On the Capacity of Wireless 1-Hop Intersession Network Coding — A Broadcast Packet Erasure Channel Approach

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***Abstract*—Motivated by practical wireless network protocols, this paper answers the following questions: Exactly (or at most) how much throughput improvement one can expect from intersession network coding (INC) in a 1-hop neighborhood over non-coding solutions; and how to achieve (or approach) the capacity. Focusing on a *two-stage* setting, this work first provides a capacity outer bound for any number of M coexisting unicast sessions and any overhearing events modeled by *broadcast packet erasure channels* that are time-wise independently and identically distributed. For $M \leq 3$, it is shown that the outer bound meets the capacity. To quantify the tightness of the outer bound for $M \geq 4$, a capacity inner bound for general M is provided. Both bounds can be computed by a linear programming solver. Numeric results show that for $4 \leq M \leq 5$ with randomly chosen channel parameters, the difference between the outer and inner bounds is within 1% for 99.4% of the times. The results in this paper can also be viewed as the generalization of index-coding capacity from wireline broadcast with binary alphabets to wireless broadcast with high-order alphabets.**

I. Introduction

The throughput benefits of network coding (NC) can be fully realized when considering not only coding within a single multicast session [1], but also coding over multiple unicast sessions, the so-called *intersession network coding* (INC) [2]. Determining the capacity of INC is notoriously challenging [3] and for general integral settings with an unbounded number of coexisting sessions, the complexity of determining the INC capacity is NP-complete [2], [4]. To mitigate its complexity, recent practical wireless INC schemes [5] focus on INC over a 1-hop neighborhood. Despite its simplified setting, the exact capacity region of 1-hop INC remains an open problem [6], [7]. Motivated by practical wireless protocols, this paper answers the following questions: Exactly (or at most) how much throughput improvement one can expect from INC in a wireless 1-hop neighborhood over non-coding solutions; and how to achieve (or approach) the corresponding INC capacity.

Most 1-hop INC schemes [5] take advantage of the broadcast nature of wireless media. Take Fig. 1(a) for example. Two unicast sessions (s_1, d_1) and (s_2, d_2) would like to send packets X_1 and X_2, respectively, through a common relay node r. Due to the interference constraints, s_1, s_2, and r cannot transmit simultaneously. Assuming that each transmitted packet always arrives the target receiver successfully, without network coding four time slots are necessary to complete the two sessions. [5] observes that if we further assume that each transmitted packet can always be overheard by more than one node, then one can complete the transmission in three time slots by sending packets X_1, X_2, and $X_1 + X_2$, and by performing decoding at the destinations using the overheard *side information* (Fig. 1(a)). This simple illustrative example quickly becomes a challenging problem when the wireless channels are not perfect. What if d_i cannot overhear all the transmissions from s_j, $j \neq i$? What if a coded packet broadcast by r cannot reach both d_1 and d_2 simultaneously? How much throughput improvement can we still expect from INC? The setting becomes even more complicated when considering the random overhearing events among M coexisting sessions (s_i, d_i), $i \in [M] \triangleq \{1, \cdots, M\}$ for general $M > 2$ values.

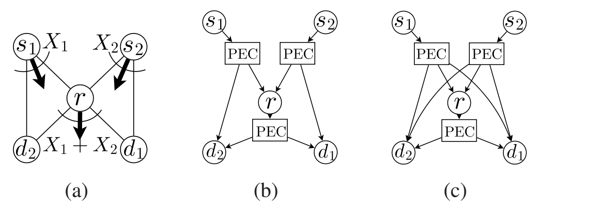


Fig. 1. (a) The optimal 1-hop INC with no channel randomness. (b) The PEC-based canonical 1-hop INC network with no direct communication between s_i and d_i. (c) A similar setting with direct (s_i, d_i) communication

This capacity problem is related to the Gaussian broadcast channel (GBC) problem with message side information (MSI), see [8] and the references therein. Namely, one can consider a two-stage scheme: In the first stage, all sources s_i, $i \in [M]$, transmit. In the second stage, the relay node r is facing a *broadcast channel problem* such that r would like to broadcast M data streams (one for each session) to M destinations, respectively, while taking advantage of the MSI available at d_i, $i \in [M]$, overheard during the first stage. The GBC capacity with general MSI is currently known only for the simplest case of $M \leq 2$ [9]. Another related setting is the *index coding* problem [10]. Index coding considers a similar broadcast w. MSI problem but focuses on noise-free binary channels and each session having 1 bit of information to transmit, which is different from the GBC settings of additive Gaussian noise with each session transmitting at a real-valued rate R_i.

This paper models the wireless 1-hop INC problem by the same 2-stage setting as in GBCs w. MSI and in index coding.

On the other hand, to capture the packet-by-packet behavior in wireless networks, we consider broadcast packet erasure channels (PECs) instead of the GBC or the noise-free channel in index coding. Each transmission symbol (a packet) is chosen from a finite field $\mathsf{GF}(q)$ for some sufficiently large q and our objective is to maximizing the achievable real-valued rate R_i, which is also different from the symbol power constraints in GBCs and the binary-alphabet, integral setting of index coding. Unlike the existing results that focus mainly on the broadcast problem, we further take into account how s_i should transmit in the first stage in order to optimize the overheard MSI that best benefits the subsequent broadcast stage.

In this paper, a capacity outer bounded is provided for any number M of coexisting unicast sessions and any overhearing events modeled by PECs that are time-wise independently and identically distributed (i.i.d.). For $M \leq 3$, it is shown that the outer bound meets the capacity, which is a step forward from the $M = 2$ capacity characterization in the GBC results. To quantify the tightness of the outer bound for $M \geq 4$, a capacity inner bound for general M is provided. Both bounds can be computed by linear programming (LP) solvers for any M value. Numeric results show that for $4 \leq M \leq 5$ with randomly chosen PEC parameters, the gap between the two bounds is within 1% for 99.4% of the times, which demonstrates the effectiveness of our capacity bounds from a practical perspective.

II. The Setting

A. Packet Erasure Channels (PECs)

A 1-to-K PEC takes an input $X \in \mathsf{GF}(q)$ and outputs a K-dimensional vector in $(\{X\}\cup\{*\})^K$, where the k-th coordinate being "$*$" denotes that the transmitted symbol X does not reach the k-th receiver (thus being erased). We also assume that there is no other type of noise, i.e., the individual output is either equal to the input X or an erasure "$*$." We consider only time-wise i.i.d. PECs of which the erasure pattern is i.i.d. for each channel usage. For example, a 1-to-2 PEC can be described by four parameters $p_{s\to 12}$, $p_{s\to 12^c}$, $p_{s\to 1^c2}$, and $p_{s\to 1^c2^c}$, which denote the probabilities that a packet sent by s is received successfully by both receivers (rx) 1 and 2, by only rx 1, by only rx 2, and by neither rx 1 nor 2, respectively. More rigorously, for any $T \in 2^{[K]}$, where $2^{[K]}$ is the power set of $[K]$, we use $p_{s\to T([K]\backslash T)^c}$ to denote the probability that a symbol sent by s is received by and only by the receivers in T. For each PEC usage, the success events among different receivers can be dependent, but the PEC must be time-wise i.i.d. over multiple channel usages.

B. The Canonical 1-Hop Relay Network With No Direct Path

We consider the following canonical 1-hop relay network. Such a network consists of M source and destination pairs (s_i, d_i), $\forall i \in [M]$ and a single relay node r. Each s_i is associated with a 1-to-M PEC that sends a symbol from s_i to $\{d_j : \forall j \in [M]\backslash i\} \cup \{r\}$, and r is associated with a 1-to-M PEC that sends a symbol from r to $\{d_j : \forall j \in [M]\}$. There is no direct communication between s_i and d_i for all $i \in [M]$. For example, suppose $M = 2$ (see Fig. 1(b)). One parameter of the s_2-PEC is $p_{s_2\to d_1^c r}$, which describes the probability that a packet sent by s_2 is heard by r but not by d_1. For simplicity, we often use $p_{s_2\to 1^c r}$ as shorthand. Similarly, if $M = 4$, $p_{r\to 12^c3^c4}$ denotes the probability that a packet sent by r is received by d_1 and d_4 but not by d_2 and d_3.

Each (s_i, d_i) session has $n \cdot R_i$ symbols $X_{i,1}, \cdots, X_{i,nR_i}$ to transmit for some integer n and fractional-valued rate R_i. All sources s_i and the relay r have full knowledge of the PEC parameters before transmission. We limit ourselves to consider only 2-stage INC schemes. That is, in the first stage, each source s_i, $i \in [M]$ uses the 1-to-M PEC for n times and sends out n coded symbols generated from the information symbols[1] $X_{i,1}, \cdots, X_{i,nR_i}$, possibly in a non-linear fashion. Due to the randomness of PECs, some packets will arrive r and/or be overheard by d_j and some will not. In the beginning of the second stage, all destinations $\{d_i : i \in [M]\}$ report to the relay r the reception status of the packets sent in the first stage so that r is fully aware of the MSI available at individual d_i. The relay r then uses its 1-to-M PEC for n times and sends out coded packets generated from the packets received by r during the first stage, possibly in a non-linear fashion. No feedback is allowed in the middle of the second stage, which models the practical consideration that the "reception reports" may only be sent infrequently [5] since per-packet acknowledgement (ARQ) may induce too much interference.[2] After the second stage, each d_i computes the *decoded symbols* $\hat{\mathbf{X}}_i \triangleq (\hat{X}_{i,1}, \cdots, \hat{X}_{i,nR_i})$ based on all the symbols received by d_i. Assume that each symbol in $\mathbf{X}_i = (X_{i,1}, \cdots, X_{i,nR_i})$ is uniformly and independently distributed over $\mathsf{GF}(q)$ for all $i \in [M]$. We can then define the achievability of any 2-stage 1-hop INC scheme as follows.

Definition 1: A rate vector $(R_1, \cdots, R_M)$ is achievable if for any $\epsilon > 0$, there exist sufficiently large n and sufficiently large underlying finite field $\mathsf{GF}(q)$ such that $\forall i \in [M]$, $\mathsf{P}\left(\hat{\mathbf{X}}_i \neq \mathbf{X}_i\right) < \epsilon$.

III. Main Results

A. The Capacity Outer Bound

We first introduce a critical definition and a central proposition for deriving the capacity outer bound.

Definition 2: We use the convention that any function that is not decreasing is called *an increasing function*. With this convention, a function $f : [0, 1] \mapsto \mathbb{R}^+$ is a ZPPLCIF if $f(\cdot)$ is a Zero-Passing (i.e., $f(0) = 0$), Piecewise Linear, Concave, Increasing Function. Note that for the following, whenever we say a function is a ZPPLCIF, we automatically imply that its domain is $[0, 1]$ and its range is non-negative reals $\mathbb{R}^+$.

[1] Since we focus only on 1-hop INC at relay r, we required that each s_i can only transmit coded symbols generated from its own information packets. That is, no INC is allowed at s_i even if s_i may overhear some packets from s_j or when s_i and s_j are situated at the same physical node. On the other hand, we allow two destinations d_i and d_j to be situated at the same physical node, which is modelled by choosing the $p_{s_i\to\cdot}$ and $p_{r\to\cdot}$ parameters so that the success events for d_i and for d_j are perfectly correlated with each other.

[2] For $M = 2$, there is a similar work that allows per-packet ARQ [11].

Consider a 1-to-3 PEC with the input symbol denoted by $X \in \mathsf{GF}(q)$ and the output symbols denoted by Y_1, Y_2, and Y_3 in $\{X\} \cup \{*\}$, respectively. Define $p_1 \triangleq \mathsf{P}(Y_1 = X) = p_{X\to 123} + p_{X\to 123^c} + p_{X\to 12^c3} + p_{X\to 12^c3^c}$ as the marginal success probability that rx 1 receives X. Similarly, define p_2 and p_3 as the marginal success probability for rx 2 and rx 3, respectively. Without loss of generality, assume that $p_1 > p_2 > p_3$, which can be achieved by relabeling the receivers.

Proposition 1 (Concavity of Mutual Information): For any n, suppose we use the i.i.d. PEC n times and let $\mathbf{X}$, $\mathbf{Y}_1$, $\mathbf{Y}_2$, and $\mathbf{Y}_3$ denote the n-dimensional vectors of the input and output symbols, respectively. Let U denote an auxiliary random variable. For any distributions P_U and $P_{\mathbf{X}|U}$, we have

$$I(U;\mathbf{Y}_1) \geq I(U;\mathbf{Y}_2) \geq I(U;\mathbf{Y}_3) \quad (1)$$

$$I(U;\mathbf{Y}_2) \geq \frac{p_2-p_3}{p_1-p_3} I(U;\mathbf{Y}_1) + \frac{p_1-p_2}{p_1-p_3} I(U;\mathbf{Y}_3). \quad (2)$$

Proposition 1 implies the following. Consider a 1-to-M PEC with M different receivers $\mathbf{Y}_1$ to $\mathbf{Y}_M$. Without loss of generality, we add two auxiliary receivers $\mathbf{Y}_0$ and $\mathbf{Y}_{M+1}$ and assume that their marginals satisfy $p_0 = 1 > p_1 > \cdots > p_M > p_{M+1} = 0$. For any fixed coding scheme from U to $\mathbf{X}$, if we define $f : [0,1] \mapsto \mathbb{R}^+$ by $f(p_i) \triangleq I(U;\mathbf{Y}_i), \forall i \in \{0,\cdots,M+1\}$ and linearly interpolating the $f(p)$ value for other $p \neq p_i$, then $f(\cdot)$ is a ZPPLCIF.

Proposition 2: [Capacity Outer Bound] Let $p_{r;i}$ denote the marginal success probability from r to d_i. Without loss of generality, also assume that $p_{r;1} > p_{r;2} > \cdots > p_{r;M} > 0$. Let $p_{s_i;r}$ denote the marginal success probability from s_i to r. For simplicity, we slightly abuse the notation such that for any $T \in 2^{[M]}$, we define $p_{s_i;T}$ as the probability that a packet sent by s_i is received by *at least* one node in T. By convention, we define $p_{s_i;\emptyset} = 0$.

A rate vector $(R_1,\cdots,R_M)$ is feasible only if there exist 2^M ZPPLCIFs $\left\{f_S(\cdot) : \forall S \in 2^{[M]}\right\}$ such that

$$\forall i \in [M],\ R_i \leq p_{s_i;r} \quad (3)$$

$$\forall i \in [M], \forall S \subset T \in 2^{[M]} \text{ satisfying } |S|+1 = |T|,$$
$$f_S(p_{r;i}) \geq f_T(p_{r;i}) + 1_{\{i\leq k, k\in T\backslash S\}} (R_k - p_{s_k;S})^+ \quad (4)$$
$$\text{and } f_\emptyset(p_{r;M}) \leq p_{r;M}, \quad (5)$$

where $1_{\{\cdot\}}$ is the indicator function and $(\cdot)^+ \triangleq \max(\cdot,0)$.

Sketch of the proof: Let us focus on the 1-to-M broadcast PEC used by r in the second stage with MSI already created in the first stage. The key step of the proof is that for any given wireless 1-hop INC scheme, we set function $f_S(\cdot)$ as

$$f_S(p_{r;i}) \triangleq I(U;\mathbf{Y}_i|\mathbf{W}_S U_S),$$

where $U = \{X_{i,j} : \forall i \in [M], j \in [nR_i]\}$ denotes all the information messages, $\mathbf{Y}_i$ denotes the n-dimensional (r,d_i) PEC output vector, $\mathbf{W}_S$ is all the side information received by $\{d_i : \forall i \in S\}$ in the first stage, and $U_S = \{X_{i,j} : \forall i \in S, j \in [nR_i]\}$ is the information messages for session d_i, $i \in S$. By Proposition 1, $f_S(\cdot)$ must be a ZPPLCIF. With the above construction of $f_S(\cdot)$, the following simplified version of (4)

$$f_S(p_{r;k}) \geq f_T(p_{r;k}) + (R_k - p_{s_k;S})^+ \quad (6)$$

can be interpreted as follows. $(R_k - p_{s_k;S})^+$ quantifies the amount of those $X_{k,j}$, $j \in [nR_k]$ that are not received by any d_i, $i \in S$. Therefore, those packets must be decodable based on the mutual information $f_S(p_{r;k})$. However, $f_T(p_{r;k}) = I(U;\mathbf{Y}_k|\mathbf{W}_{S\cup\{k\}}U_{S\cup\{k\}})$ can also be viewed as the amount of interference (from d_k's perspective) that is caused by those (s,d_i) sessions with $i \notin (S\cup\{k\})$, but is not *cancelable* by the side information $\mathbf{W}_{\{k\}}$ received by d_k. As a result, by Fano's inequality, decodability implies that (6) must hold.

Since $f_\emptyset(p_{r;M}) = I(U;\mathbf{Y}_M)$, (5) is simply a restatement that the total mutual information at d_M cannot exceed the total number of successfully received time slots. (3) quantifies the maximal rate that s_i can covey to the relay node r. A detailed proof can be completed following the above reasoning. ■

Proposition 3: The capacity outer bound in Proposition 2 is indeed the capacity region for the cases of $M \leq 3$.

Proposition 3 generalizes the results in [12], which considers only $M = 2$. Proposition 2 can be viewed as generalization of the index-coding outer bound [10] for the wireless setting.

B. The Capacity Inner Bound For General M Values

We need the following new notation when describing the capacity inner bound. For any arbitrary permutation π on $[M]$, $\pi : [M] \mapsto [M]$, we define $(M+1)$ sets S_j^π for $j \in [M+1]$ such that for each j, we have $S_j^\pi \triangleq \{\pi(j') : j' = j,\cdots,M\}$. By definition, S_1^π and S_{M+1}^π are $[M]$ and $\emptyset$, respectively. Note that there are $M!$ different distinct permutations π on $[M]$. For the capacity inner bound, we consider $(M+1)!$ ZPPLCIFs, denoted by $\left\{f_{\pi,S_j^\pi}(\cdot) : \forall\pi, \forall j \in [M+1]\right\}$. For example, if $M = 2$, we have two permutations $\pi_1 = (1,2)$ and $\pi_2 = (2,1)$. Thus $S_1^{\pi_1} = \{1,2\}$, $S_2^{\pi_1} = \{2\}$, $S_3^{\pi_1} = \emptyset$; $S_1^{\pi_2} = \{1,2\}$, $S_2^{\pi_2} = \{1\}$, $S_3^{\pi_1} = \emptyset$. We thus consider 6 ZPPLCIFs $f_{\pi_1,\{1,2\}}$, $f_{\pi_1,\{2\}}$, $f_{\pi_1,\emptyset}$, $f_{\pi_2,\{1,2\}}$, $f_{\pi_2,\{1\}}$, and $f_{\pi_2,\emptyset}$.

Proposition 4 (Capacity Inner Bound): Following the same settings and definitions of Proposition 2, a rate vector $(R_1,\cdots,R_M)$ can be achieved by a *linear network code* if there exist $M!(2^{M+1}-M-2)$ non-negative variables:

$$\Big\{y_{k,T,(\pi,S_j^\pi)} \geq 0 : \forall k \in [M], \forall T \in 2^{[M]}, \forall\pi, \forall j \in [M+1],$$
$$\text{satisfying } k \notin S_j^\pi \text{ and } T \subseteq S_j^\pi\Big\}, \quad (7)$$

and $M2^{M-1}$ non-negative variables:

$$\Big\{z_{k,T} \geq 0 : \forall k \in [M], \forall T \in 2^{[M]}, \text{ satisfying } k \notin T\Big\}, \quad (8)$$

and $(M+1)!$ ZPPLCIFs $\left\{f_{\pi,S_j^\pi} : \forall\pi, \forall j \in [M+1]\right\}$, such that jointly the following inequalities are satisfied:

$$\forall i \in [M],\ R_i \leq p_{s_i;r} \quad (9)$$

$$\forall\pi, \forall i,j,k \in [M], \quad f_{\pi,S_{j+1}^\pi}(p_{r;i}) \geq f_{\pi,S_j^\pi}(p_{r;i})$$
$$+ 1_{\{i\leq k, k\notin S_{j+1}^\pi\}} \sum_{\{\forall T\in 2^{[M]} : T\subseteq S_{j+1}^\pi\}} y_{k,T,(\pi,S_{j+1}^\pi)} \quad (10)$$

$$\forall k \in [M], \forall T \in 2^{[M]}, k \notin T,$$
$$\sum_{\{\forall \pi, \forall j \in [M+1]: S_j^\pi \supseteq T, k \notin S_j^\pi\}} y_{k,T,(\pi,S_j^\pi)} \geq z_{k,T} \quad (11)$$
$$\forall k \in [M], \forall S \in 2^{[M]}, k \notin S,$$
$$\sum_{\{\forall T \in 2^{[M]}: T \supseteq S, k \notin T\}} z_{k,T} \geq (R_k - p_{s_k;S})^+ \quad (12)$$
$$\sum_{\forall \pi} f_{\pi,\emptyset}(p_{r;M}) \leq p_{r;M}. \quad (13)$$

C. Casting the Outer and Inner Bounds As Linear Programs

We close this section by showing how to convert the outer and inner bounds in Propositions 2 and 4 as a problem of determining whether a LP problem is feasible.

Lemma 1: For any ZPPLCIF $f(\cdot)$ that changes its slope only at M points: $(p_i, f(p_i))$ for $i \in [M]$ and $p_1 > \cdots > p_M$, there exist M non-negative values x_h for $h \in [M]$ such that

$$\forall p \in [0,1],\ f(p) = \sum_{h=1}^{M} x_h \cdot \min(p, p_h). \quad (14)$$

The converse is also true. That is, for any M non-negative values x_h, $\forall h \in [M]$, any $f(\cdot)$ satisfying (14) must be a ZPPLCIF that changes its slope only at $(p_i, f(p_i))$ for $i \in [M]$.

Take Proposition 4 for example. Since Proposition 4 focuses only on the values of $f_{\pi,S_j^\pi}(\cdot)$ at the points $p_{r;1}$ to $p_{r;M}$, the existence of such ZPPLCIFs in Proposition 4 is equivalent to the existence of ZPPLCIFs that change the slopes only at the points $p_{r;1}$ to $p_{r;M}$. Lemma 1 allows us to convert each ZPPLCIF in Proposition 4 into a set of M non-negative x variables. Checking the existence of such $(M+1)!$ ZPPLCIFs is thus equivalent to checking the existence of $M(M+1)!$ x variables such that jointly the x, y, and z variables satisfying the linear constraints in Proposition 4.

Remark: For any fixed M value, the complexity of checking the bounds does not depend on n, the length of the sessions, even though there are $\sum_{i=1}^{M} nR_i$ symbols, $X_{i,1}, \cdots, X_{i,nR_i}, i \in [M]$, to be coded together. The capacity region can thus be easily evaluated for small M, say $M \leq 6$.

IV. Alternative Settings

A. Direct Communication Between s_i and d_i

In a wireless network, it is possible that a packet sent by s_i may jump two hops and directly arrive d_i although the two-hop jump happens with a smaller probability than a 1-hop jump. The MORE protocol [13], [14] takes advantage of this spatial diversity and drastically improves the throughput even without INC. One natural question is thus what is the ultimate capacity when combining both the 2-hop jump and the INC at the relay node. Our results in Section III can be generalized to handle the combination of 2-hop jumps and INC.

Consider a new network model such that each s_i is associated with a 1-to-$(M+1)$ PEC that reaches $\{d_j : \forall j \in [M]\} \cup \{r\}$, namely, we allow direct communication from s_i to d_i (see Fig. 1(c)). The capacity of any 2-stage policy under the new model can bounded as follows.

Proposition 5 (Bounding the Cap. w. Direct Commun.): **Outer bound:** $\mathbf{R} = (R_1, \cdots, R_M)$ is feasible only if there exist two non-negative vectors $\mathbf{R}_{\text{direct}}$, the direct transmission rate, and $\mathbf{R}_{\text{relay}}$, the relay rate, such that $\forall i \in [M]$,

$$R_i = R_{\text{direct},i} + R_{\text{relay},i} \text{ and } R_{\text{direct},i} \leq p_{s_i;d_i}; \quad (15)$$

and $\mathbf{R}_{\text{relay}}$ satisfies the constraints in Proposition 2 with the PEC parameters of the canonical no-direct-path model being set as follows.

$$\forall i \in [M], \forall T \in 2^{[M]}, i \notin T,$$
$$p'_{s_i \to T^c([M]\backslash(T\cup\{i\}))r} = p_{s_i \to T^c i^c([M]\backslash(T\cup\{i\}))r} \quad (16)$$
$$p'_{s_i \to T^c([M]\backslash(T\cup\{i\}))r^c} = p_{s_i \to T^c i^c([M]\backslash(T\cup\{i\}))r^c} \quad (17)$$
$$\forall T \in 2^{[M]}, \quad p'_{r \to T^c([M]\backslash T)} = p_{r \to T^c([M]\backslash T)}. \quad (18)$$

Inner Bound: $\mathbf{R} = (R_1, \cdots, R_M)$ can be achieved by a linear network code if the sub-rate vectors $\mathbf{R}_{\text{direct}}$ and $\mathbf{R}_{\text{relay}}$ satisfy (15), and $\mathbf{R}_{\text{relay}}$ satisfies the constraints in Proposition 4 with the PEC parameter values being set according to (16) to (18).

B. Unequal Numbers of Time Slots for $\{s_i\}$ and r

Propositions 2 and 4 include as a special case the setting in which each s_i and r have different numbers of slots, n_i and n_r, for transmission. That is, given n_i and n_r we simply need to choose p_i, p_r, and n' such that $n_i = p_i \cdot n'$ and $n_r = p_r \cdot n'$, and consider a canonical setting with n' slots for each node, and new success probabilities of the s_i-PEC and r-PEC being scaled[3] by p_i and p_r, respectively: i.e., set $p'_{s_i;T} = p_i \cdot p_{s_i;T}$ and $p'_{r;i} = p_r \cdot p_{r;i}$ for all $i \in [M]$ and $T \in 2^{[M]}$.

By this generalization, one can dynamically allocate the available time slots among different nodes to further improve the throughput. This joint exploration of the 1-hop INC capacity and cross-layer optimization was first studied in [12].

V. Examples and Numerical Results

A. The Case of $M = 3$

We briefly demonstrate how to apply Proposition 2 to the case of $M = 3$ based on the canonical 1-hop setting with no direct communication between s_i and d_i. We have 8 ZPPLCIFs to consider: $f_S(\cdot), \forall S \in 2^{[M]}$. Using the techniques discussed in Section III-C, these functions correspond to 24 non-negative x variables: $x_{S,h}, \forall S \in 2^{[M]}, \forall h \in [M]$. For the main constraint (4), we have $M \cdot \sum_{k=0}^{M} k\binom{M}{k} = 36$ different combinations of (i, S, T) to consider. For example, for $(i, S, T) = (1, \{2\}, \{2,3\})$, we have

$$\begin{aligned} &x_{\{2\},1} \min(p_1, p_1) + x_{\{2\},2} \min(p_1, p_2) + x_{\{2\},3} \min(p_1, p_3) \\ &\geq x_{\{2,3\},1} \min(p_1, p_1) + x_{\{2,3\},2} \min(p_1, p_2) + x_{\{2,3\},3} \min(p_1, p_3) \\ &\quad + (R_3 - p_{3;2})^+, \end{aligned}$$

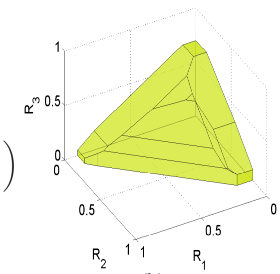
where we use p_i as shorthand for $p_{r;i}$ for all $i \in [M]$, respectively. Similarly, the other constraint (5) translates to

$$x_{\emptyset,1} \min(p_3, p_1) + x_{\emptyset,2} \min(p_3, p_2) + x_{\emptyset,3} \min(p_3, p_3) \leq p_3.$$

[3]The total sum of the probabilities is not one after scaling. Since all our derivations depend only on the relative magnitude instead of the absolute value, all the propositions hold verbatim.

$$
\begin{aligned}
&R_i \le p_{s_i;r},\ \forall i \in [M] \\
&R_1 \le p_{r;1} - \max\left((R_2 - p_{s_2;1})^+ + (R_3 - p_{s_3;\{1,2\}})^+, (R_2 - p_{s_2;\{1,3\}})^+ + (R_3 - p_{s_3;1})^+\right) \\
&R_2 \le p_{r;2} - \max\left(\frac{p_{r;2}}{p_{r;1}}\left((R_1 - p_{s_1;2})^+ + (R_3 - p_{s_3;\{1,2\}})^+\right), \frac{p_{r;2}}{p_{r;1}}(R_1 - p_{s_1;\{2,3\}})^+ + (R_3 - p_{s_3;2})^+,\right. \\
&\qquad \left.\frac{p_{r;1} - p_{r;2}}{p_{r;1} - p_{r;3}}\left(\frac{p_{r;3}}{p_{r;1}}(R_1 - p_{s_1;\{2,3\}})^+ + (R_3 - p_{s_3;2})^+\right) + \frac{p_{r;2} - p_{r;3}}{p_{r;1} - p_{r;3}}\left((R_1 - p_{s_1;2})^+ + (R_3 - p_{s_3;\{1,2\}})^+\right)\right) \\
&R_3 \le p_{r;3} - \max\left(\frac{p_{r;3}}{p_{r;1}}\left((R_1 - p_{s_1;3})^+ + (R_2 - p_{s_2;\{1,3\}})^+\right), \frac{p_{r;3}}{p_{r;1}}(R_1 - p_{s_1;\{2,3\}})^+ + \frac{p_{r;3}}{p_{r;2}}(R_2 - p_{s_2;3})^+\right).
\end{aligned}
$$

(a)



(b)

Fig. 2. (a) The capacity region of 3-user 1-hop INC. Without loss of generality, we assume $p_{r;1} > p_{r;2} > p_{r;3}$ by relabeling the three users. (b) Illustration of an example capacity region.

Constraint (3) remains unchanged. Totally, we have $3 + 36 + 1$ constraints on 3 rate variables $R_i \ge 0$ and 24 auxiliary variables $x_{S,h} \ge 0$. If preferred, we can also combine several linear constraints to remove the auxiliary variables $x_{S,h}$. The end results become an equivalent set of linear constraints involving only R_i, which is represented in Fig. 2(a). Fig. 2(b) illustrates one example[4] of the 3-D capacity region of (R_1, R_2, R_3). As seen in Fig. 2(b), the capacity region of $M = 3$ is non-trivial and contains many new facets that are governed by the linear inequalities in Fig. 2(a).

B. Numerical Results for General $M \ge 4$

For general $M \ge 4$, we use numerical experiments to evaluate the tightness of the outer and the inner bounds. To that end, we first assume the independence between receivers so that we can specify the PECs by their marginal success probabilities. We then assume that $p_{s_i;r} = 1$ for all $i \in [M]$ and assume that all other $p_{s_i;j}$ are independently and uniformly distributed over $[0, 1]$. The $p_{r;i}$ are chosen by sorting M independent uniform random variables on $[0, 1]$ in the descending order. To capture the M-dimensional capacity *region*, we first choose a search direction $\vec{v} = (v_1, \cdots, v_M)$ uniformly randomly from an M-dimensional unit ball. With the chosen values of $p_{\cdot;\cdot}$ and $\vec{v}$, we use a LP solver to find the largest t_{outer} such that $(R_1, \cdots, R_M) = (v_1 \cdot t_{\text{outer}}, \cdots, v_M \cdot t_{\text{outer}})$ satisfies Proposition 2, and find the largest t_{inner} such that $(R_1, \cdots, R_M) = (v_1 \cdot t_{\text{inner}}, \cdots, v_M \cdot t_{\text{inner}})$ satisfies Proposition 4. The deficiency is then defined as defi $\triangleq \frac{t_{\text{outer}} - t_{\text{inner}}}{t_{\text{outer}}}$. We then repeat the above experiment for 10^5 times. For both the cases of $M = 4$ and 5, defi is less than 1% for at least 99.4% of the times. This shows the effectiveness of the proposed outer and inner bound pair from a practical perspective.

VI. Conclusion

In this work, we have studied the capacity of 2-stage wireless intersession network coding schemes over a 1-hop neighborhood modelled by packet erasure channels with M coexisting unicast sessions. The capacity for $M = 3$ has been fully characterized. For $M \ge 4$, the capacity has been bounded by a new pair of outer and inner bounds, the tightness of which is within 1% for at least 99.4% of the times for $M \in \{4, 5\}$.

[4]For $M = 3$, each PEC has 2^3 parameters and there are totally $(M + 1) \times 2^M = 32$ such parameters. The exact values of the 32 parameters are omitted due to the limit of space.

Acknowledgment

This work was supported in parts by NSF grants CCF-0845968 and CNS-0905331.

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