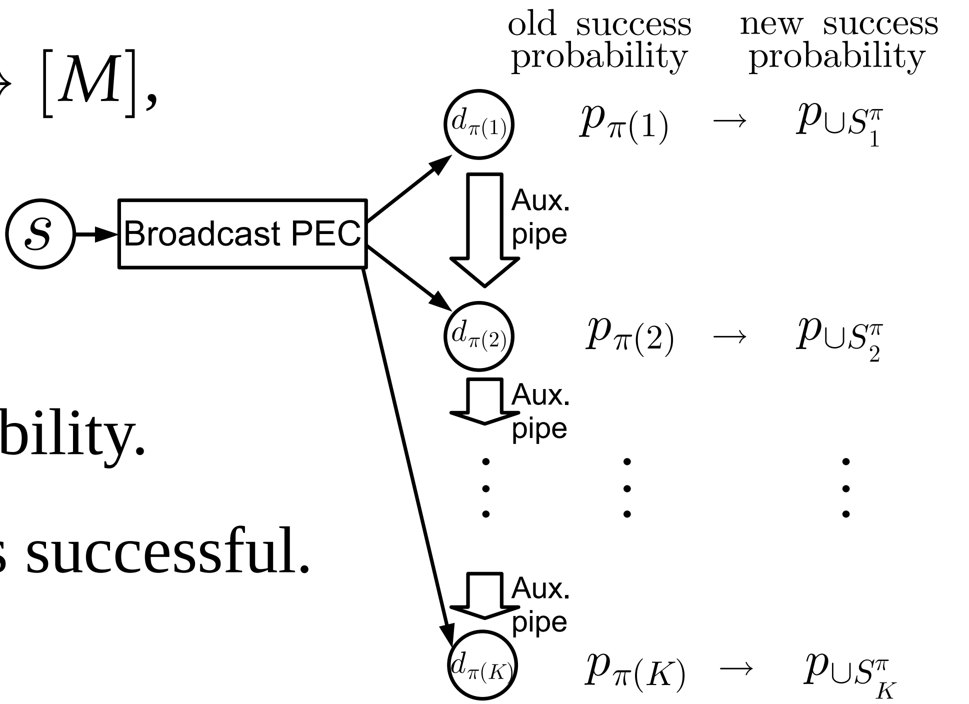

Simple Cap. Outer Bound

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Cap. of the original CH with feedback
$\prec$ Cap. of the new CH with feedback
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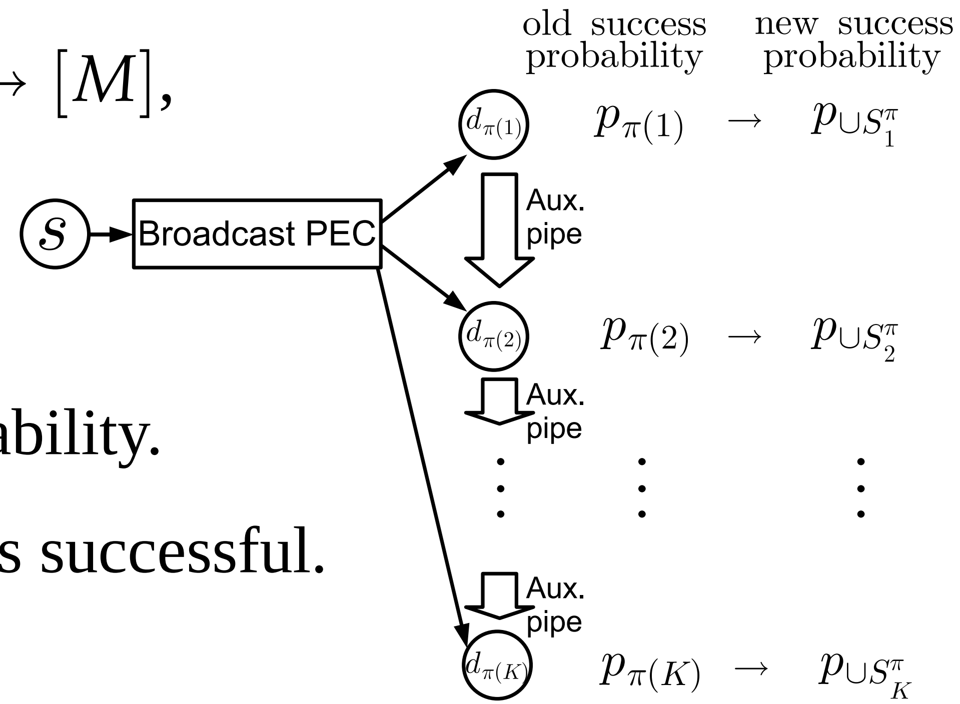

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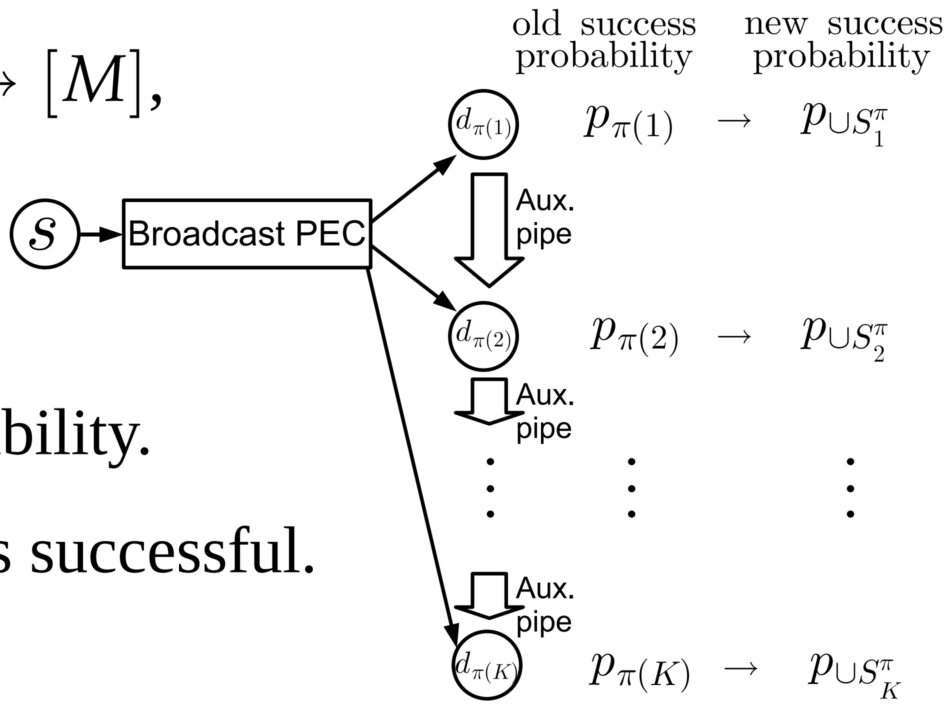

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- A capacity outer bound is thus $\forall \pi, \quad \sum_{k=1}^{M} \frac{R_{\pi(k)}}{p_{\cup S_k^\pi}} \leq 1.$

Cap. Inner Bound?

How to achieve the outer bound: $\forall \pi, \sum_{k=1}^{M} \frac{R_{\pi(k)}}{p_{\cup S_k^\pi}} \leq 1$

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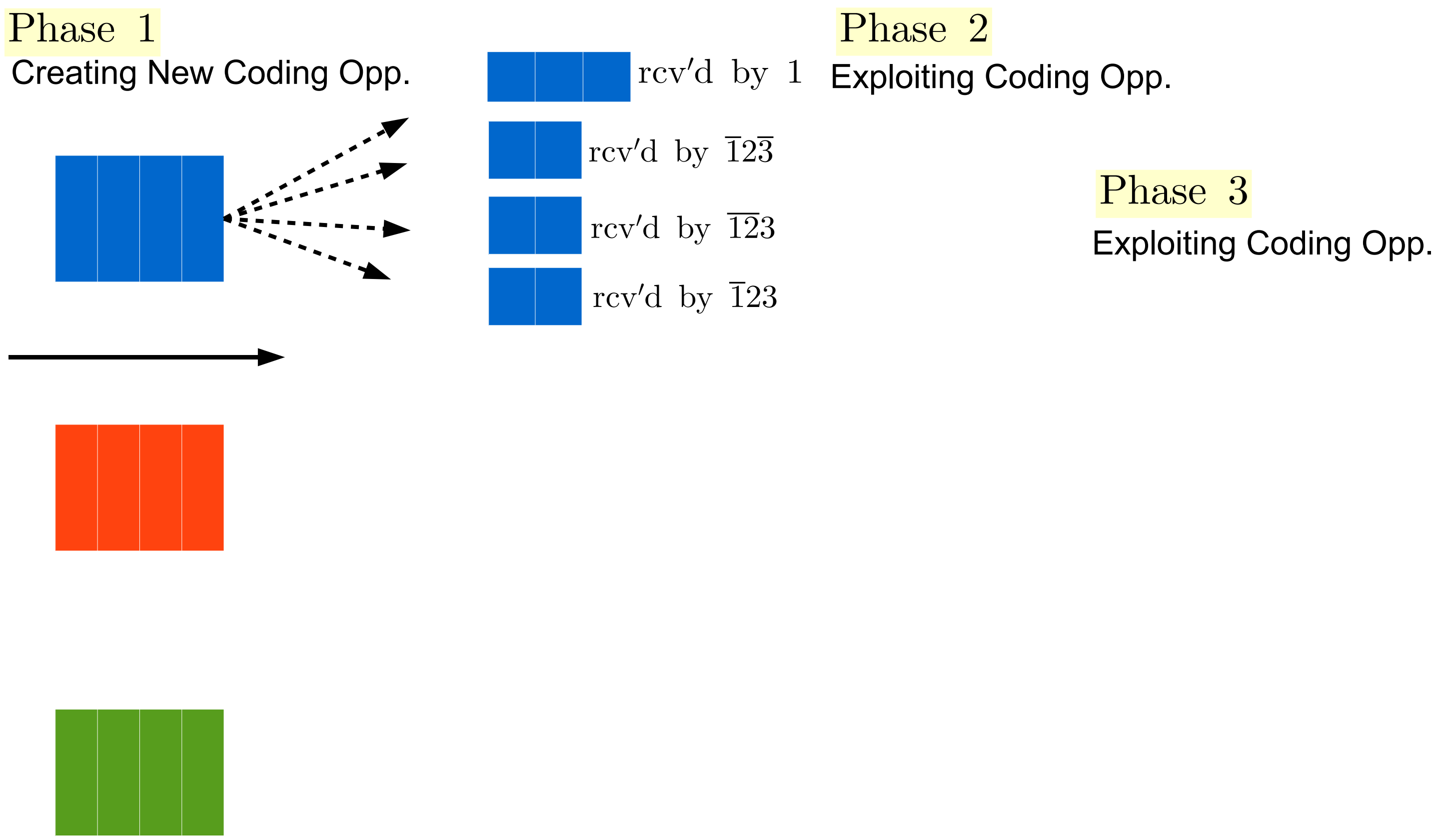

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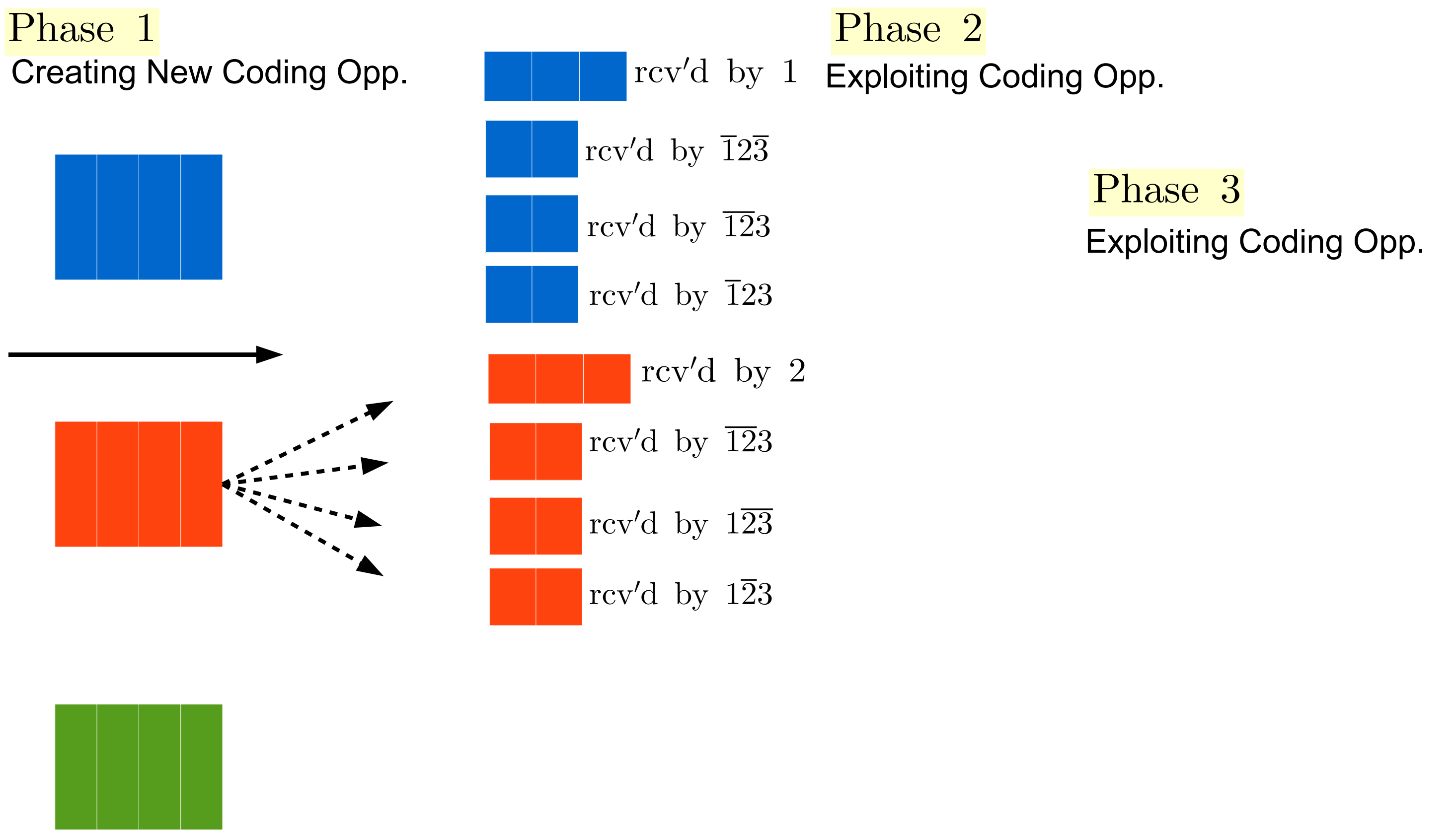

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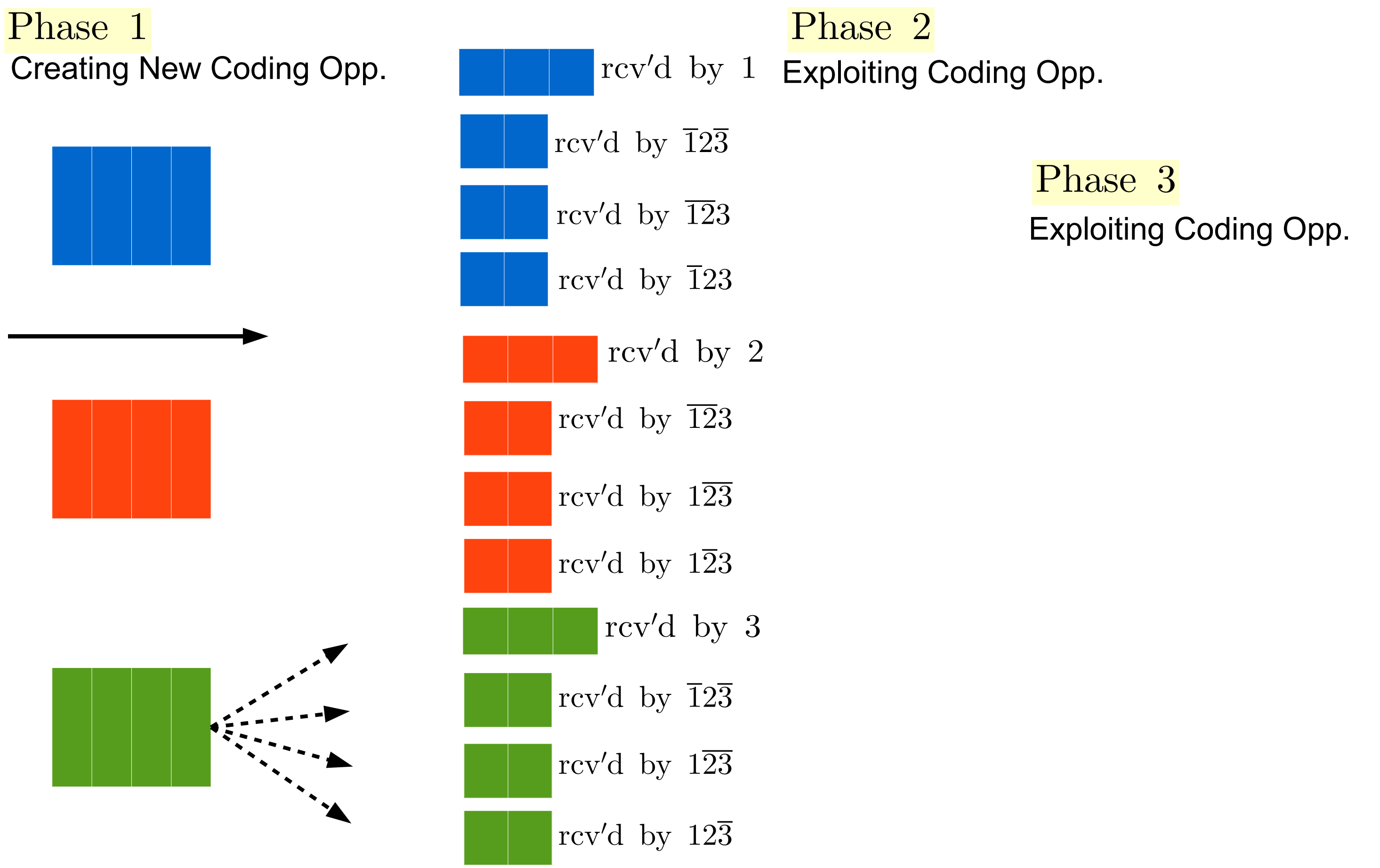

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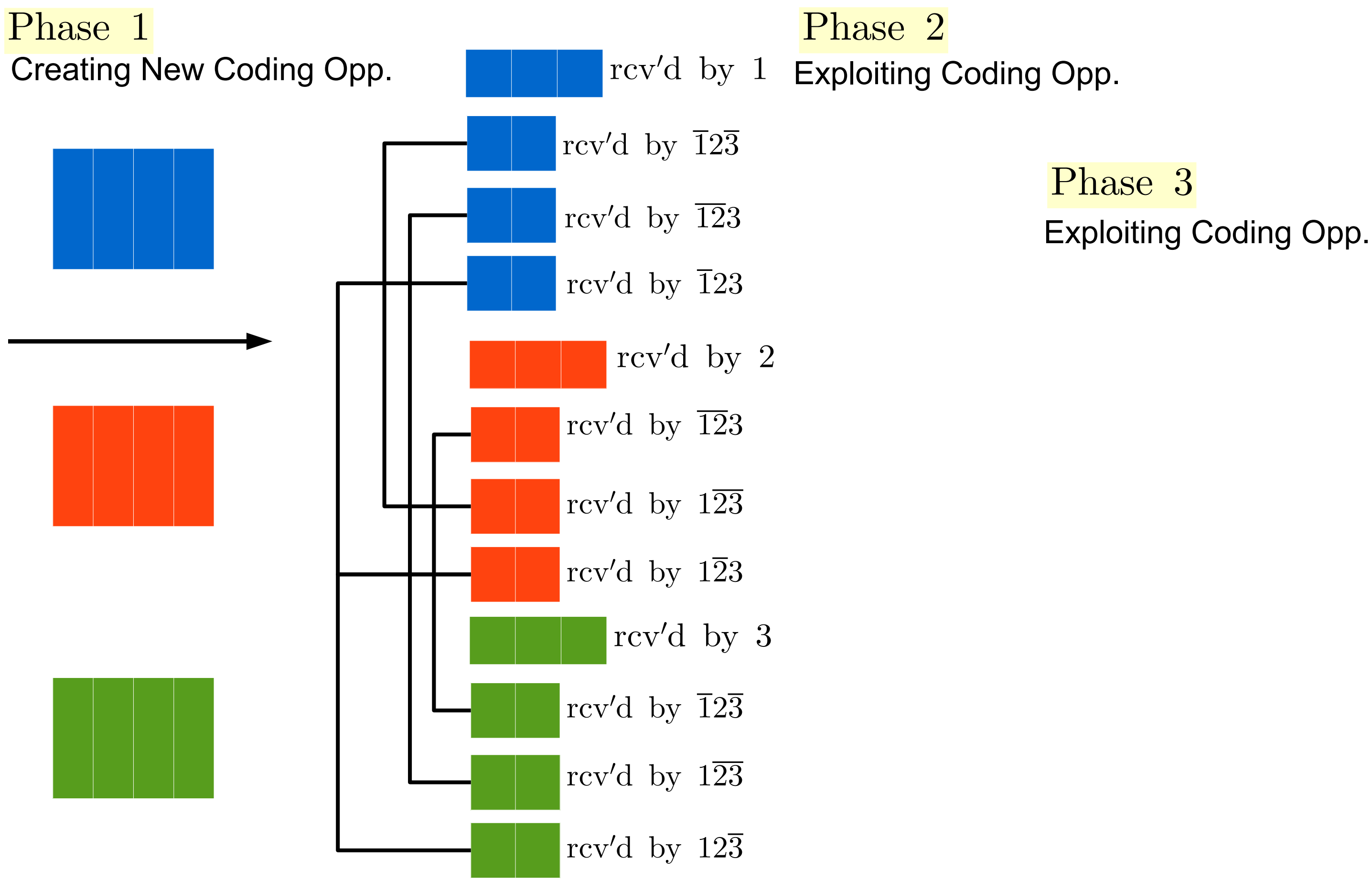

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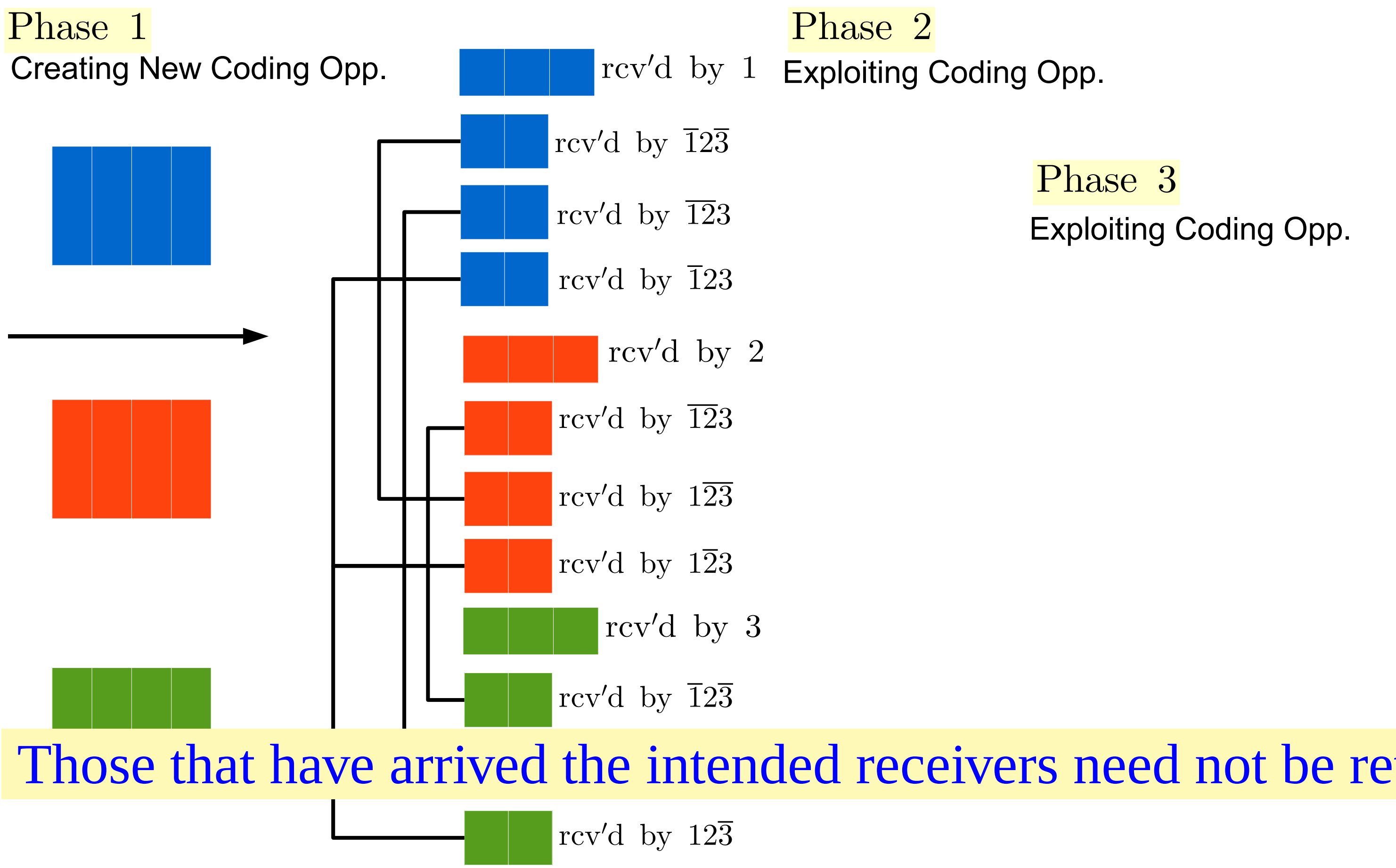

Those that have arrived the intended receivers need not be retransmitted!

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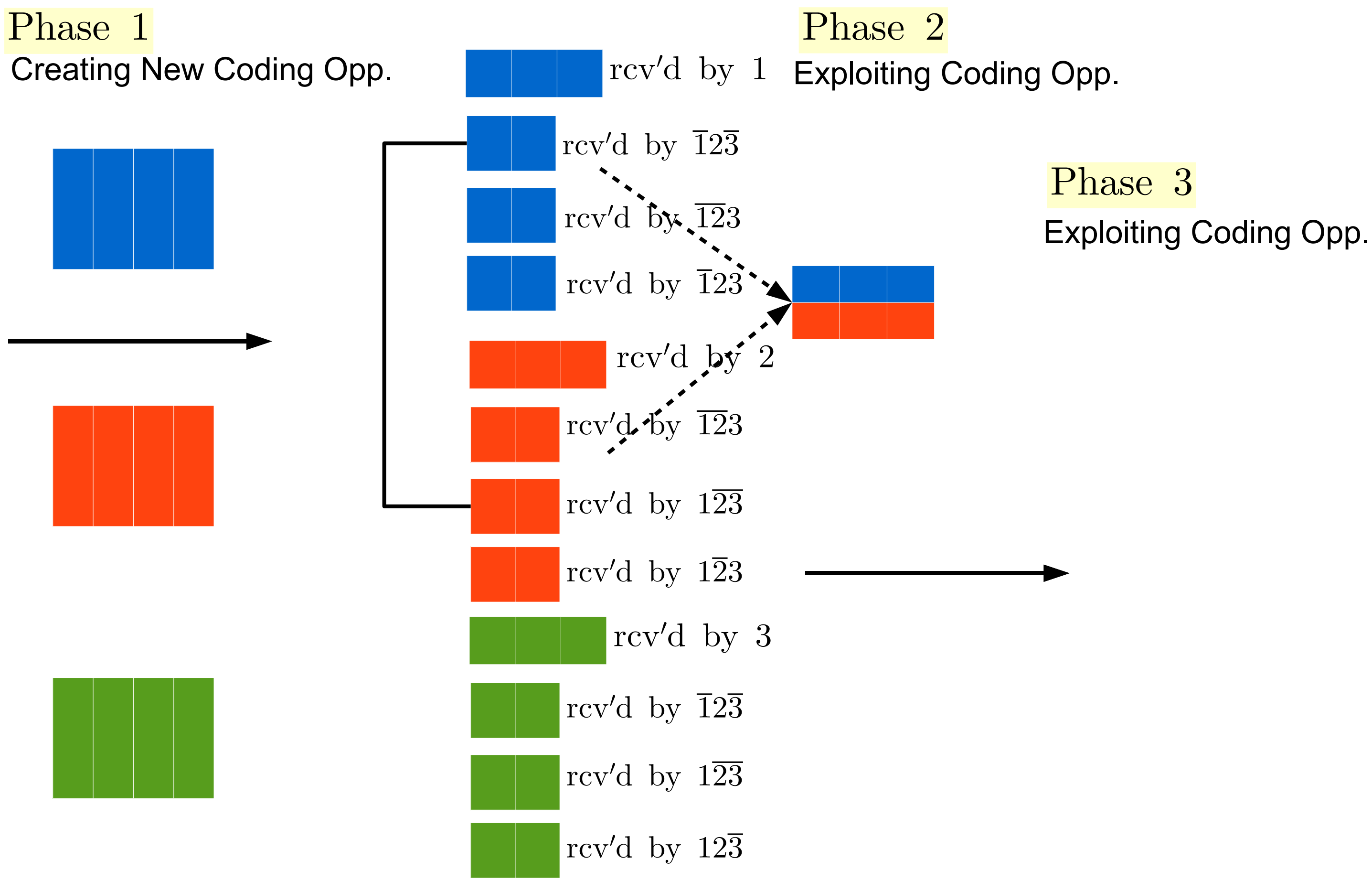

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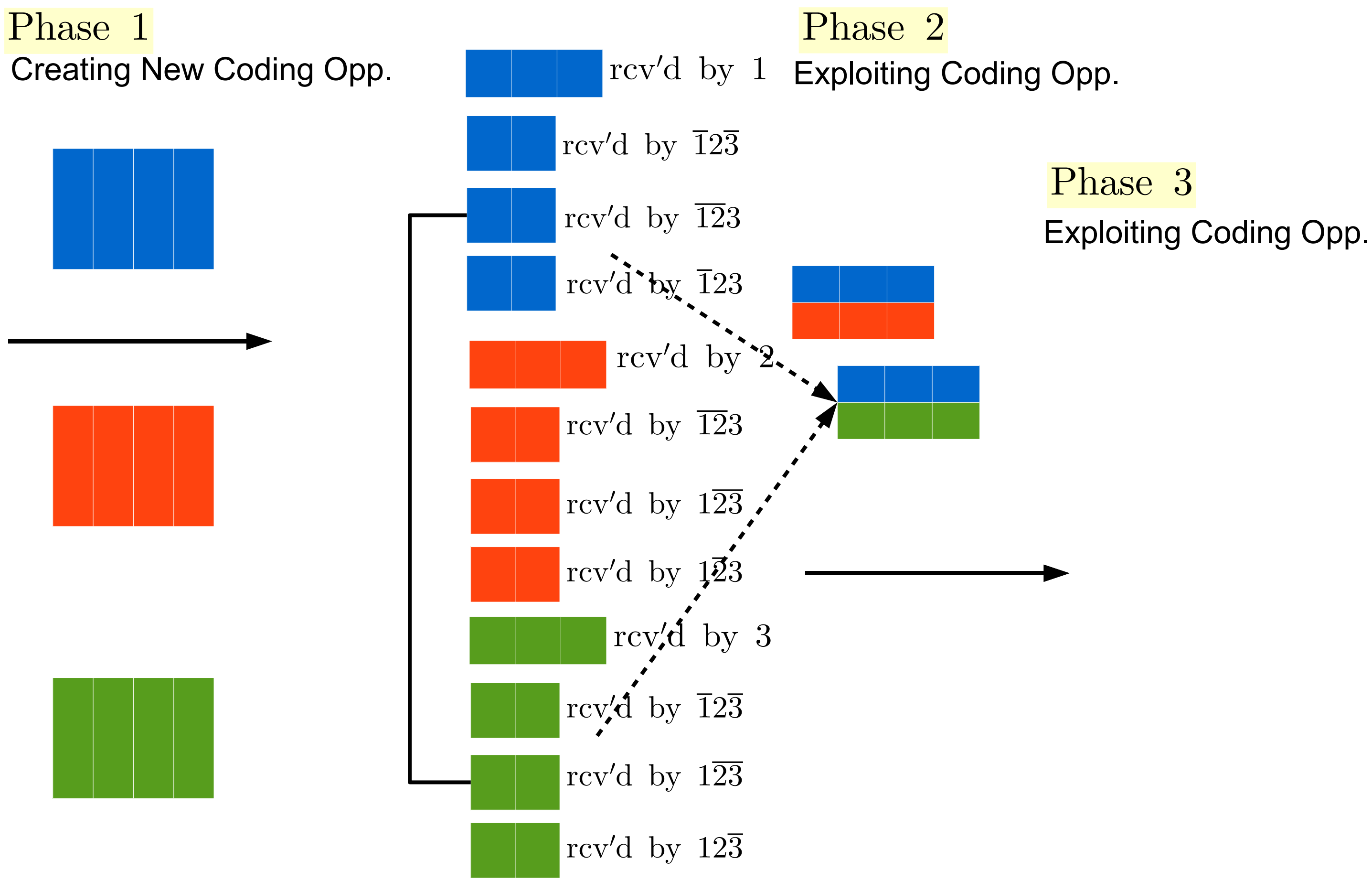

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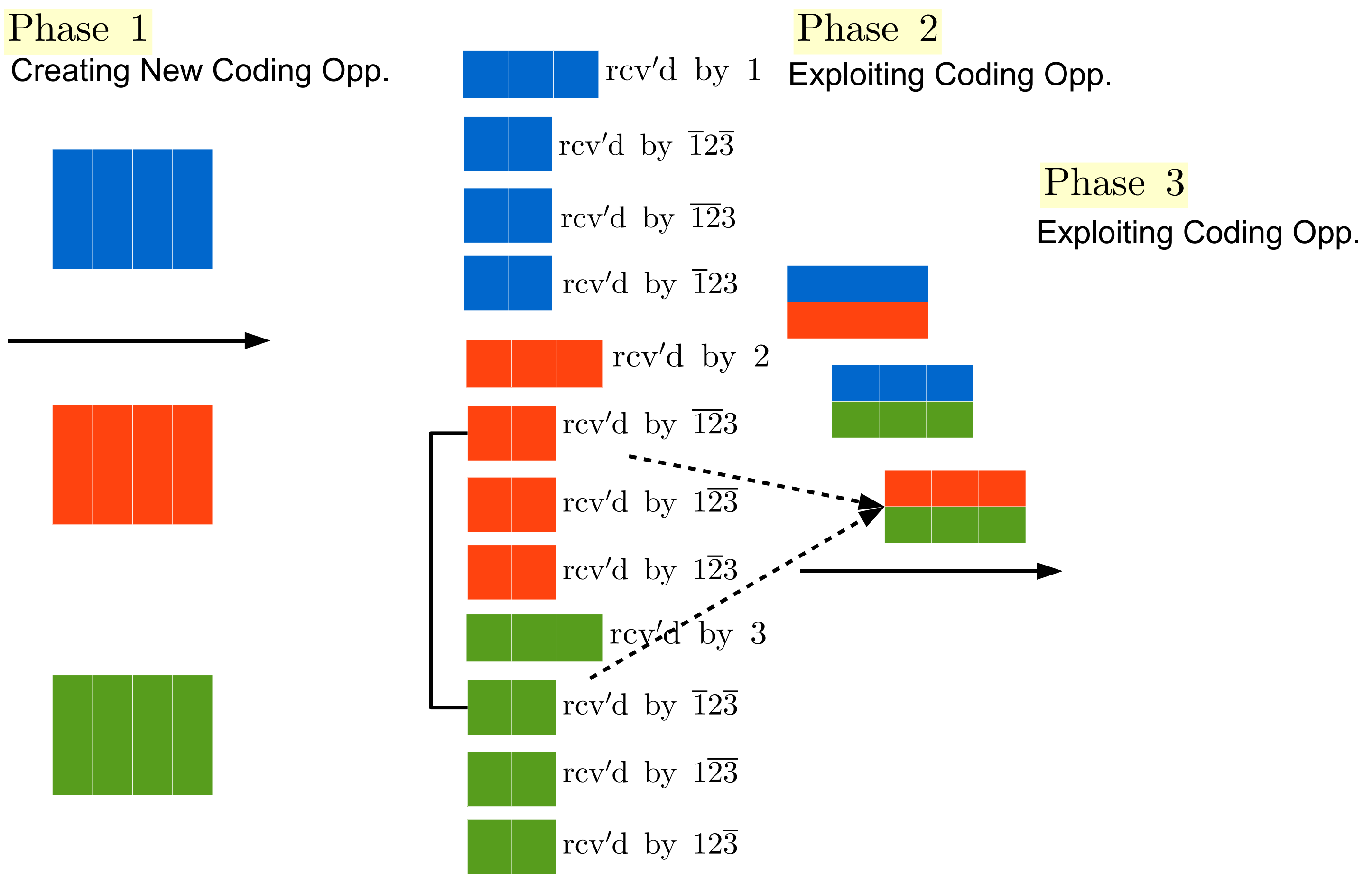

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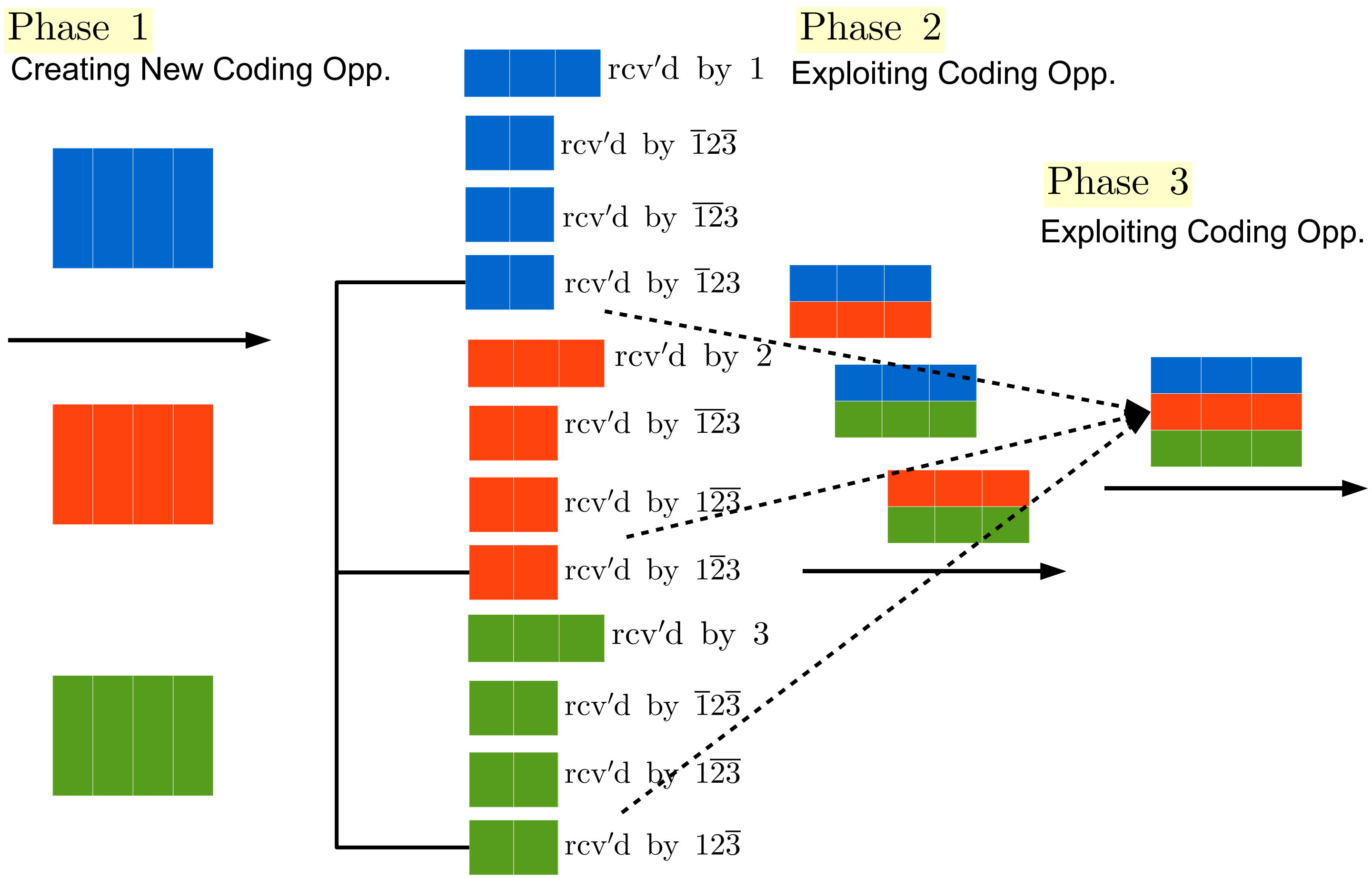

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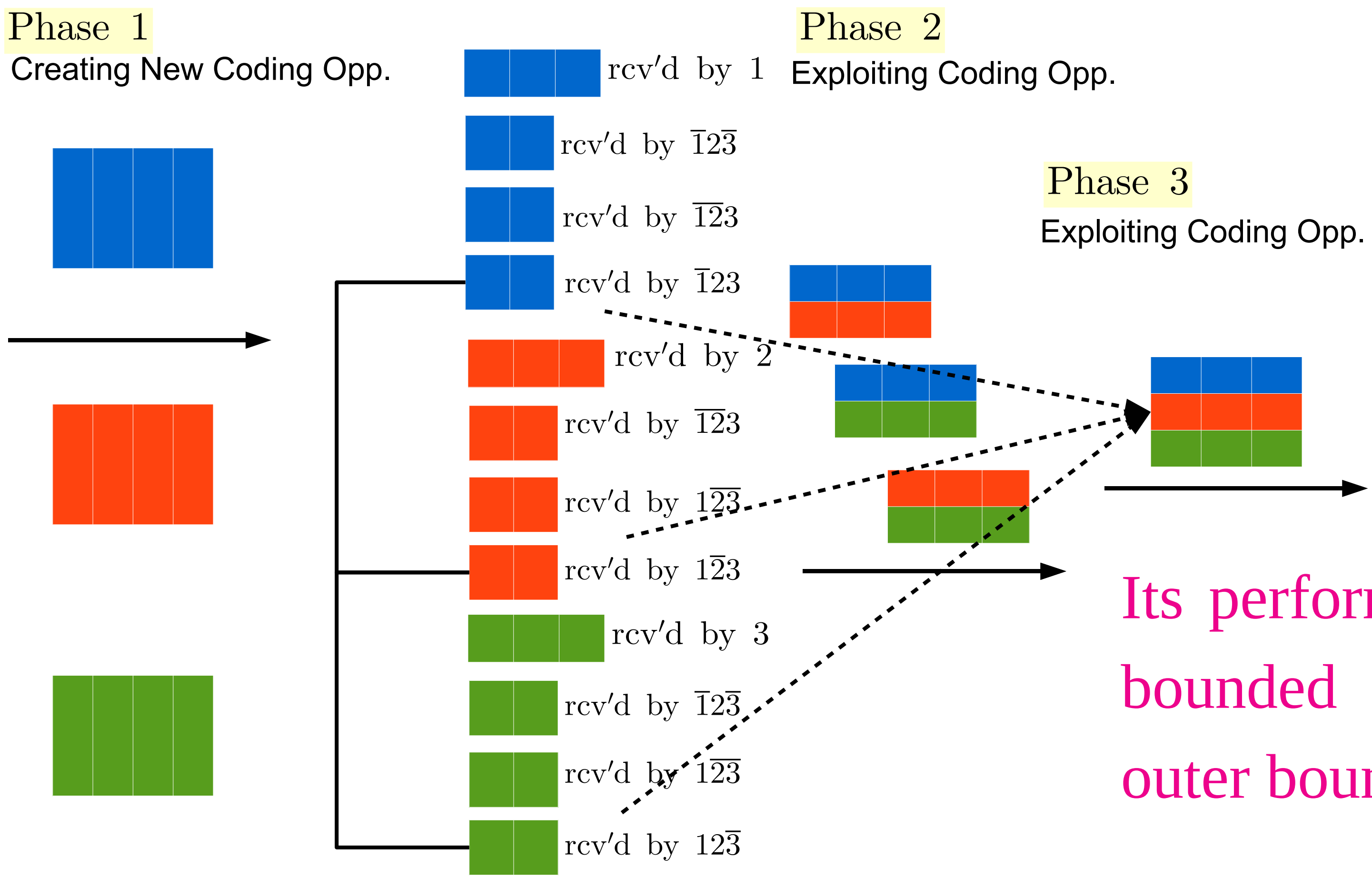
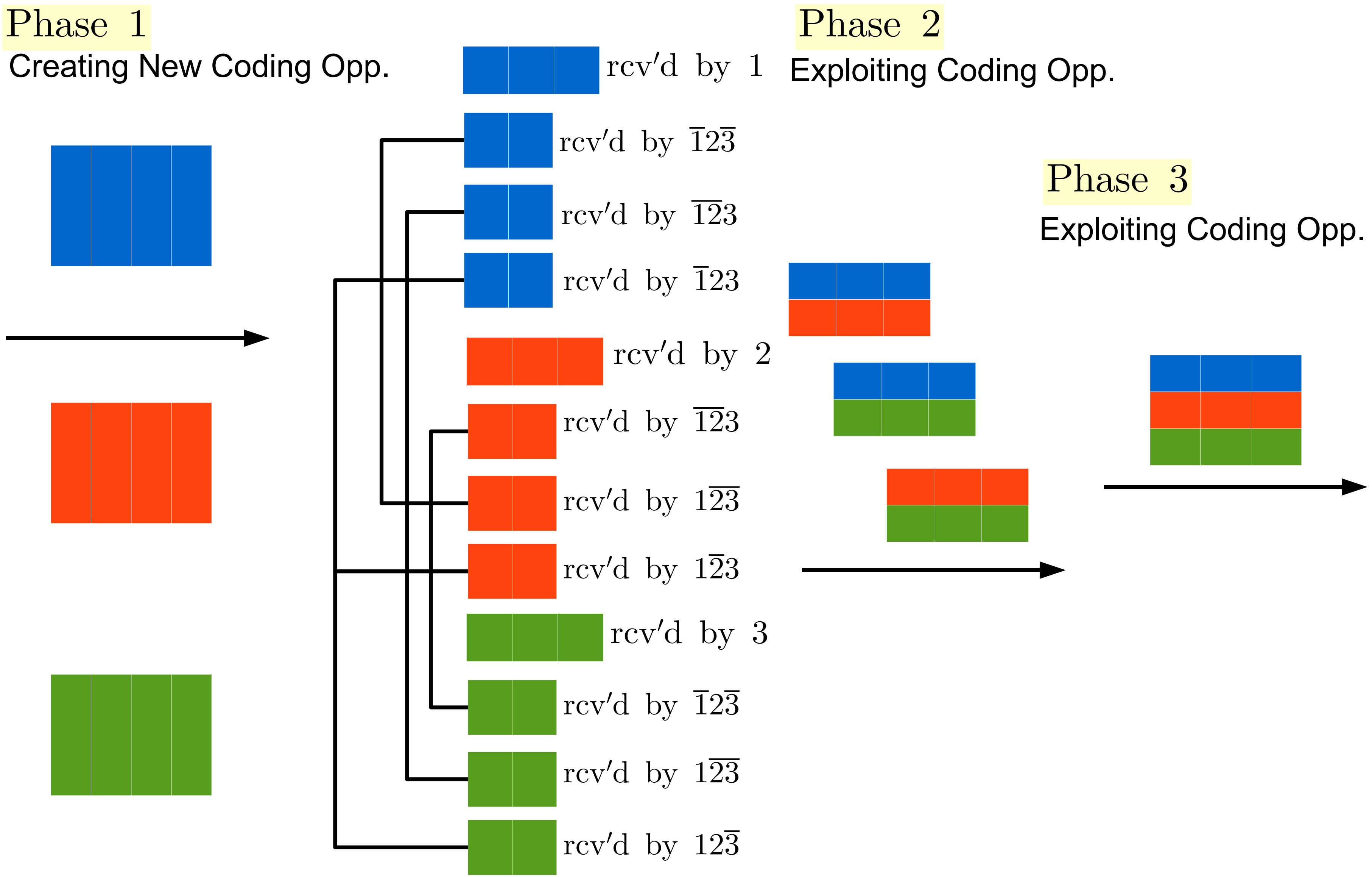


Its performance is strictly bounded away from the outer bound.

What Went Wrong?



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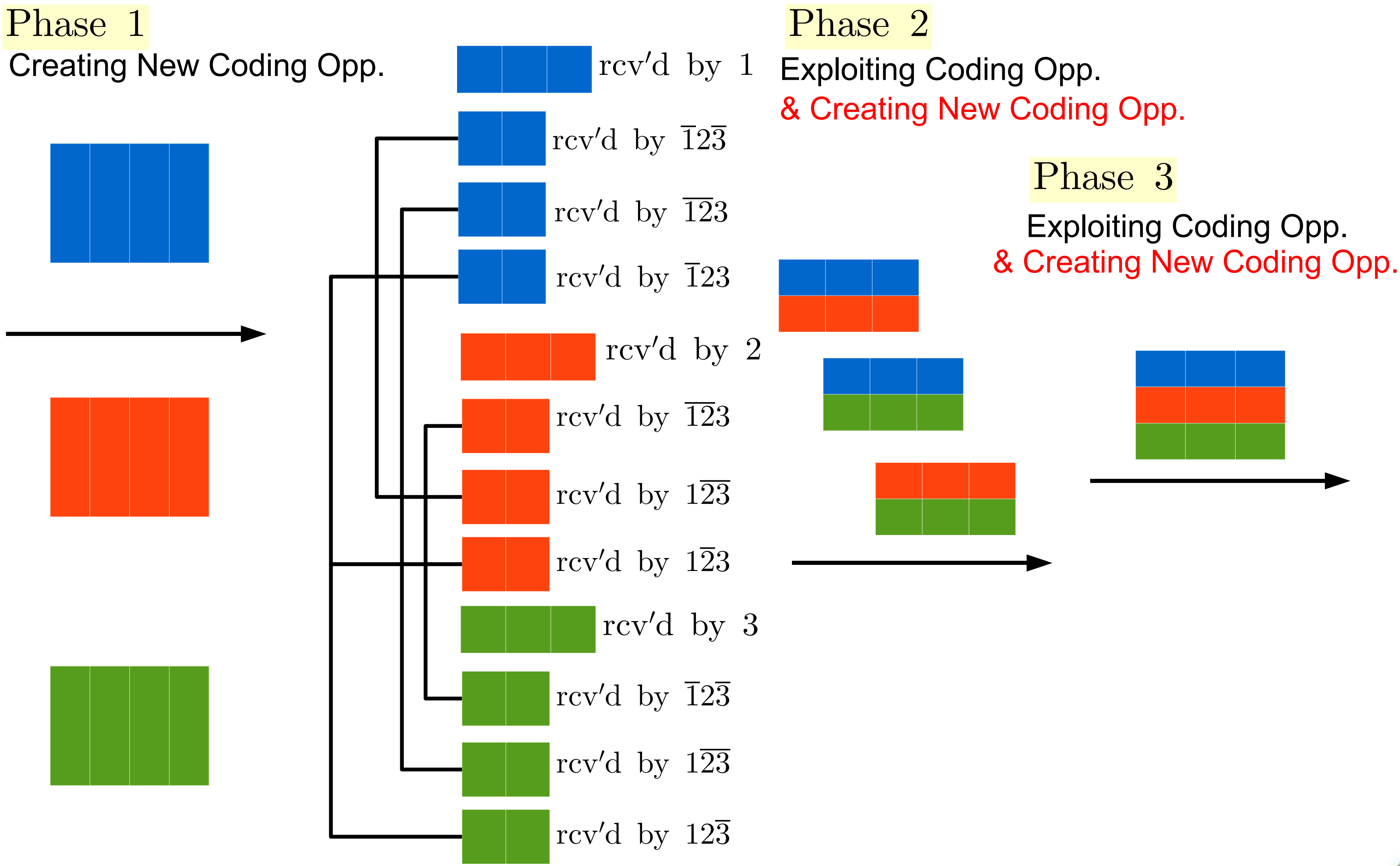

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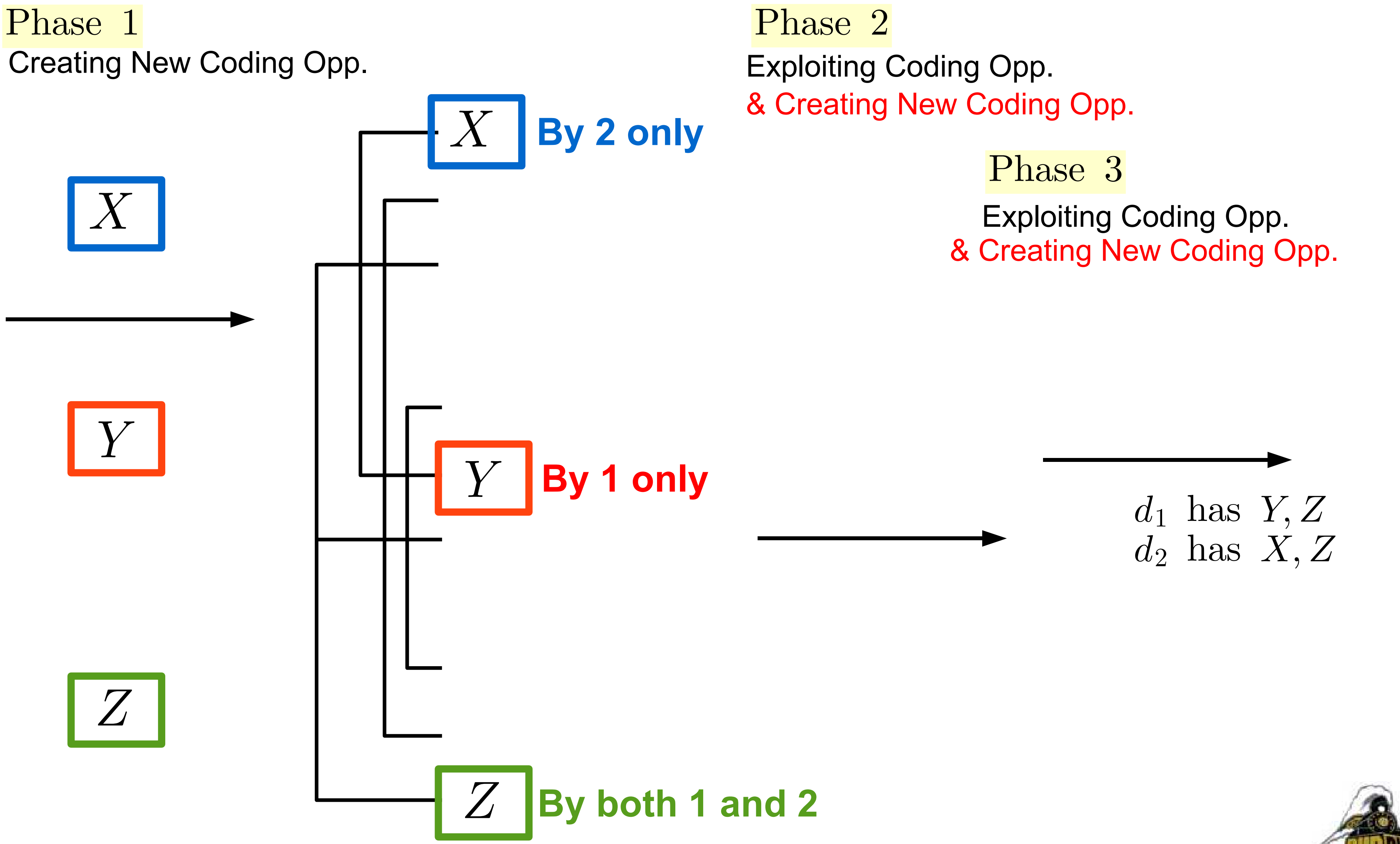

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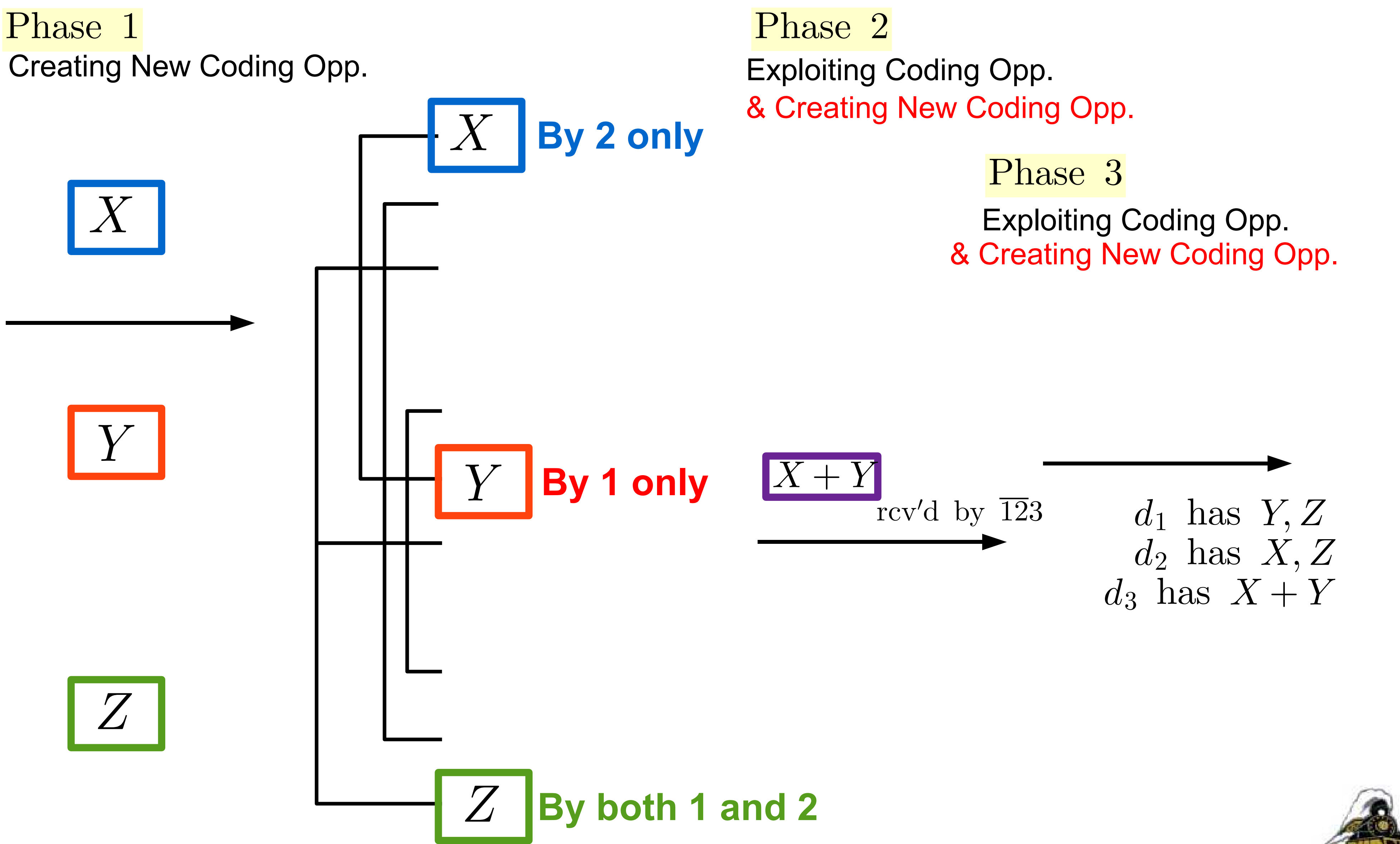
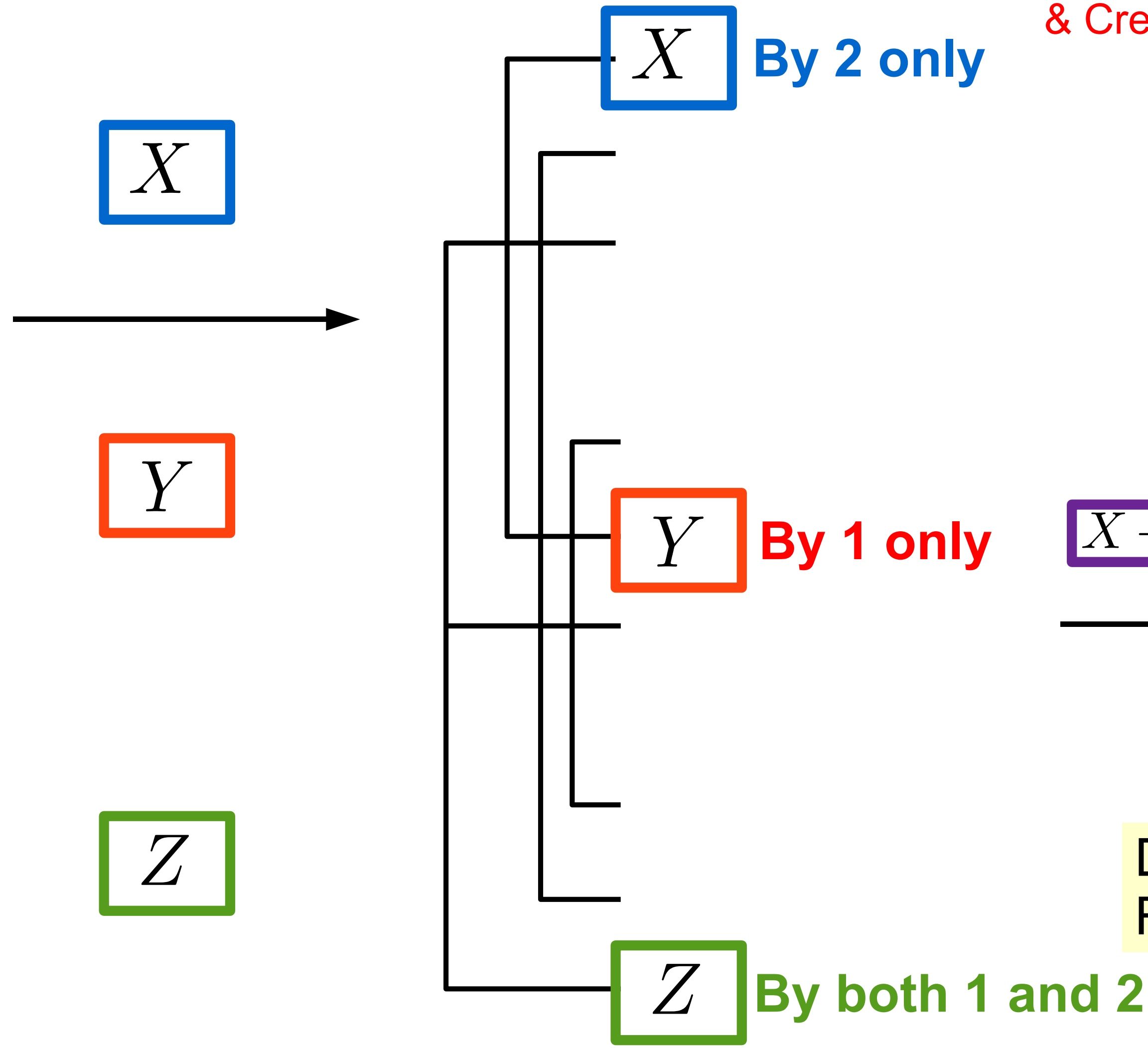


What Went Wrong?

Phase 1

Creating New Coding Opp.



Phase 2

Exploiting Coding Opp.

& Creating New Coding Opp.

Phase 3

Exploiting Coding Opp.

& Creating New Coding Opp.

$X+Y$ rcv'd by $\overline{1}23$

d_1 has Y, Z

d_2 has X, Z

d_3 has $X+Y$

Discard it => Suboptimal

Recoup it for new coding opp.

What Went Wrong?

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Creating New Coding Opp.

Phase 2
Exploiting Coding Opp.
& Creating New Coding Opp.

Phase 3
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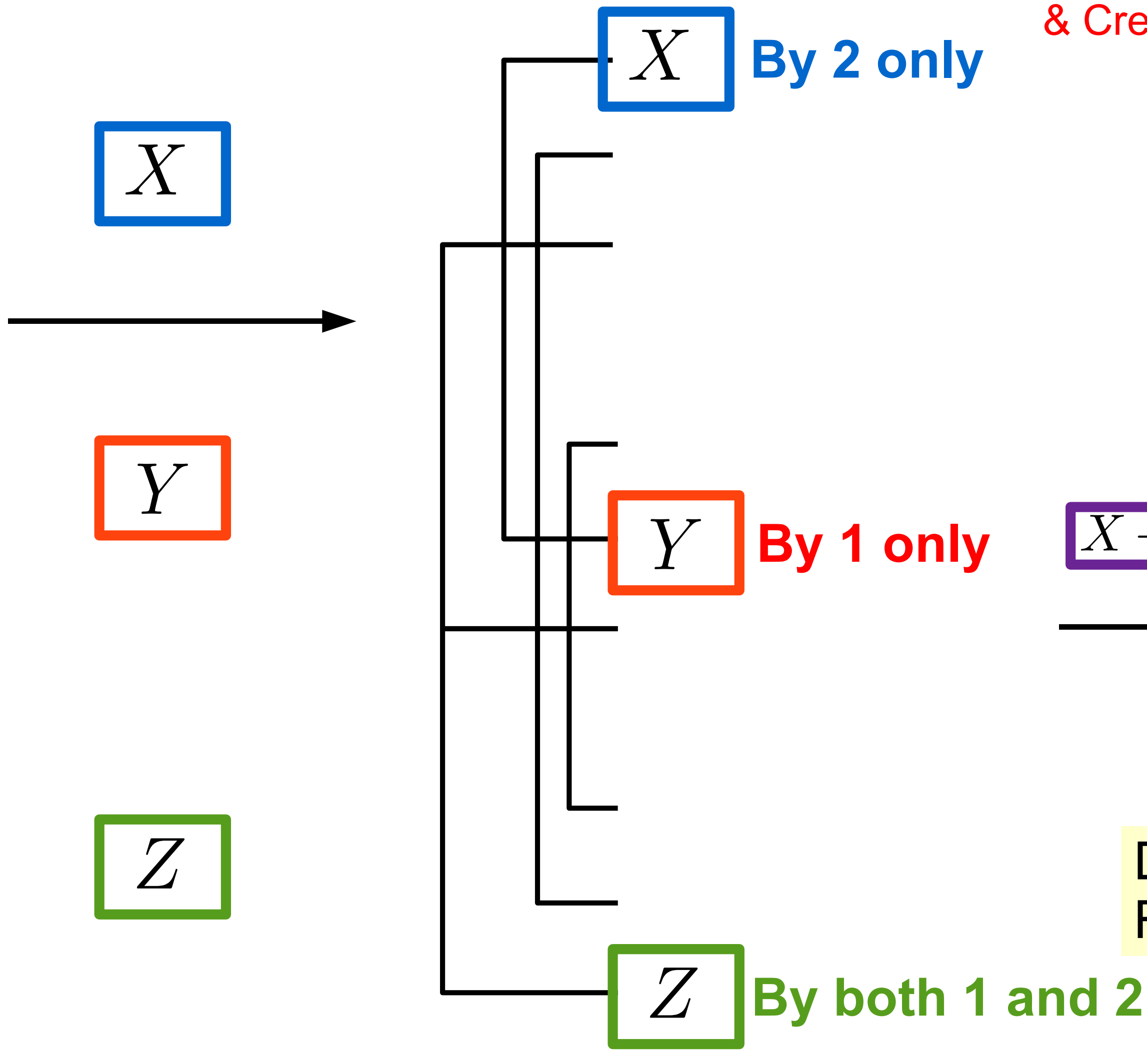

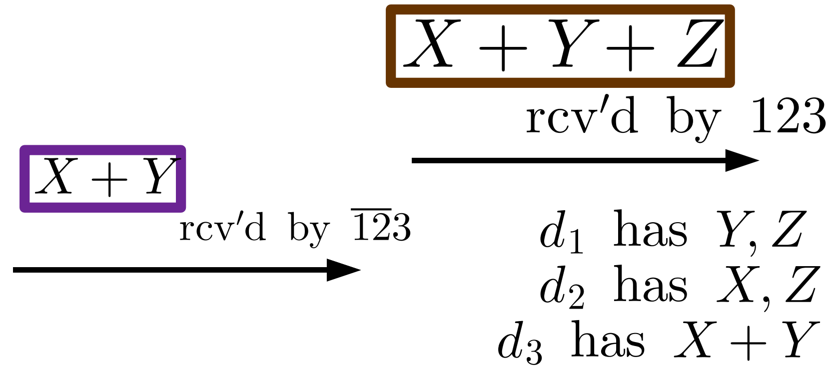

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New Cap. Inner Bound

- We need code alignment [W. ISIT10] in order to recoup the overheard coding opportunities during Phases 2 to M.
- That is, the overheard coding vector $[X + Y]$ has to remain aligned in the subsequent mixing stages.
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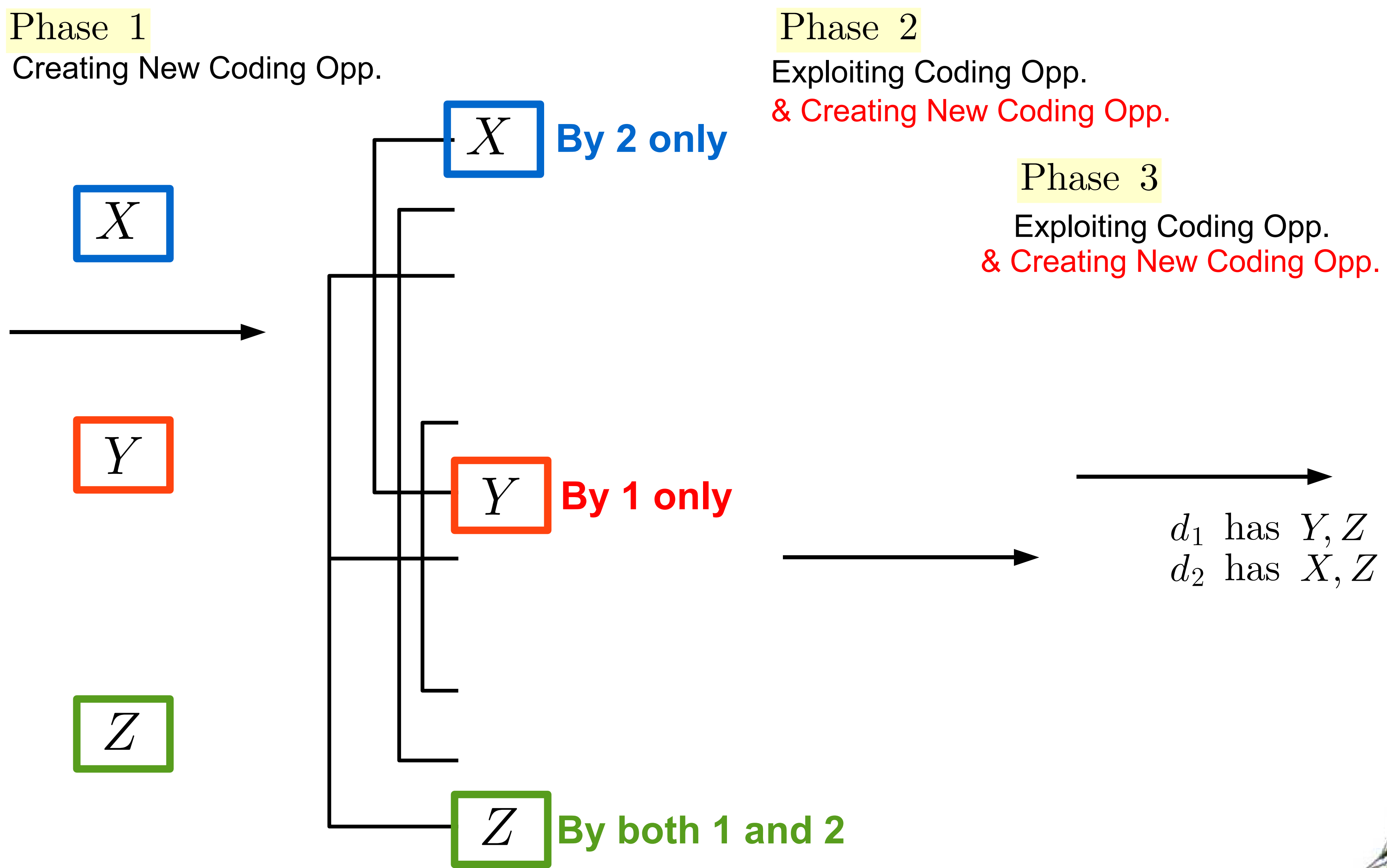
- We propose a new Packet Evolution scheme.
- Each information packet (payload) is expanded to

  (payload, overhearing status, representative coding vector)
  - overhearing status keeps evolving to create more coding opportunities.
  - representative coding vector keeps evolving to ensure code alignment.

The Packet Evolution Scheme



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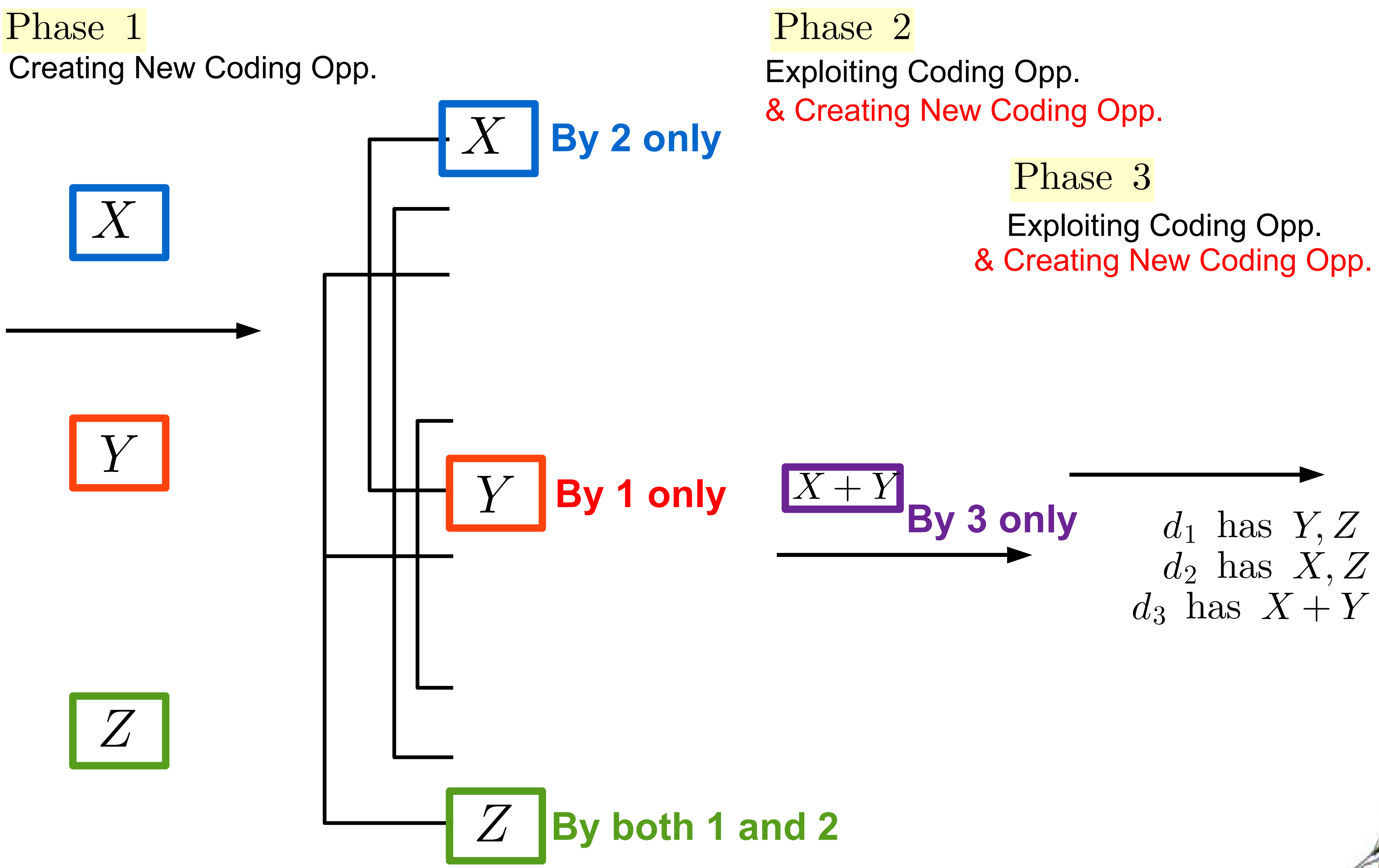

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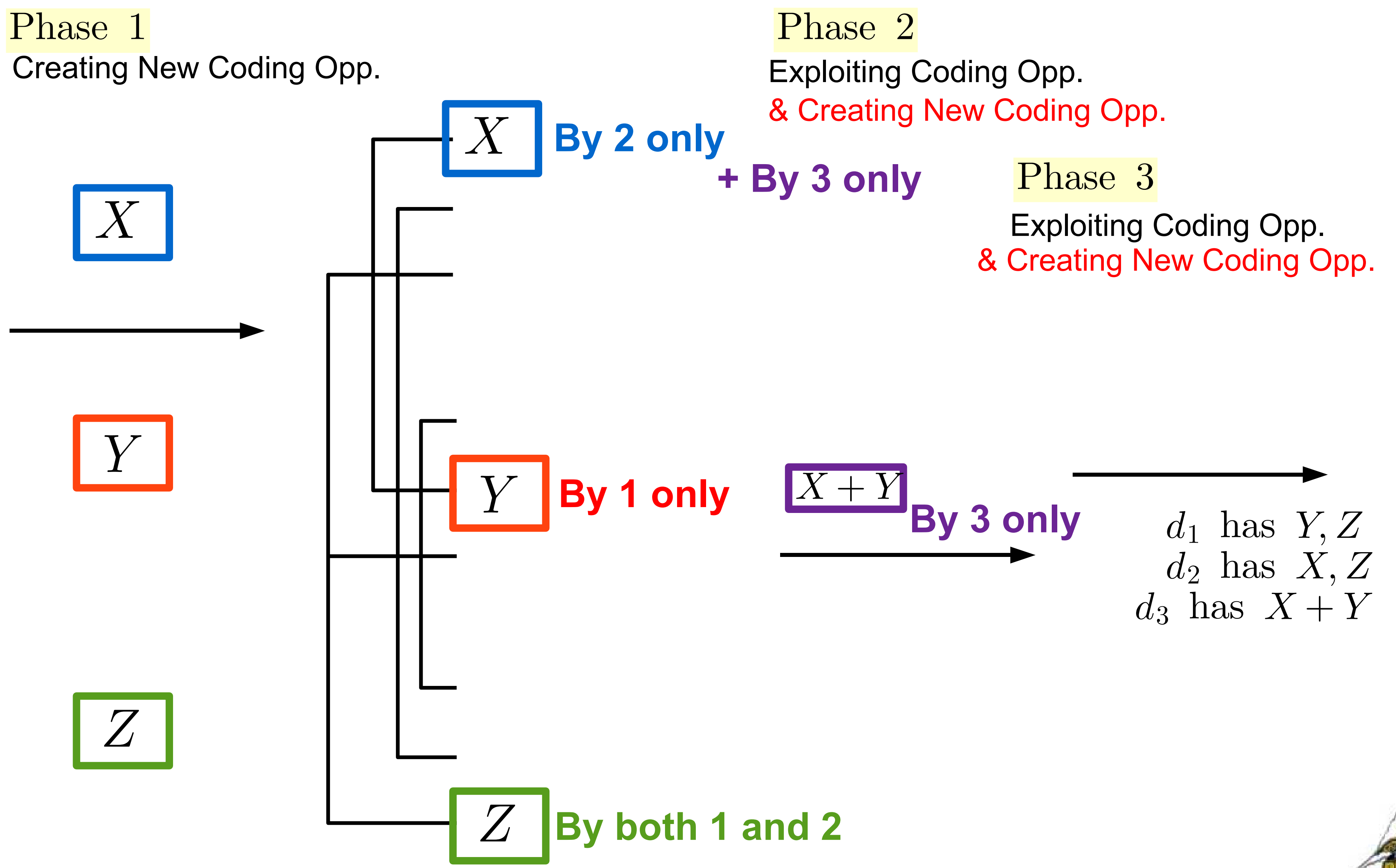

The Packet Evolution Scheme

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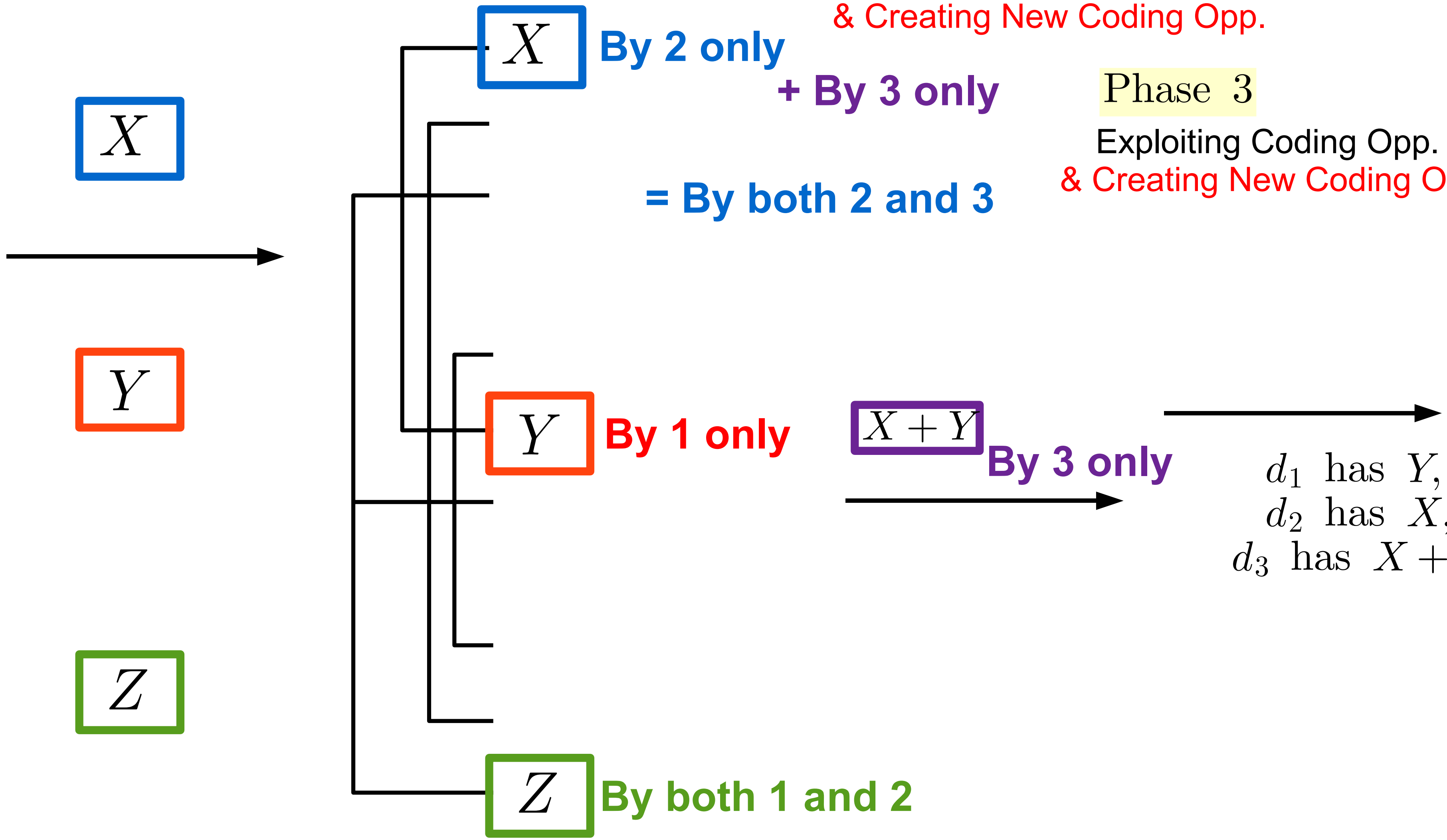

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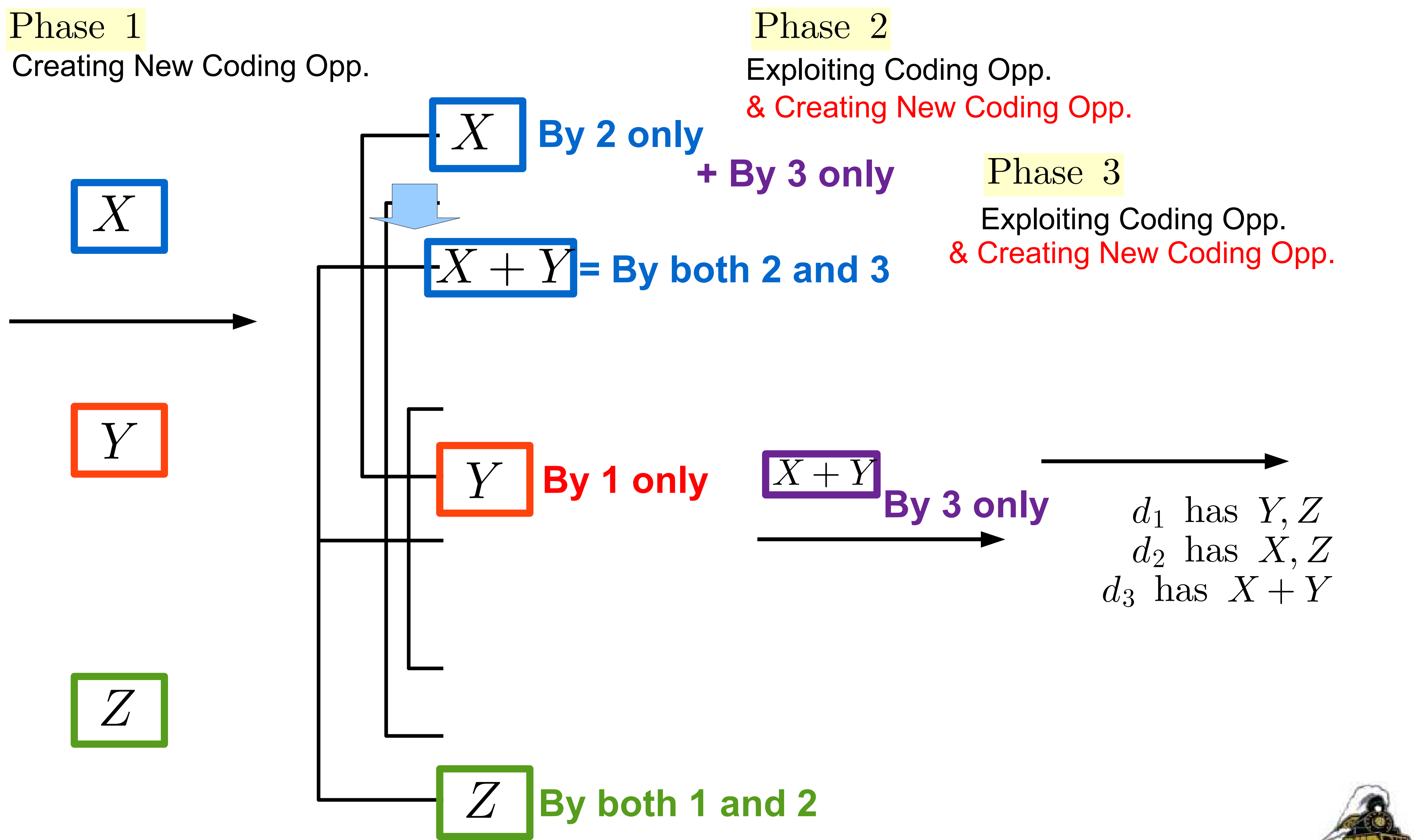

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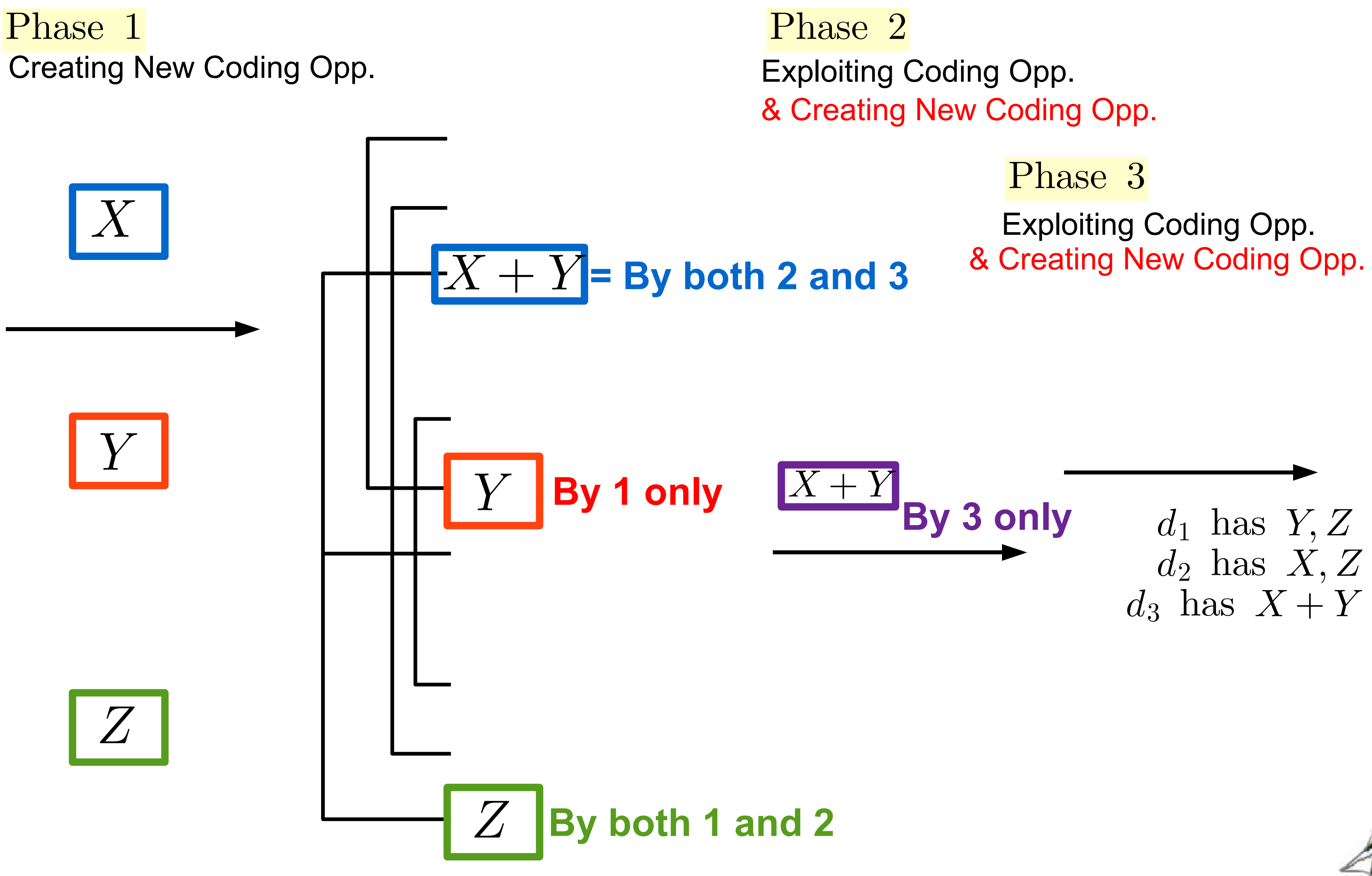

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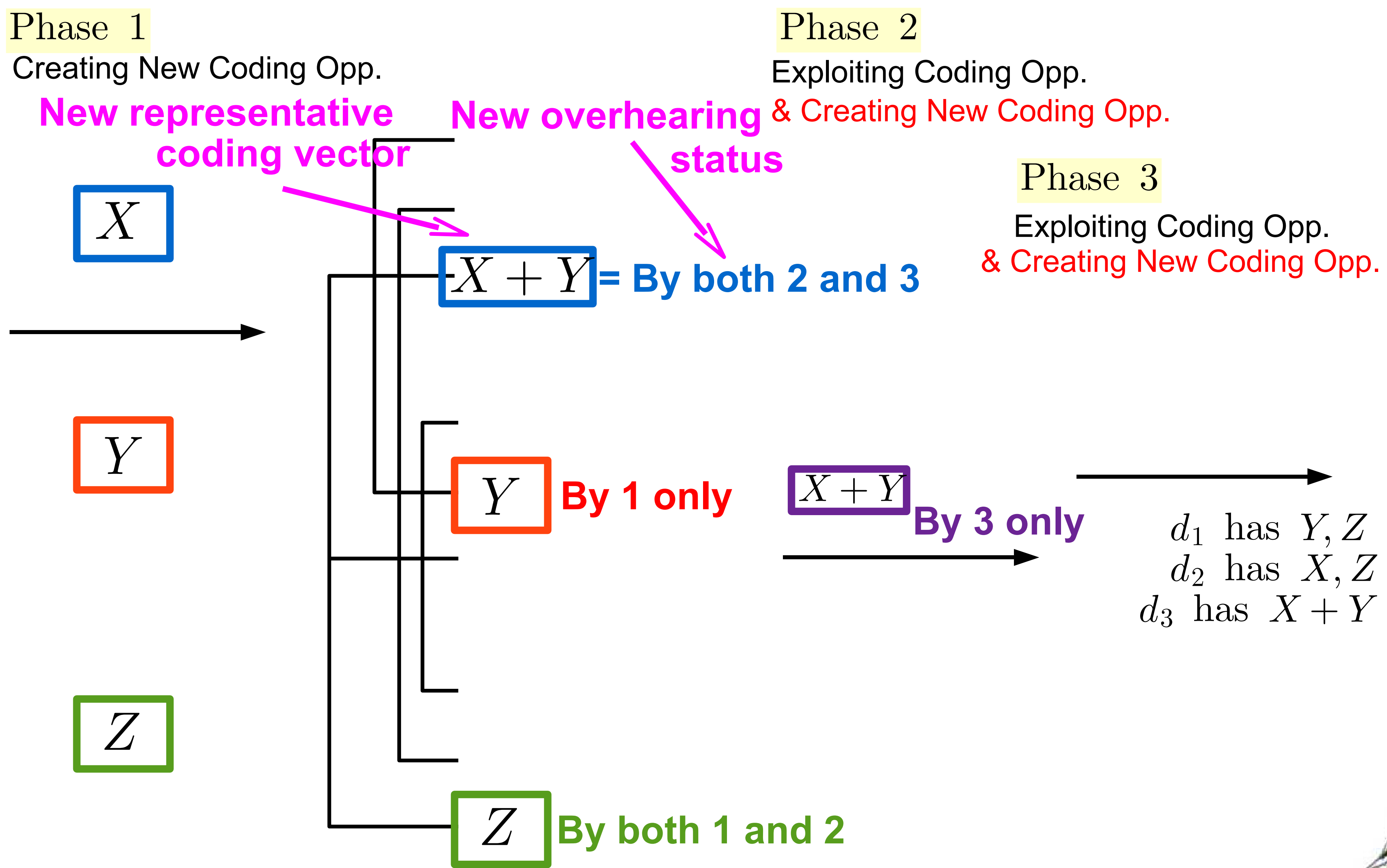

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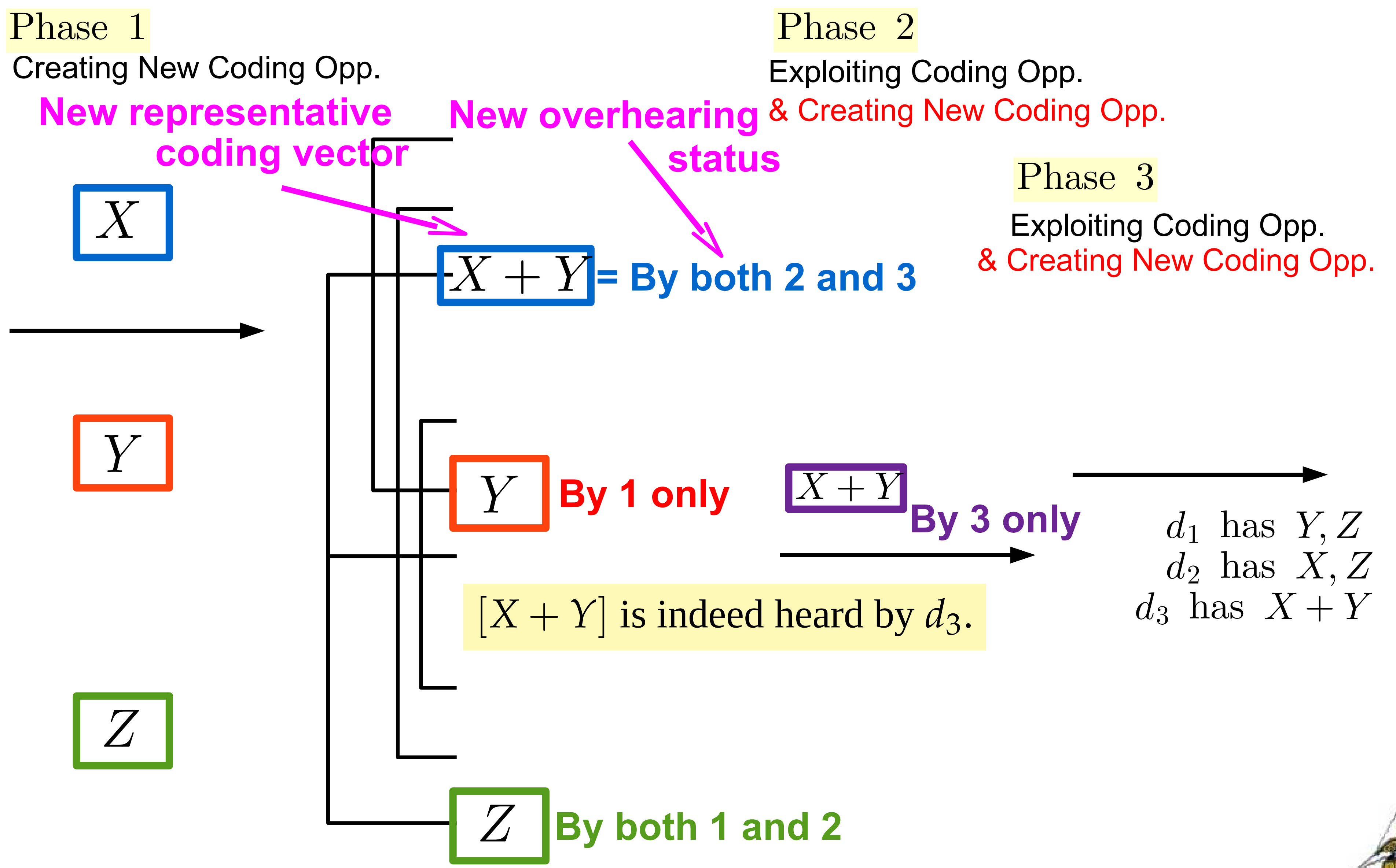

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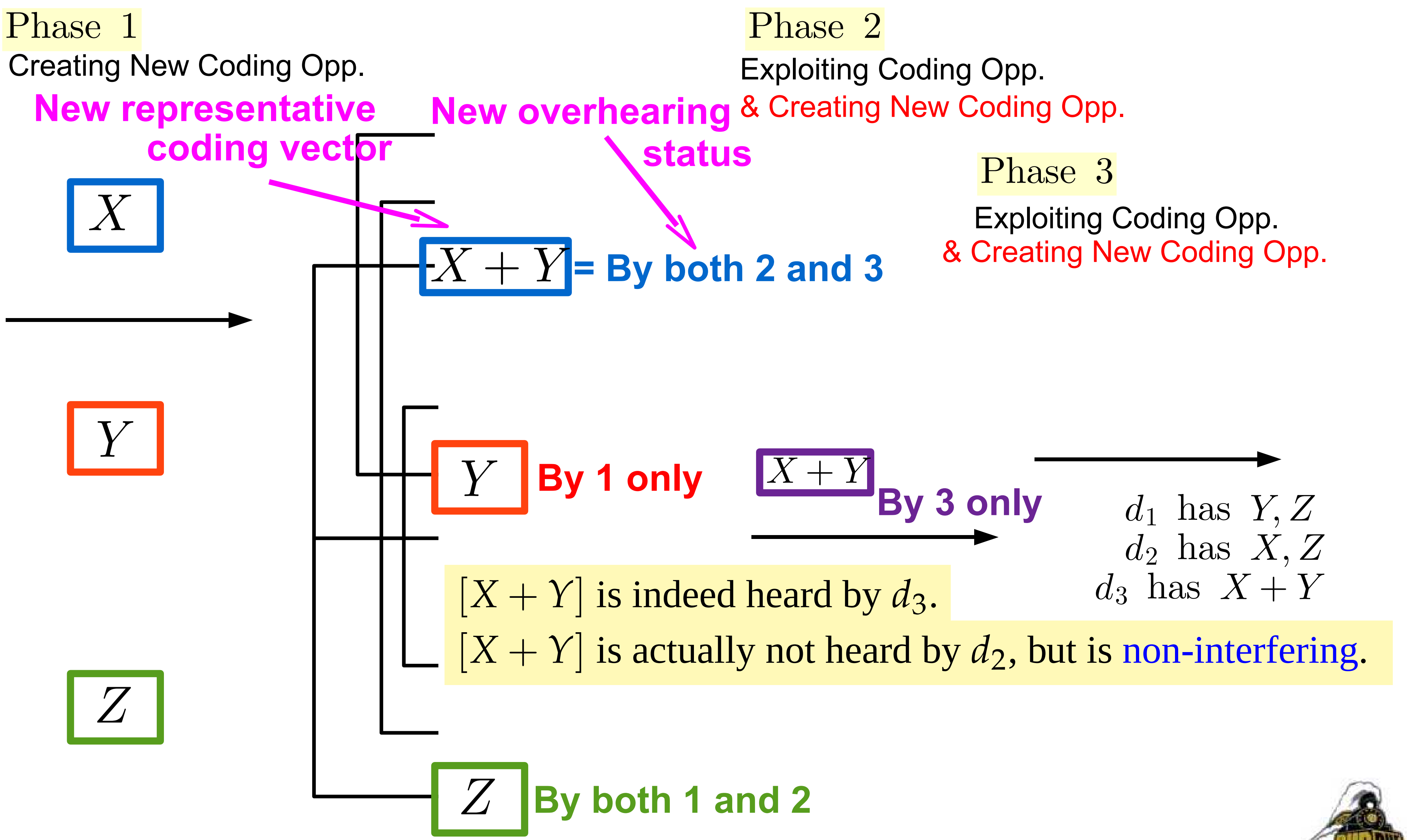

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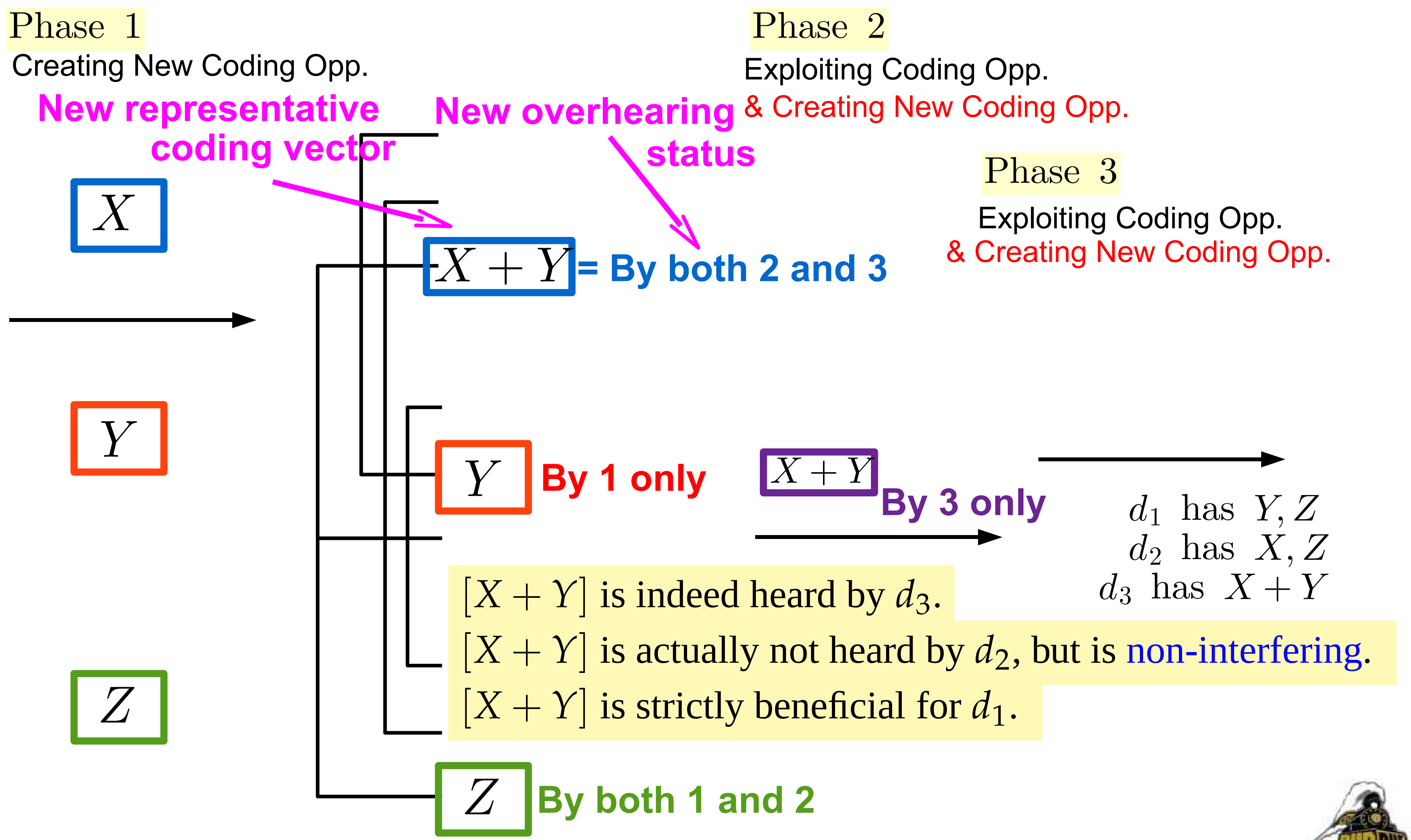

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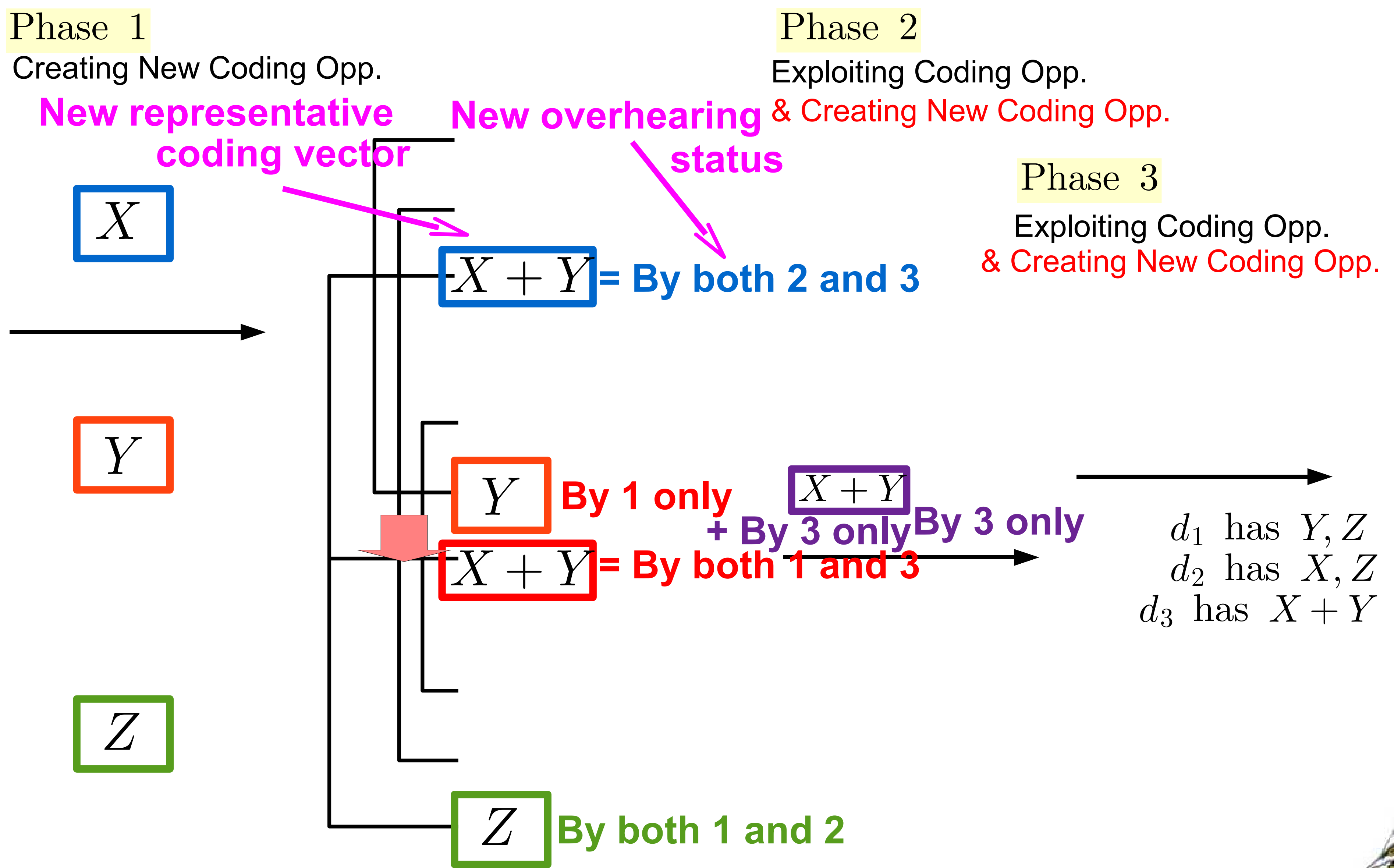

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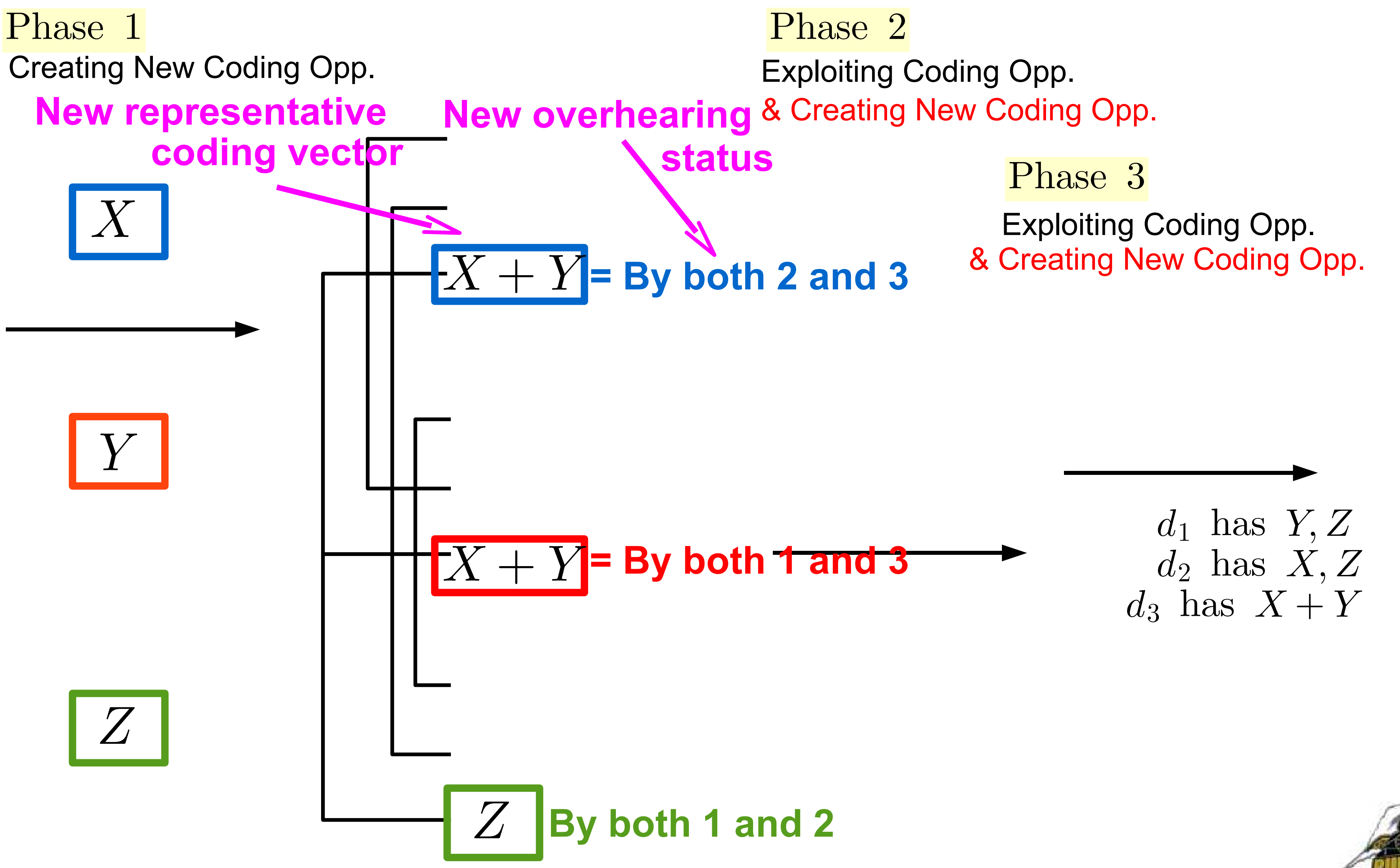

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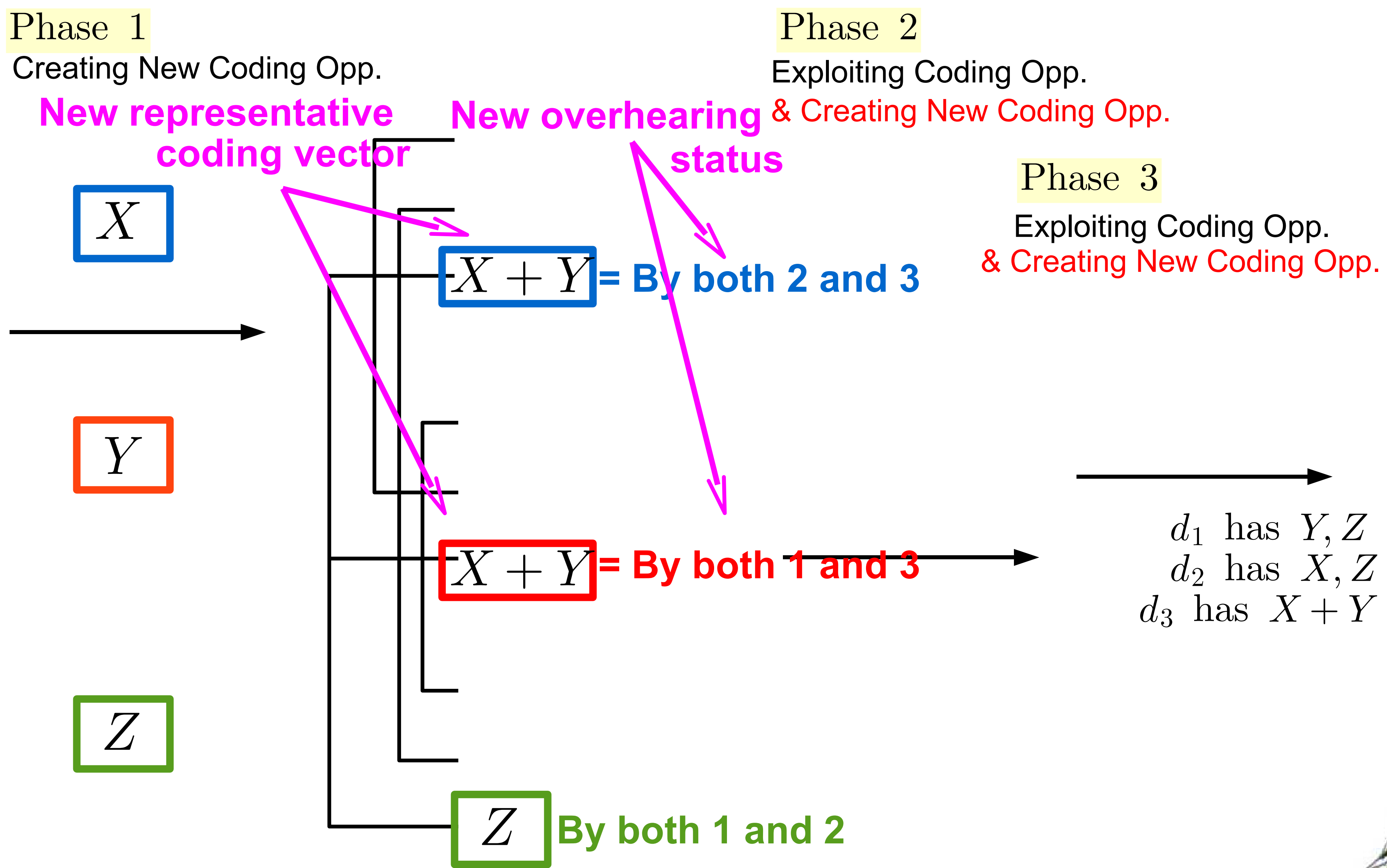

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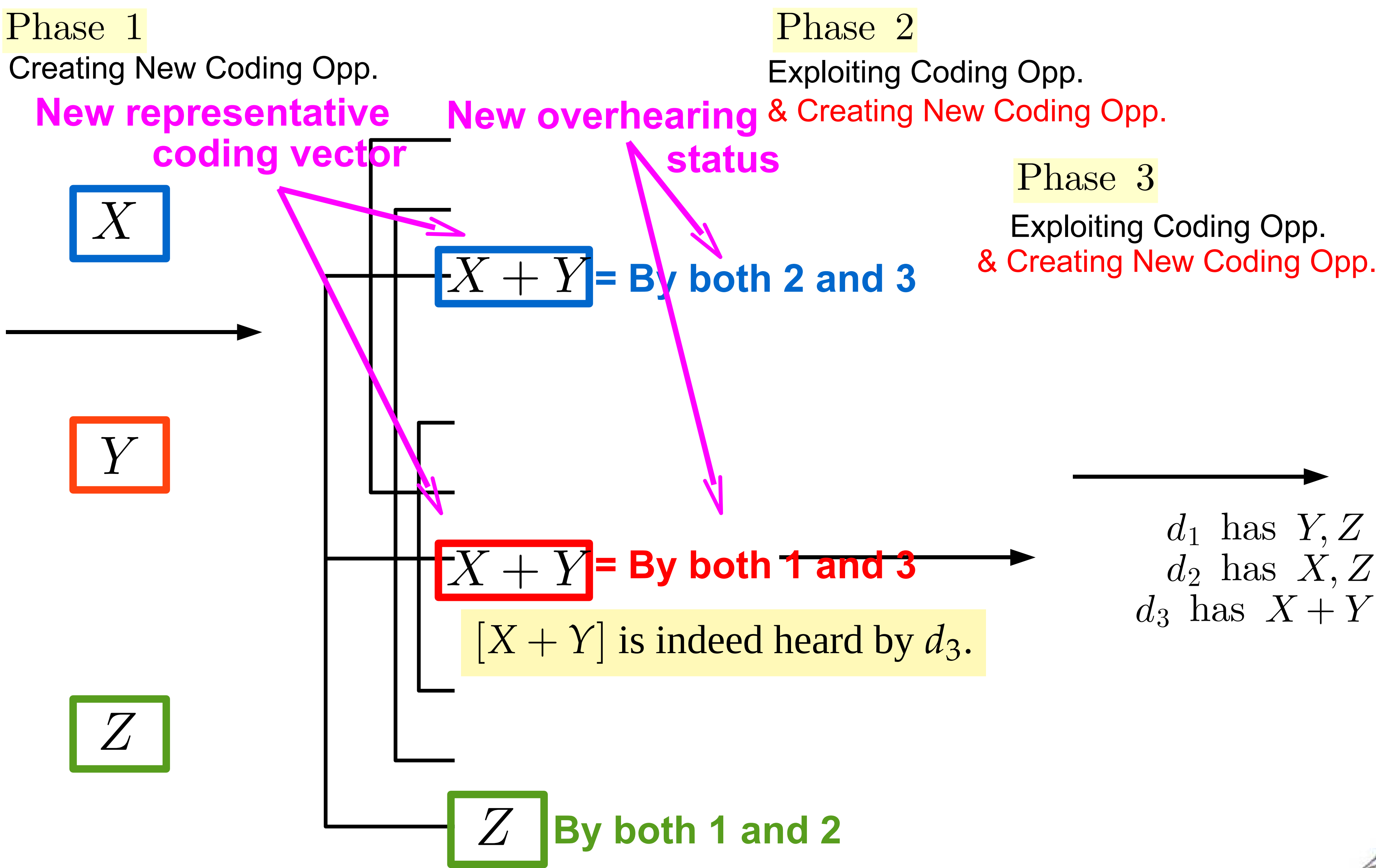

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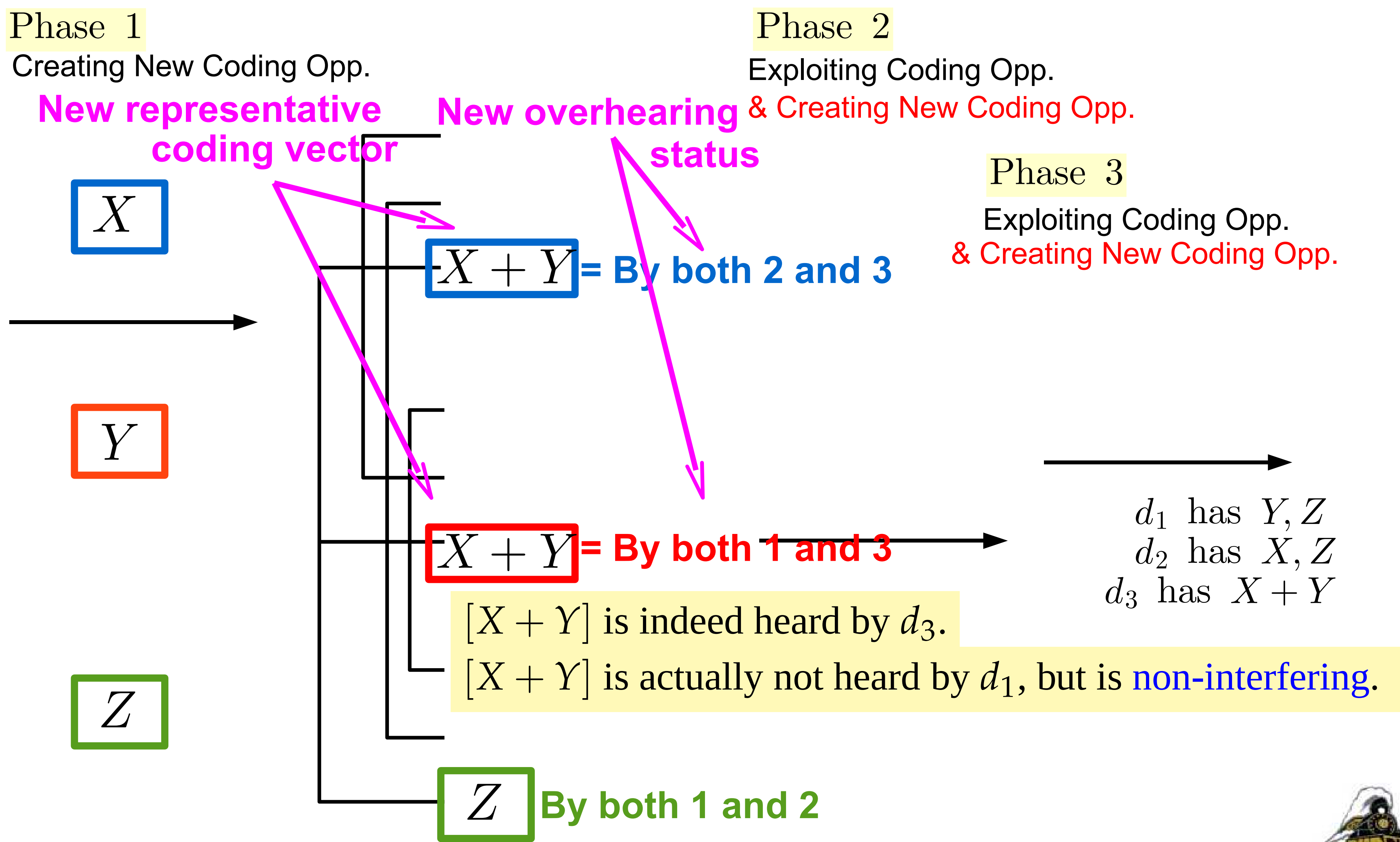

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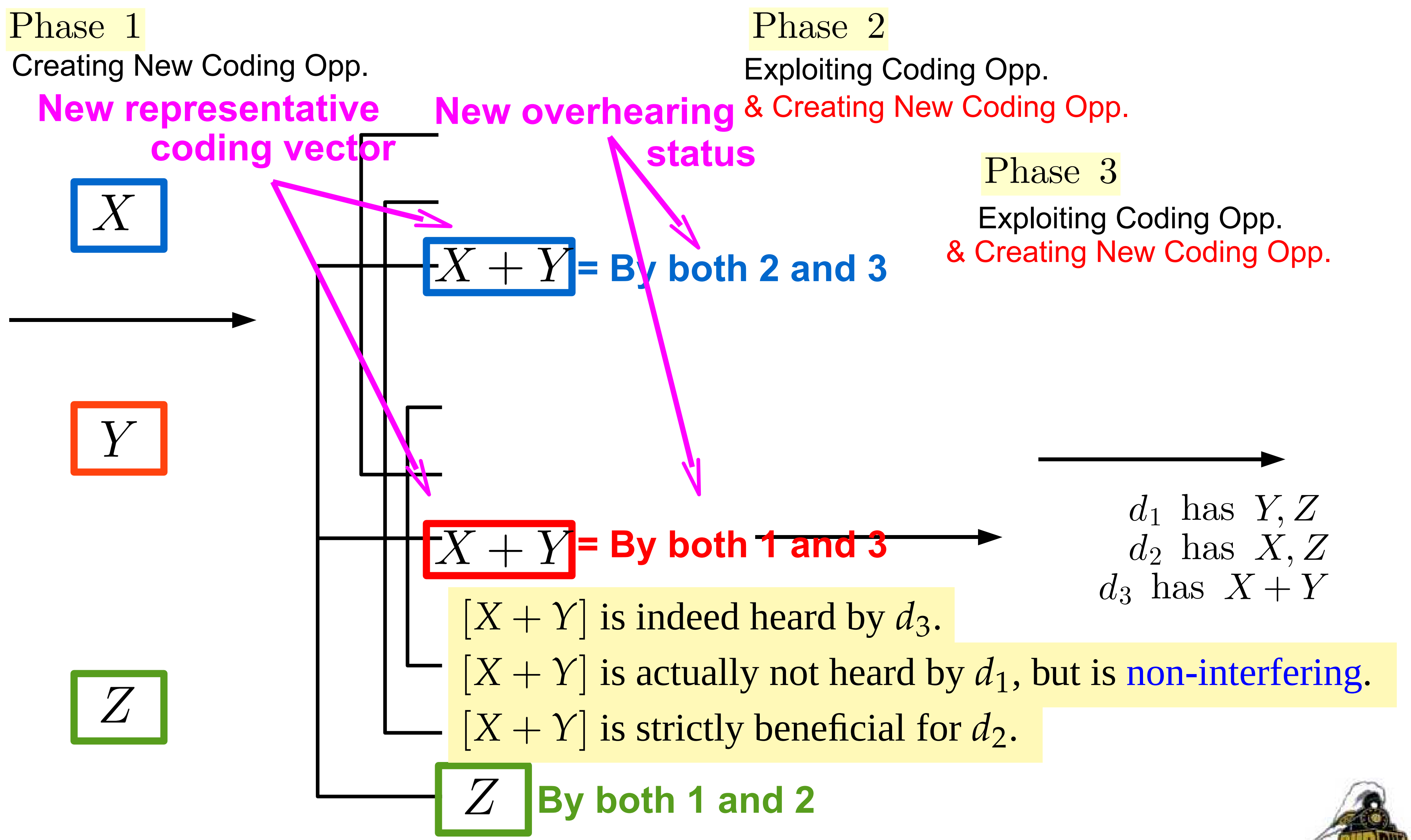

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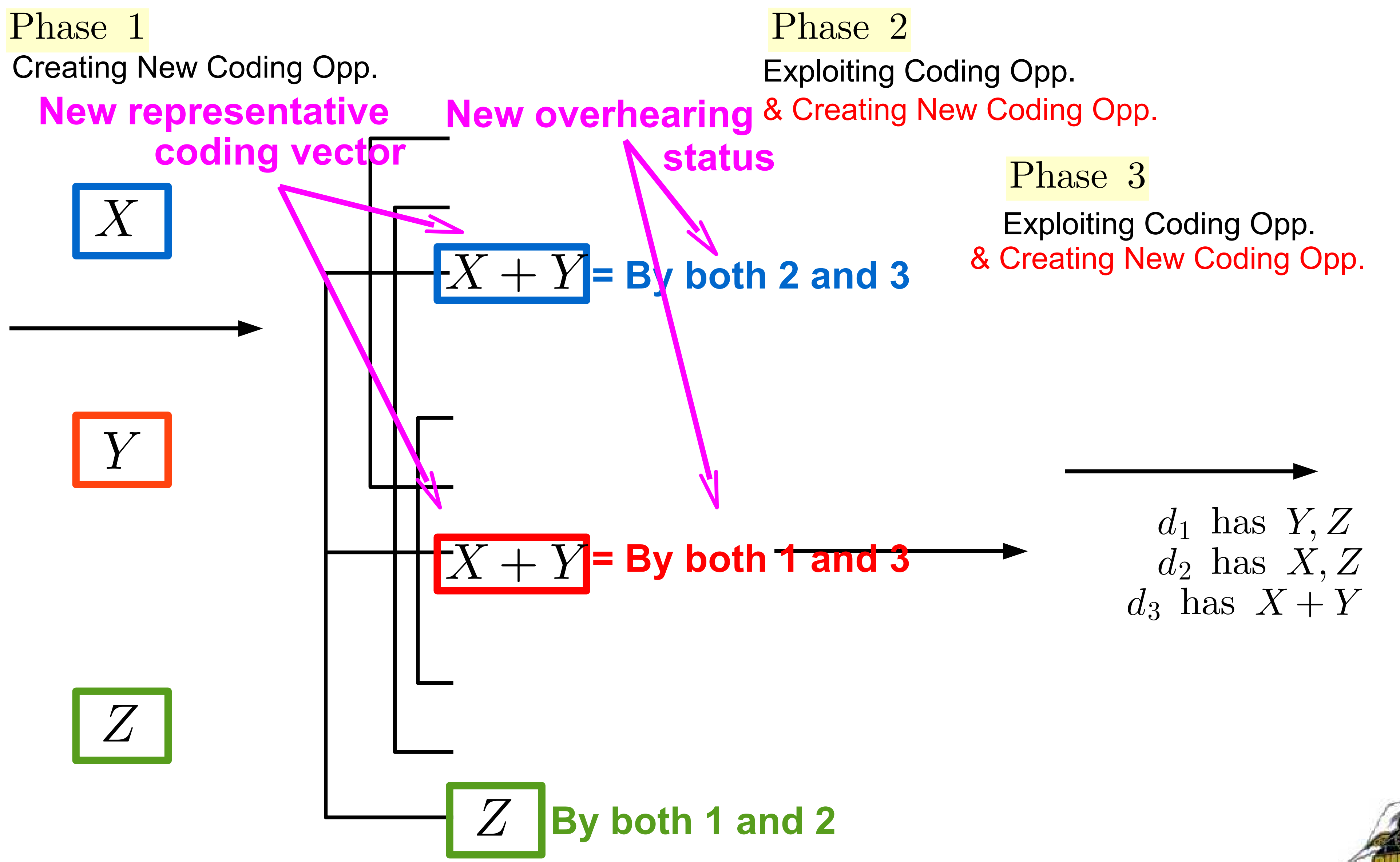

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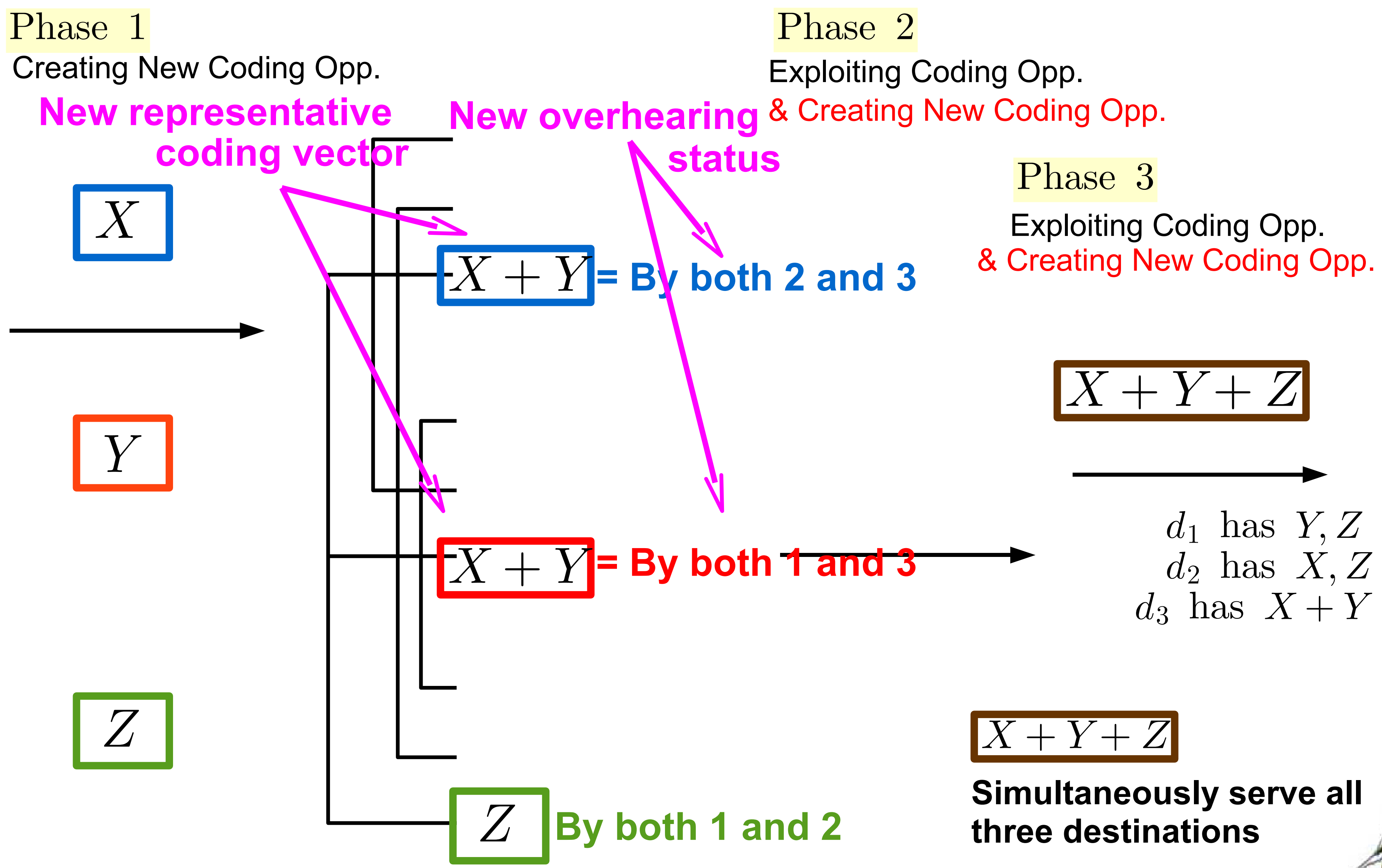

An Example of $M = 4$

- $M = 4$ sessions, each d_k has $X_{k,1}$ to $X_{k,100}$ packets.
- Each $X_{k,l}, \forall k \in [4], l \in [100]$ has an overhearing status $S(X_{k,l}) \subseteq \{1,2,3,4\}$, and a representative coding vector $\mathbf{v}(X_{k,l})$ being a 400-dimensional vector.

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$$
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Create more coding Opp.

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  - Suppose we plan to encode sessions 1, 2, 3 together. We choose three packets X_{1,l_1}, X_{2,l_2}, and X_{3,l_3} such that $S(X_{1,l_1}) = \{2,3\}$, $S(X_{2,l_2}) = \{1,3\}$, and $S(X_{3,l_3}) = \{1,2\}$
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Achieve Code Alignment

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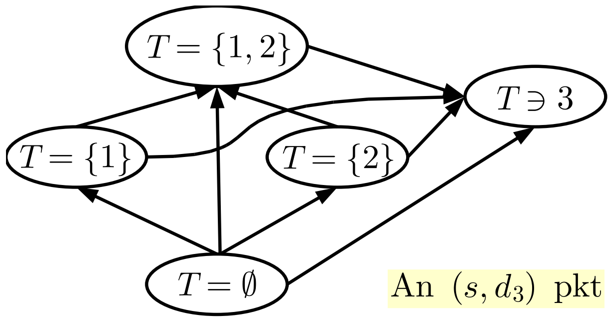
Analysis of The PE Schemes

- In the packet evolution scheme, each packet evolves independently.



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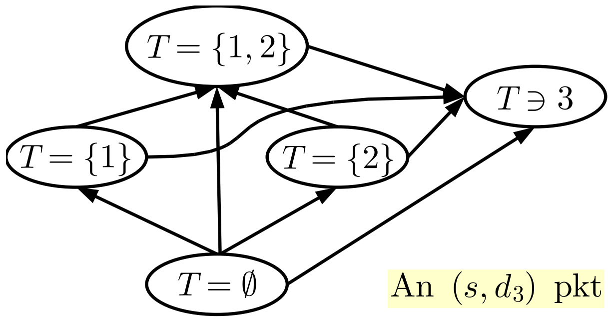
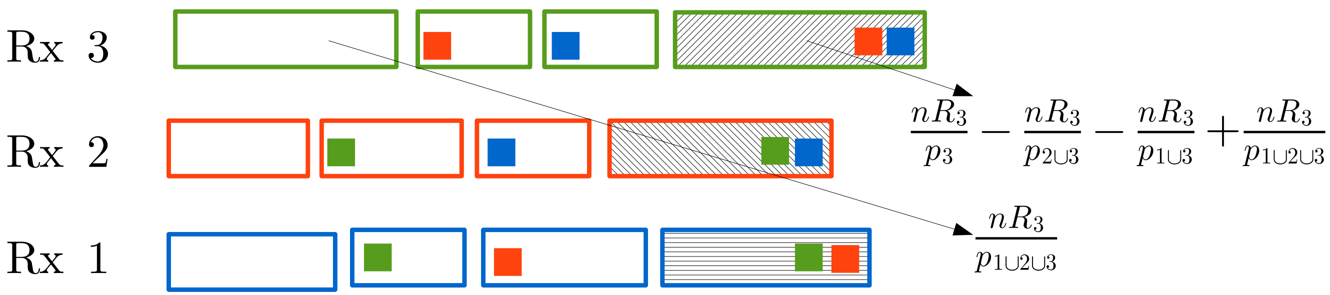


- We can quantify the number of slots that a packet has overhearing status T.



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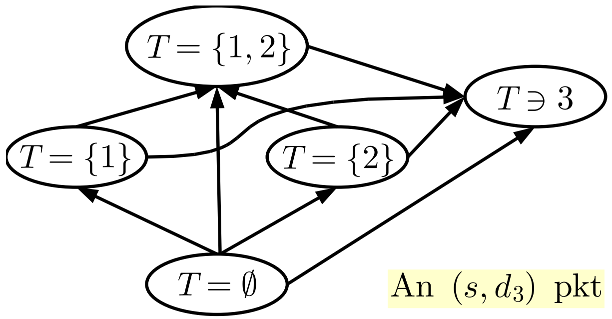

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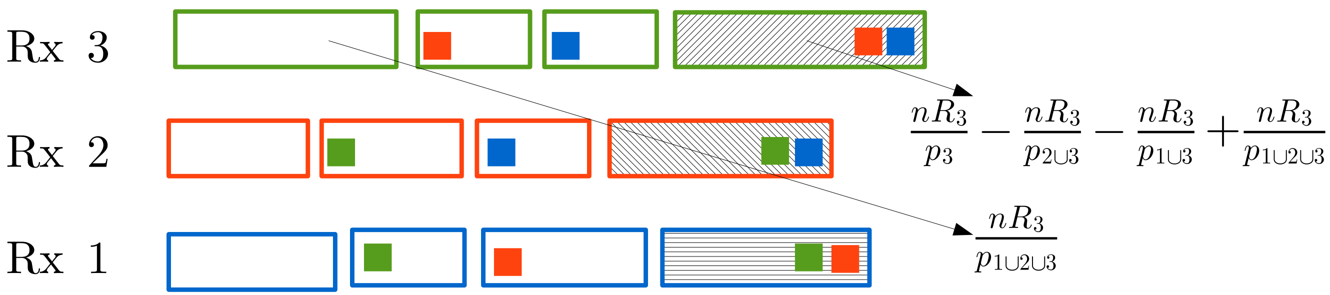

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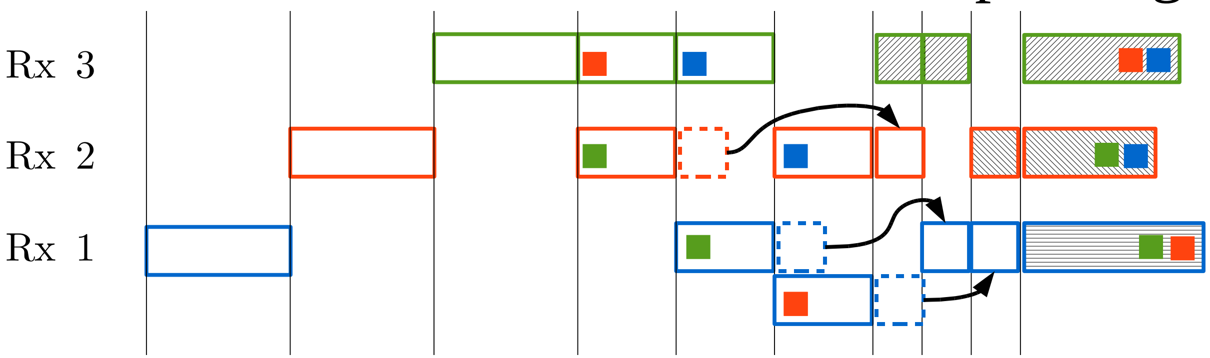

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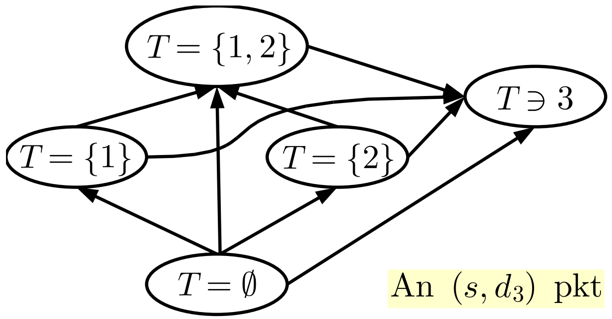

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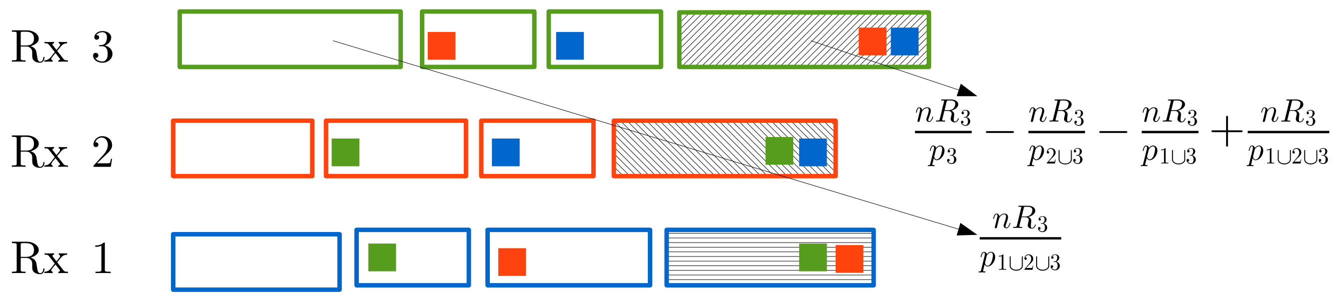
The joint success prob. $p_{S\overline{\{1,2,3\}\backslash S}}$ affects the duration of each status, and thus how to pack them.



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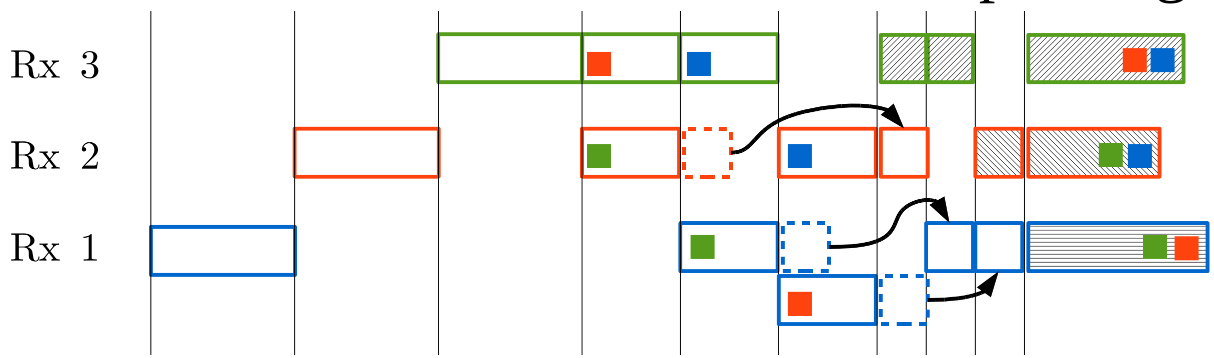
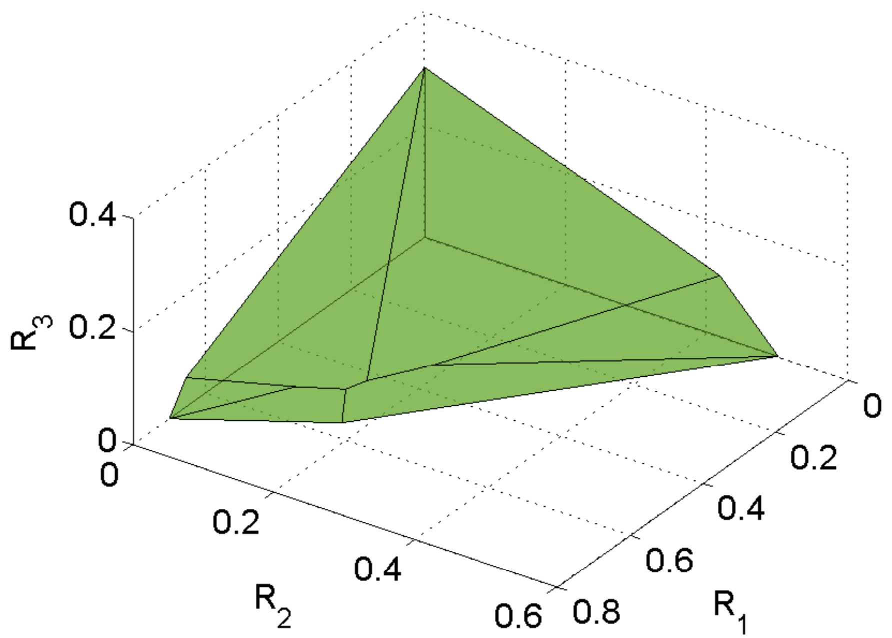


Capacity Results $M = 3$

Based on the Packet Evolution method, we have:

Proposition 1 *Consider any 1-to-3 broadcast PEC with channel output feedback with arbitrary parameters* $p_{S\overline{(\{1,2,3\}\backslash S)}}$ *for all* $S \subseteq \{1,2,3\}$.

The capacity region is indeed $\forall \pi, \sum_{k=1}^{M} \frac{R_{\pi(k)}}{p_{\cup S_k^\pi}} \leq 1$.



6 facets $\Leftrightarrow$ 6 different permutations π

Capacity Results $M \geq 4$

Outer bound: $\forall \pi, \quad \sum_{k=1}^{M} \frac{R_{\pi(k)}}{p_{\cup S_k^{\pi}}} \leq 1.$

Settings with general M>3 values	Capacity inner bound results
General $p_{S\overline{[M]\backslash S}}$	
Spatially symmetric broadcast PECs	
Spatially independent broadcast PECs	

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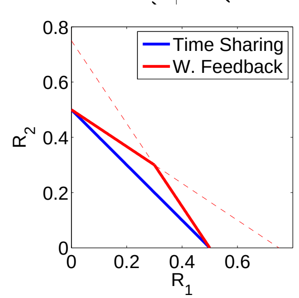

Numerical Evaluation

$$\forall \pi, \sum_{k=1}^{M} \frac{R_{\pi(k)}}{p_{\cup S_k^{\pi}}} \leq 1.$$

Symmetric spatially independent PECs: $p_1 = p_2 = \cdots = p_M = p$

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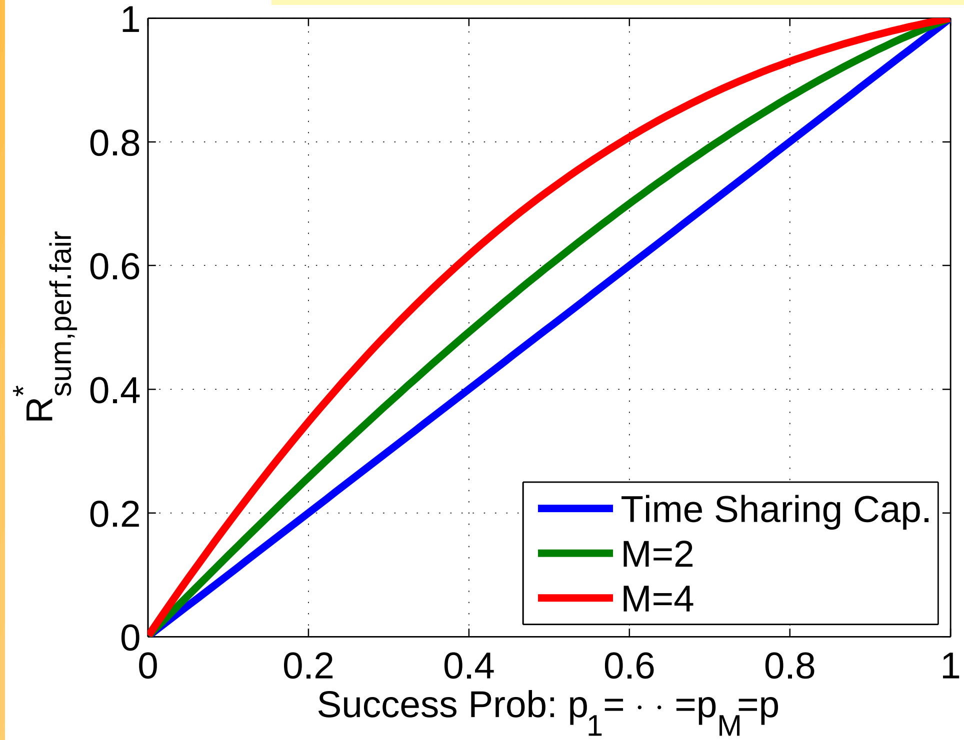
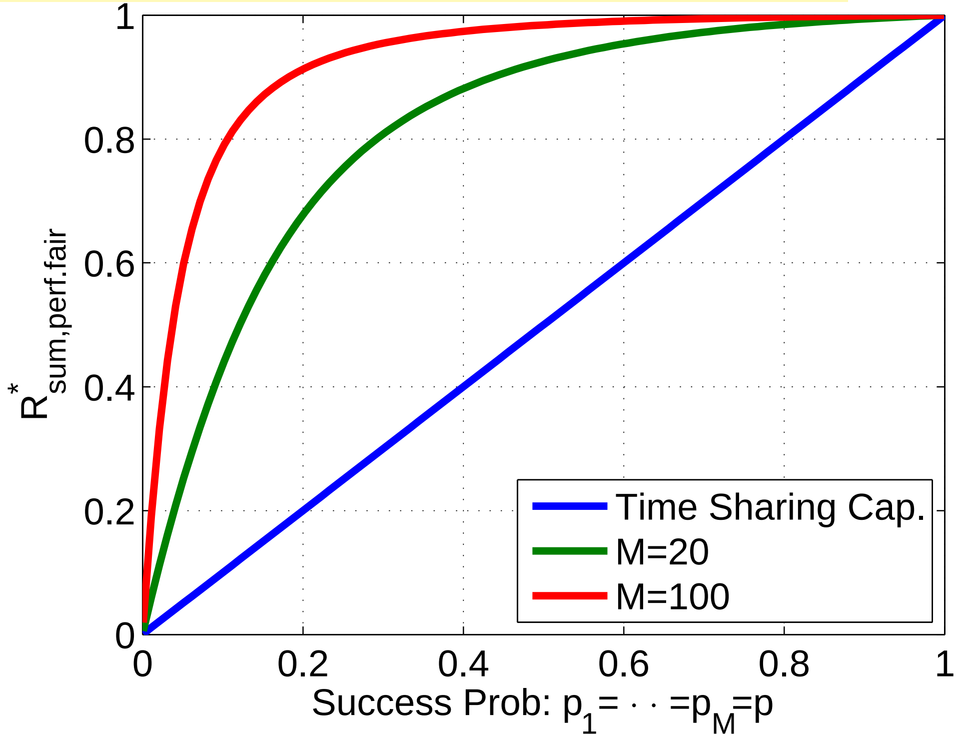
Sum rate capacity $\sum_{k=1}^{M} R_k$ vs. marginal success prob. p.





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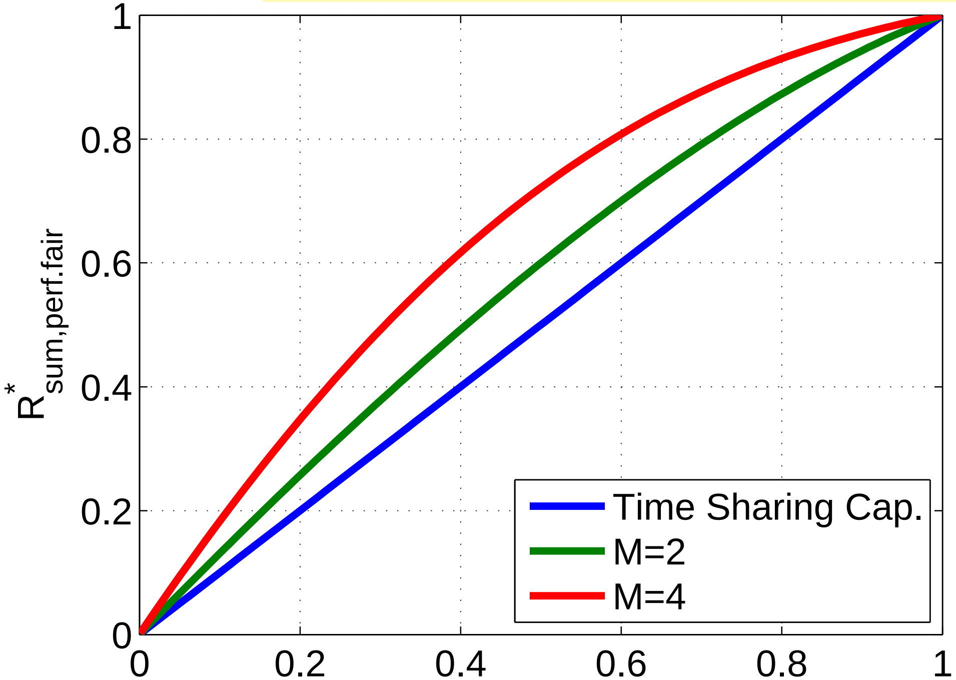
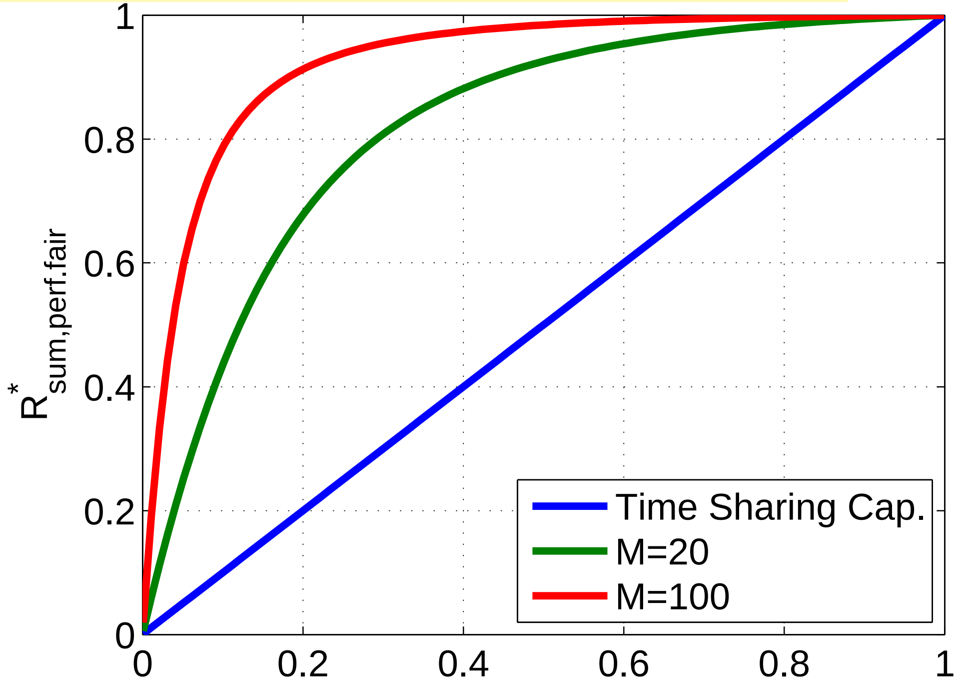
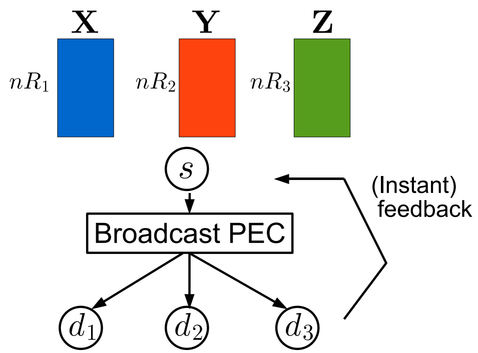




Corollary : When $M \to \infty$, the channel becomes effectively noiseless. [Larsson *et al.* 06]

Summary



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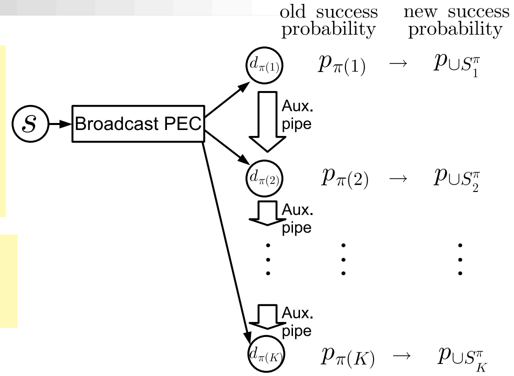
Tight for $M = 3$



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One-Sided Fairness

Definition 2 *The one-sidedly fair region Λ_{osf} contains all rate vectors $(R_1, \cdots, R_M)$ satisfying*

$$\forall i, j \text{ satisfying } p_i < p_j, \text{ we have } R_i(1 - p_i) \geq R_j(1 - p_j).$$

Remark 1: A perfectly fair vector $(R, \cdots, R)$ belongs to Λ_{osf}.

Remark 2: A proportionally fair vector $(p_1 R, \cdots, p_M R)$ belongs to Λ_{osf} if $\min\{p_k : \forall k \in [M]\} \geq 0.5$.