General Capacity Region For The Fully-Connected 3-node Packet Erasure Network

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***Abstract*—This work considers the fully-connected 3-node packet erasure network: For each time slot, with some probabilities a packet sent by any node i may be received by both of the other nodes j and k; received only by node j (or node k); or received by neither nodes. Interference is avoided by enforcing that at most one node can transmit in each time slot. We assume that node i can always reach node j, possibly with the help of the third node k, for any $i \neq j$ pairs (thus the term fully-connected). One example of this model is any Wi-Fi network with 3 nodes within the hearing range of each other.**

We consider the most general traffic demands. Namely, there are six private-information flows with rates $(R_{1\to2}, R_{1\to3}, R_{2\to1}, R_{2\to3}, R_{3\to1}, R_{3\to2})$, respectively, and three common-information flows with rates $(R_{1\to23}, R_{2\to31}, R_{3\to12})$, respectively. We characterize the 9-dimensional Shannon capacity region within a gap that is inversely proportional to the packet size (bits). The gap can be attributed to exchanging reception status (ACK) and can be further reduced to 0 if we allow ACK to be transmitted via a separate control channel. For normal-sized packets, say 12000 bits, our results have thus effectively characterized the capacity region. Technical contributions of this work include a new converse for many-to-many network communications and a new capacity-approaching simple linear network coding scheme.

***Index Terms*—Packet Erasure Networks, Channel Capacity, Network Coding**

I. Introduction

Recently, Linear Network Coding (LNC) has emerged as a promising technique in the Network Information Theory (NIT) developments. For the single-multicast traffic, LNC strictly outperforms non-coding solutions and can achieve the capacity for *error-free networks* and *random erasure networks* [1]. Recent wireless testbeds [2] have also demonstrated 50-200% LNC throughput gain over the traditional 802.11 protocols.

Despite the above promising results, our NIT understanding is still nascent for networks with general traffic patterns. Shannon [3] first provided an inner and an outer bounds for the 2-node network with co-existing information flows of opposite directions. The Shannon's setting was later generalized under the names of the 3-terminal communication channels [4] and the discrete memoryless network channels [5].

There are at least two difficulties when characterizing the network capacity. Firstly, the information transfer from node A to node B may alter the channel of another transmission. For example, due to lack of full-duplex hardware, transmission from B to A may be impossible when A is sending information to B. Such a dependence among the point-to-point channels within a network was succinctly characterized by the 2-way model in [3]. Secondly, if there are multiple co-existing flows in a multi-hop network, then each node sometimes has to assume different roles (say, being a sender and/or being a relay) simultaneously. An optimal solution thus needs to balance the roles of each node either through scheduling [6] or through ingenious ways of coding and cooperation [5], [7]. Due to the inherent hardness, the network capacity region is known only for some scenarios, most of which involve only 1-hop transmissions, say broadcast or multiple access channels, and/or with a small number of co-existing flows in parallel directions (i.e., flows not forming cycles).

In this work, we study the 3-node network, Fig. 1(a), with the most general traffic requirements. Namely, there are six co-existing private-information flows with rates $(R_{1\to2}, R_{1\to3}, R_{2\to1}, R_{2\to3}, R_{3\to1}, R_{3\to2})$, respectively, in all possible directions; and there are three co-existing common-information flows with rates $(R_{1\to23}, R_{2\to31}, R_{3\to12})$, respectively, from a node to the other two nodes. We are interested in characterizing the corresponding 9-dimensional Shannon capacity region. To simplify the analysis, we consider a simple but non-trivial noisy channel model, the random packet erasure network (PEN). That is, each node is associated with its own broadcast packet erasure channel (PEC) such that each node can choose a symbol $X \in \mathbb{F}_q$ from some finite field $\mathbb{F}_q$, transmits X, and a random subset of the other two nodes will receive the packet, see Fig. 1(b). The symbol X is sometimes called a packet of size $\log_2(q)$ bits. We assume time-sharing among all three nodes so that interference is fully avoided. In this way, we can concentrate on the topological effects and the broadcast-channel diversity gain within the network.

We assume one of the following two scenarios. **Scenario 1**: Motivated by the throughput benefit of the causal packet ACKnowledgment feedback for erasure networks [6], [8]–[11], we assume that the reception status is causally available to the entire network through a separate control channel for free. Such assumption can be justified by the fact that the length of ACK/NACK is 1 bit, much smaller than the size of a regular packet. **Scenario 2**: We assume that any ACK signal, if there is any, has to be sent through the regular forward channels along with information messages. Thus, any achievability scheme needs to balance the amount of information and control messages and decides *when* and *how* to send them. In Scenario 2, we further assume that node i can always reach node j, possibly with the help of the third node k, for any $i \neq j$ (thus the term *fully-connected*). Note that *the fully-connectedness is assumed only in Scenario 2*. When the casual ACK is available for free (Scenario 1), our results do not need the fully-connectedness assumption.

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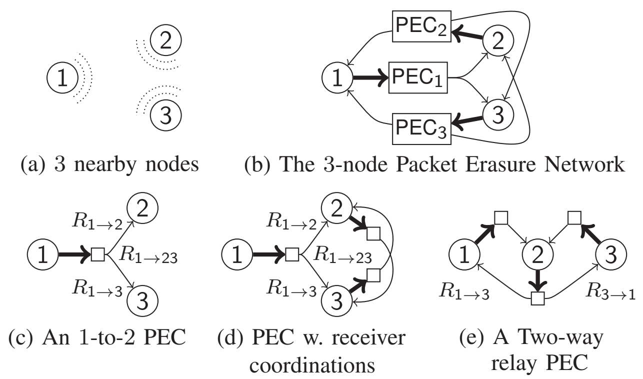

(a) 3 nearby nodes (b) The 3-node Packet Erasure Network

(c) An 1-to-2 PEC (d) PEC w. receiver coordinations (e) A Two-way relay PEC

Fig. 1. Illustrations of 3-node Packet Erasure Network (PEN) and its variants.

Contributions: For Scenario 1, we characterize the exact 9-dim. Shannon capacity region. For Scenario 2, the capacity region is characterized with a gap inversely proportional to $\log_2(q)$. The technical contributions include a new converse for many-to-many network communications and a new capacity-approaching simple LNC scheme.

It is worth noting that the considered 3-node PEN contains many important practical scenarios as sub-cases. **Example 1:** If we set broadcast PECs of nodes 2 and 3 to be always erasure (i.e., neither nodes can transmit anything), then Fig. 1(b) collapses to Fig. 1(c), the 2-receiver broadcast PEC. The capacity region $(R_{1\to 2}, R_{1\to 3}, R_{1\to 23})$ derived in our Scenario 1 is identical to the existing results in [8]. **Example 2:** Instead of setting the PECs of nodes 2 and 3 to all erasure, we set $R_{2\to 1}$, $R_{2\to 3}$, $R_{3\to 1}$, $R_{3\to 2}$, $R_{2\to 31}$, $R_{3\to 12}$ to be zeros. In this case, node 2 can still potentially help relay the packets destined for node 3 and vice versa, see Fig. 1(d). This work then characterizes the Shannon capacity[1] $(R_{1\to 2}, R_{1\to 3}, R_{1\to 23})$ of a broadcast PEC with receiver cooperation. **Example 3:** If we set $R_{1\to 2}$, $R_{2\to 1}$, $R_{2\to 3}$, $R_{3\to 2}$, $R_{1\to 23}$, $R_{2\to 31}$, $R_{3\to 12}$ to be zeros and prohibit any direct communication between nodes 1 and 3, Fig. 1(b) now collapses to Fig. 1(e), in which node 2 is a two-way relay for unicast flows $1\to 3$ and $3\to 1$. The results in this work thus characterizes the Shannon capacity region $(R_{1\to 3}, R_{3\to 1})$ of this two-way relay network Fig. 1(e).

Summary: Most existing works on PENs studied either ≤ 2 flows [2], [6], [8], [10], [11] or all flows originating from the same node [6], [8], [9], [11]–[13]. By characterizing the most general 9-dim. capacity, this work significantly improves our understanding for communications over 3-node PENs.

II. Problem Formulation

Throughout this work, we use (i, j, k) to represent one of three cyclically shifted tuples $\{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$. We define the 9-dimensional rate vector $\vec{R} \triangleq (R_{1\to 2}, R_{1\to 3}, R_{2\to 1}, R_{2\to 3}, R_{3\to 1}, R_{3\to 2}, R_{1\to 23}, R_{2\to 31}, R_{3\to 12})$.

Assume time-slotted transmissions. Within a total budget of n time slots, node i would like to send $nR_{i\to h}$ packets (private-information messages), denoted by a row vector $\mathbf{W}_{i\to h}$, to node $h \neq i$, and would like to send $nR_{i\to jk}$ packets (common-information messages), denoted by a row vector $\mathbf{W}_{i\to jk}$, to the other two nodes simultaneously. Each packet is chosen independently and uniformly randomly from $\mathbb{F}_q$.

For any time slot $t \in \{1, \cdots, n\}$, define the *channel output vector* $\mathbf{Z}(t) \triangleq (Z_{1\to 2}(t), Z_{1\to 3}(t), Z_{2\to 1}(t), Z_{2\to 3}(t), Z_{3\to 1}(t), Z_{3\to 2}(t)) \in \{1, \varepsilon\}^6$, where $Z_{i\to h}(t) = 1$ and ε represents whether node h can receive the transmission from node i or not, respectively. For any node i, the corresponding scheduling decision $\sigma_i(t) = 1$ represents that node i decides to transmit at time t and $\sigma_i(t) = 0$ represents not transmitting. We assume that any transmission is completely destroyed if there are ≥ 2 nodes transmitting simultaneously. For example, suppose node i decides to transmit a packet $X_i(t) \in \mathbb{F}_q$ in time t (thus $\sigma_i(t) = 1$). Then, only when $Z_{i\to h}(t) = 1$ and $\sigma_j(t) = \sigma_k(t) = 0$ will node h receive $Y_{i\to h}(t) = X_i(t)$. In all other cases, node h receives an erasure $Y_{i\to h}(t) = \varepsilon$. To highlight this interference model, we sometimes write

$$Y_{i\to h}(t) = X_i(t) \circ Z_{i\to h}(t) \circ 1_{\{\sigma_i(t)=1, \sigma_j(t)=\sigma_k(t)=0\}}. \quad (1)$$

We further assume that the 3-node PEN is memoryless and stationary,[2] i.e., we allow arbitrary joint distribution for the 6 coordinates of $\mathbf{Z}(t)$ but assume that $\mathbf{Z}(t)$ is i.i.d. over the time axis t. We use $p_{i\to jk} \triangleq \mathsf{Prob}(Z_{i\to j}(t) = 1, Z_{i\to k}(t) = 1)$ to denote the probability that $X_i(t)$ is successfully received by both nodes j and k; and use $p_{i\to j\overline{k}}$ to denote the probability $\mathsf{Prob}(Z_{i\to j}(t) = 1,\ Z_{i\to k}(t) = \varepsilon)$ that $X_i(t)$ is received by node j but not by node k. Probability $p_{i\to \overline{j}k}$ is defined symmetrically. Define $p_{i\to j\vee k} \triangleq p_{i\to \overline{j}k} + p_{i\to jk} + p_{i\to j\overline{k}}$ as the probability that at least one of nodes j and k receives the packet, and define $p_{i\to j} \triangleq p_{i\to jk} + p_{i\to j\overline{k}}$ (resp. $p_{i\to k}$) as the marginal reception probability from node i to node j (resp. node k). We also assume that the random process $\{\mathbf{Z}(t) : \forall t\}$ is independent of any information messages.

Define $\mathbf{W}_{i*} \triangleq \mathbf{W}_{i\to j} \cup \mathbf{W}_{i\to k} \cup \mathbf{W}_{i\to jk}$ as the collection of all messages originated from node i. Similarly, define $\mathbf{W}_{*i} \triangleq \mathbf{W}_{j\to i} \cup \mathbf{W}_{j\to ki} \cup \mathbf{W}_{k\to i} \cup \mathbf{W}_{k\to ij}$ as the collection of all messages intended to node i. Sometimes we slightly abuse the notation and define $\mathbf{W}_{\{i,j\}*} \triangleq \mathbf{W}_{i*} \cup \mathbf{W}_{j*}$ as the collection of messages originated from both nodes i and j. Similar notation can be used to define $\mathbf{Y}_{*i}(t) \triangleq \{Y_{j\to i}(t), Y_{k\to i}(t)\}$ and $\mathbf{Y}_{i*}(t) \triangleq \{Y_{i\to j}(t), Y_{i\to k}(t)\}$ as the collection of all symbols received and transmitted by node i during time t, respectively. We also use brackets $[\cdot]_1^t$ to denote the collection from time 1 to t. For example, $[\mathbf{Y}_{*i}, \mathbf{Z}]_1^{t-1}$ is shorthand for the collection $\{Y_{j\to i}(\tau), Y_{k\to i}(\tau), \mathbf{Z}(\tau) : \forall \tau \in \{1, \cdots, t-1\}\}$.

Recall that causal ACK can be transmitted for free in Scenario 1 but has to go through the forward channel when in Scenario 2. We first focus on the formulation of Scenario 2.

Given any $\vec{R}$, a communication scheme is described by $3n$ scheduling functions: $\forall t \in \{1, \cdots, n\}$ and $\forall i \in \{1, 2, 3\}$,

$$\sigma_i(t) = f^{(t)}_{\mathsf{SCH},\, i}([\mathbf{Y}_{*i}]_1^{t-1}); \quad (2)$$

plus $3n$ encoding functions: $\forall t \in \{1, \cdots, n\}$ and $\forall i \in \{1, 2, 3\}$,

$$X_i(t) = f^{(t)}_i(\mathbf{W}_{i*}, [\mathbf{Y}_{*i}]_1^{t-1}); \quad (3)$$

[1] In [6], the LNC capacity is characterized instead of the most general Shannon capacity.

[2] The 3-node PEN is a special case of the discrete memoryless network [5].

plus 3 decoding functions: $\forall\, i \in \{1,2,3\}$,

$$\hat{\mathbf{W}}_{*i} = g_i(\mathbf{W}_{i*}, [\mathbf{Y}_{*i}]_1^n). \tag{4}$$

To refrain from using the timing-channel[3] techniques [14], we also require the following equality

$$I([\sigma_1, \sigma_2, \sigma_3]_1^n\,;\, \mathbf{W}_{\{1,2,3\}*}) = 0, \tag{5}$$

where $I(\cdot\,;\,\cdot)$ is the mutual information.

Namely, at every time t, each node decides whether it wants to transmit or not based on what it has received in the past, see (2). Note that the received symbols $[\mathbf{Y}_{*i}]_1^{t-1}$ may contain both the message information and the control information. (5) ensures that only the control information is used to determine $\sigma_i(t)$ and thus the messages $\mathbf{W}$ cannot be sent through the timing of the scheduling.[4] Once each node decides whether to transmit or not,[5] it encodes $X_i(t)$ based on its information messages and what it has received from other nodes, see (3). In the end of time n, each node decodes its desired packets based on its messages and what it has received, see (4).

Definition 1: Fix the distribution of $\mathbf{Z}(t)$ and finite field $\mathbb{F}_q$. A 9-dimensional rate vector $\vec{R}$ is achievable if for any $\epsilon > 0$ there exists a communication scheme with sufficiently large n such that $\mathsf{Prob}(\hat{\mathbf{W}}_{*i} \neq \mathbf{W}_{*i}) < \epsilon$ for all $i \in \{1,2,3\}$. The capacity region is the closure of all achievable $\vec{R}$.

A. Comparison between Scenarios 1 and 2

The difference between Scenarios 1 and 2 is that the former allows the use of causal ACK for free. As a result, for Scenario 1, we simply need to insert the *causal* network-wide channel status information $[\mathbf{Z}]_1^{t-1}$ in the input arguments of (2) and (3), respectively; and insert the *overall* channel status information $[\mathbf{Z}]_1^n$ in (4). The formulation of Scenario 1 thus becomes as follows: $\forall\, t \in \{1,\cdots,n\}$ and $\forall\, i \in \{1,2,3\}$,

$$\sigma_i(t) = \overline{f}^{(t)}_{\mathsf{SCH},\,i}([\mathbf{Y}_{*i}, \mathbf{Z}]_1^{t-1}), \tag{6}$$

$$X_i(t) = \overline{f}^{(t)}_i(\mathbf{W}_{i*}, [\mathbf{Y}_{*i}, \mathbf{Z}]_1^{t-1}), \tag{7}$$

$$\text{and } \hat{\mathbf{W}}_{*i} = \overline{g}_i(\mathbf{W}_{i*}, [\mathbf{Y}_{*i}, \mathbf{Z}]_1^n), \tag{8}$$

while we still impose no-timing channel information (5). Obviously, with more information to use, the capacity region under Scenario 1 is a superset of that of Scenario 2, which is why we use overlines in the above function descriptions.

Following this observation, we will outer bound the (larger) capacity of Scenario 1 and inner bound the (smaller) capacity of Scenario 2.

III. Main Results

A. Capacity outer bound of 3-node Packet Erasure Network

Proposition 1: For any fixed $\mathbb{F}_q$, a 9-dimensional $\vec{R}$ is achievable under[6] Scenario 1 only if there exist 3 non-negative variables $\tau^{(i)}$ for all $i \in \{1,2,3\}$ such that jointly they satisfy the following three groups of linear inequalities:

- Group 1, termed the *time-sharing condition*, has 1 inequality:

$$\sum_{\forall\, i \in \{1,2,3\}} \tau^{(i)} \leq 1. \tag{9}$$

- Group 2, termed the *broadcast cut-set condition*, has 3 inequalities: For all $i \in \{1,2,3\}$,

$$R_{i\to j} + R_{i\to k} + R_{i\to jk} \leq \tau^{(i)} p_{i\to j\vee k}. \tag{10}$$

- Group 3, termed the *3-way multiple-access cut-set condition*, has 3 inequalities: For all $i \in \{1,2,3\}$,

$$\begin{aligned} R_{j\to i} + R_{j\to ki} + R_{k\to i} + R_{k\to ij} &\leq \tau^{(j)} p_{j\to i} + \tau^{(k)} p_{k\to i} \\ &\quad - \left(\frac{p_{j\to i}}{p_{j\to k\vee i}} R_{j\to k} + \frac{p_{k\to i}}{p_{k\to i\vee j}} R_{k\to j} \right). \end{aligned} \tag{11}$$

The proof of Proposition 1 is derived by entropy-based analysis and thus considers arbitrary, possibly non-linear schemes in (5) to (8). It includes the results in [6], [8] as special cases.

The intuition is as follows. Each variable $\tau^{(i)}$ counts the expected frequency (normalized over the time budget n) that node i is scheduled for transmission. As a result, (9) holds naturally. (10) is a simple cut-set condition for broadcasting from node i. One main contribution of this work is the derivation of the new 3-way multiple-access outer bound in (11). The LHS of (11) contains all the information destined for node i. The term $\tau^{(j)} p_{j\to i} + \tau^{(k)} p_{k\to i}$ on the RHS of (11) is the amount of time slots that either node j or node k can communicate with node i. As a result, it resembles a multiple-access cut condition of a typical cut-set argument. What is special in our setting is that, since node j may have some private-information for node k and vice versa, sending those private-information has a penalty on the multiple access channel from nodes $\{j,k\}$ to node i. The remaining term on the RHS of (11) quantifies such penalty that is inevitable regardless of what kind of coding schemes being used. The detailed proof is omitted due to space limit.

B. A Linear Network Coding Capacity Achieving Scheme

We first focus on Scenario 2, where we assume that node i can always reach node j, possibly with the help of the third node k, for any $i \neq j$ pairs, thus the term fully-connected 3-node PEN. To formalize our discussion, we define

Definition 2: In Scenario 2, we assume the 3-node PEN is *fully-connected* in the sense that the given channel reception probabilities satisfy either $p_{i_1\to i_2} > 0$ or $\min(p_{i_1\to i_3}, p_{i_3\to i_2}) > 0$ for all distinct $i_1, i_2, i_3 \in \{1,2,3\}$.

Namely, node i_1 must be able to communicate with node i_2 either through direct communication (i.e., $p_{i_1\to i_2} > 0$) or through relaying (i.e., $\min(p_{i_1\to i_3}, p_{i_3\to i_2}) > 0$). We also need the following new math operator.

[3] One justification is that the rate of the timing channel is at most 3 bits per slot, which is negligible compared to a normal packet size 12000 bits.

[4] E.g., one (not necessarily optimal) way to encode is to divide a packet $X_i(t)$ into the header and the payload. The messages $\mathbf{W}_{i*}$ will be embedded in the payload while the header contains control information such as ACK. If this is indeed the way we encode, then (5) requires that scheduling depend only on the control information in the header, not the messages in the payload.

[5] If two nodes i and j want to transmit simultaneously, then our channel model (1) automatically leads to full collision and erases both transmissions.

[6] Proposition 1 is naturally an outer bound for Scenario 2, see Section II-A.

Definition 3: For any 2 non-negative values a and b, the operator $\mathsf{nzmin}\{a,b\}$, standing for non-zero minimum, is defined as:

$$\mathsf{nzmin}\{a,b\} = \begin{cases} \max(a,b) & \text{if } \min(a,b) = 0, \\ \min(a,b) & \text{if } \min(a,b) \neq 0, \end{cases}$$

i.e., $\mathsf{nzmin}\{a,b\}$ is the minimum of the strictly positive entries.

Proposition 2: For any fixed $\mathbb{F}_q$, a 9-dimensional $\vec{R}$ is LNC-achievable in Scenario 2 if there exist 15 non-negative variables $t^{(i)}_{[\mathsf{u}]}$ and $\{t^{(i)}_{[\mathsf{c},l]}\}_{l=1}^{4}$ for all $i \in \{1,2,3\}$ such that jointly they satisfy the following three groups of linear conditions:

- Group 1, termed the *time-sharing condition*, has 1 inequality:

$$\sum_{\forall i \in \{1,2,3\}} t^{(i)}_{[\mathsf{u}]} + t^{(i)}_{[\mathsf{c},1]} + t^{(i)}_{[\mathsf{c},2]} + t^{(i)}_{[\mathsf{c},3]} + t^{(i)}_{[\mathsf{c},4]} \leq 1 - t_{\mathsf{FB}}, \quad (12)$$

where t_{FB} is a constant defined as

$$t_{\mathsf{FB}} \triangleq \sum_{\forall i \in \{1,2,3\}} \frac{3}{\log_2(q) \cdot \mathsf{nzmin}\{p_{i\to j}, p_{i\to k}\}}. \quad (13)$$

- Group 2 has 3 inequalities: For all $(i,j,k) \in \{(1,2,3), (2,3,1), (3,1,2)\}$,

$$R_{i\to j} + R_{i\to k} + R_{i\to jk} < t^{(i)}_{[\mathsf{u}]} p_{i\to j\vee k}. \quad (14)$$

- Group 3 has 6 inequalities: For all $(i,j,k) \in \{(1,2,3), (2,3,1), (3,1,2)\}$,

$$\begin{aligned} \Big(R_{i\to j} + R_{i\to jk}\Big)\frac{p_{i\to \bar{j}k}}{p_{i\to j\vee k}} < \Big(t^{(i)}_{[\mathsf{c},1]} + t^{(i)}_{[\mathsf{c},3]}\Big) \cdot p_{i\to j} \\ + \Big(t^{(k)}_{[\mathsf{c},2]} + t^{(k)}_{[\mathsf{c},3]}\Big) \cdot p_{k\to j}, \end{aligned} \quad (15)$$

$$\begin{aligned} \Big(R_{i\to k} + R_{i\to jk}\Big)\frac{p_{i\to j\bar{k}}}{p_{i\to j\vee k}} < \Big(t^{(i)}_{[\mathsf{c},1]} + t^{(i)}_{[\mathsf{c},4]}\Big) \cdot p_{i\to k} + \\ + \Big(t^{(j)}_{[\mathsf{c},2]} + t^{(j)}_{[\mathsf{c},4]}\Big) \cdot p_{j\to k}. \end{aligned} \quad (16)$$

Proposition 3: Continue from Proposition 2, if we focus on Scenario 1 instead, then $\vec{R}$ is LNC-achievable if there exist 15 non-negative variables $t^{(i)}_{[\mathsf{u}]}$ and $\{t^{(i)}_{[\mathsf{c},l]}\}_{l=1}^{4}$ for all $i \in \{1,2,3\}$ such that (12), (14) to (16) hold while we set $t_{\mathsf{FB}} = 0$ in (13).

In short, the constant term t_{FB} in (13) quantifies the overhead of sending ACK through the forward erasure channel in Scenario 2 and can be set to 0 in Scenario 1. We now characterize the capacity for the 3-node PEN under Scenario 1.

Proposition 4: The outer bound in Proposition 1 and the achievable region in Proposition 3 match for all the possible channel parameters $\{p_{i\to jk}, p_{i\to \bar{j}k}, p_{i\to j\bar{k}}\}$. They thus describe the corresponding 9-dimensional Shannon capacity region.

The proof of Proposition 4 is by pure algebraic arguments and thus omitted due to space limit.

From the above discussions, one can see that even for the more practical Scenario 2, in which there is no dedicated feedback control channels, Proposition 2 is indeed capacity-approaching when the 3-node PEN is fully-connected. The gap to the outer bound is inversely proportional to $\log_2(q)$ and diminishes to zero if the packet size q is large enough.

In the following, a sketch of the proof of Proposition 3 (Scenario 1) is explained while the detailed construction for Proposition 2 (Scenario 2) is omitted due to space limit.

Sketch of Proposition 3: We only provide the so-called *first-order analysis* for the achievability of a LNC solution.

We assume that all nodes know the channel reception probabilities, the total time budget n, and the rate vector $\vec{R}$ they want to achieve in the beginning of time 0. As a result, each node can compute the same 15 non-negative values $t^{(i)}_{[\mathsf{u}]}$ and $\{t^{(i)}_{[\mathsf{c},l]}\}_{l=1}^{4}$ for all $i \in \{1,2,3\}$ satisfying Proposition 3.

We propose the following 2-stage scheme. Stage 1: Each node, say node i, has $n(R_{i\to j} + R_{i\to k} + R_{i\to jk})$ unicasts and multicast packets (i.e., $\mathbf{W}_{i*}$) that need to be sent to other nodes j and k. Assume that those packets are ordered in group-wise as $\mathbf{W}_{i\to j}$, $\mathbf{W}_{i\to k}$, and then $\mathbf{W}_{i\to jk}$, and they are indexed by $l = 1$ to $n(R_{i\to j} + R_{i\to k} + R_{i\to jk})$. Then in the beginning of time 1, node 1 chooses the first packet (index 1) and repeatedly sends it uncodedly until at least one of nodes 2 and 3 receives it. Whether it is received or not can be known causally by network-wide feedbacks $\mathbf{Z}(t-1)$. Then node 1 picks the next indexed packet and repeat the same process until each of these $n(R_{1\to 2} + R_{1\to 3} + R_{1\to 23})$ packets is heard by at least one of nodes 2 and 3. By simple analysis, see [11], node 1 can finish the transmission in $nt^{(i)}_{[\mathsf{u}]}$ slots since (14).[7] We repeat this process for nodes 2 and 3, respectively. Stage 1 can be finished in $n(\sum_i t^{(i)}_{[\mathsf{u}]})$ slots.

After Stage 1, the status of all packets is summarized as follows. Each of $\mathbf{W}_{i\to j}$ packets is heard by at least one of nodes j and k. Those that have already been heard by node j, the intended destination, is delivered successfully and thus will not be considered for future operations. We denote those $\mathbf{W}_{i\to j}$ packets that are overheard by node k only (not by node j) as $\mathbf{W}^{(k)}_{i\to j}$. In average, there are $nR_{i\to j}\frac{p_{i\to \bar{j}k}}{p_{i\to j\vee k}}$ number of $\mathbf{W}^{(k)}_{i\to j}$ packets. Symmetrically, we also have $nR_{i\to k}\frac{p_{i\to j\bar{k}}}{p_{i\to j\vee k}}$ number of $\mathbf{W}^{(j)}_{i\to k}$ packets that were intended for node k but were overheard only by node j in Stage 1.

Similarly for the common-information packets $\mathbf{W}_{i\to jk}$, each packet was heard by at least one of nodes j and k during Stage 1. Those that have been heard by both nodes j and k, is delivered successfully and thus will not be considered in Stage 2. We similarly denote those $\mathbf{W}_{i\to jk}$ packets that are heard by node k only (not by node j) as $\mathbf{W}^{(k)}_{i\to jk}$. In average, there are $nR_{i\to jk}\frac{p_{i\to \bar{j}k}}{p_{i\to j\vee k}}$ number of $\mathbf{W}^{(k)}_{i\to jk}$ packets. Symmetrically, we also have $nR_{i\to jk}\frac{p_{i\to j\bar{k}}}{p_{i\to j\vee k}}$ number of $\mathbf{W}^{(j)}_{i\to jk}$ packets that were heard only by node j in Stage 1.

Stage 2 is the LNC phase, in which each node i will send a linear combination of overheard packets that it exactly knows as described above. We claim that there are (at least) 4 possible ways of sending LNC packets. That is, for each time t, node i send a linear combination $X_i(t) = [\tilde{W}_j + \tilde{W}_k]$ with 4 possible ways of choosing the individual packets $\tilde{W}_j$ and $\tilde{W}_k$:

$$[\mathsf{c},1]: \quad \tilde{W}_j \in \mathbf{W}^{(k)}_{i\to j} \cup \mathbf{W}^{(k)}_{i\to jk} \quad \text{and} \quad \tilde{W}_k \in \mathbf{W}^{(j)}_{i\to k} \cup \mathbf{W}^{(j)}_{i\to jk},$$

$$[\mathsf{c},2]: \quad \tilde{W}_j \in \mathbf{W}^{(i)}_{k\to j} \cup \mathbf{W}^{(i)}_{k\to ij} \quad \text{and} \quad \tilde{W}_k \in \mathbf{W}^{(i)}_{j\to k} \cup \mathbf{W}^{(i)}_{j\to ki},$$

$$[\mathsf{c},3]: \quad \tilde{W}_j \in \mathbf{W}^{(k)}_{i\to j} \cup \mathbf{W}^{(k)}_{i\to jk} \quad \text{and} \quad \tilde{W}_k \in \mathbf{W}^{(i)}_{j\to k} \cup \mathbf{W}^{(i)}_{j\to ki},$$

$$[\mathsf{c},4]: \quad \tilde{W}_j \in \mathbf{W}^{(i)}_{k\to j} \cup \mathbf{W}^{(i)}_{k\to ij} \quad \text{and} \quad \tilde{W}_k \in \mathbf{W}^{(j)}_{i\to k} \cup \mathbf{W}^{(j)}_{i\to jk}.$$

[7] By the law of large numbers, we can ignore the randomness of the events and treat them as deterministic when n is sufficiently large.

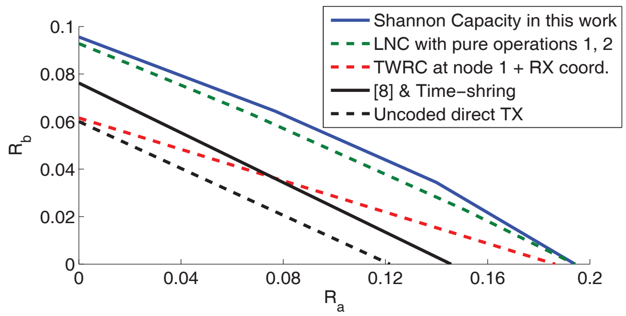

Fig. 2. Comparison of the capacity region with different achievable rates

To further explain this LNC stage, we observe that choice $[\mathrm{c}, 1]$ is the standard LNC operation for the 2-receiver broadcast channels [8] since node i sends a linear sum that benefits both nodes j and k simultaneously, i.e., the sum of two packets, each overheard by an undesired receiver. Choice $[\mathrm{c}, 2]$ is the standard LNC operation for the 2-way relay channels, since node i, as a relay for the 2-way traffic from $j \to k$ and from $k \to j$, respectively, mixes the packets from two opposite directions and sends their linear sum. Choices $[\mathrm{c}, 3]$ and $[\mathrm{c}, 4]$ are the new "hybrid" cases that are proposed in this work, for which we can mix part of the broadcast traffic and part of the 2-way traffic. One can easily prove that transmitting such a linear mixture again benefits both nodes simultaneously.

Since each node i has 4 possible coding choices, we perform coding choice $[\mathrm{c}, l]$ for exactly $n \cdot t^{(i)}_{[\mathrm{c}, l]}$ times for $l = 1$ to 4. Since $\mathbf{W}^{(k)}_{i\to j} \cup \mathbf{W}^{(k)}_{i\to jk}$ participates in coding choices $[\mathrm{c}, 1]$ and $[\mathrm{c}, 3]$ of node i and coding choices $[\mathrm{c}, 2]$ and $[\mathrm{c}, 3]$ of node k, (15) guarantees that we can finish sending all $\mathbf{W}^{(k)}_{i\to j} \cup \mathbf{W}^{(k)}_{i\to jk}$ packets and they will all successfully arrive at node j, the intended destination.[8] Symmetrically, (16) guarantees that we can finish sending all $\mathbf{W}^{(j)}_{i\to k} \cup \mathbf{W}^{(j)}_{i\to jk}$ packets to their intended destination node k in the end of Stage 2. Finally, (12) guarantees that we can finish Stages 1 and 2 in the allotted n time slots. The sketch of the proof is complete. □

C. Numerical Evaluation

Consider a 3-node network with marginal channel success probabilities $p_{1\to 2} = 0.35$, $p_{1\to 3} = 0.8$, $p_{2\to 1} = 0.6$, $p_{2\to 3} = 0.5$, $p_{3\to 1} = 0.3$, and $p_{3\to 2} = 0.75$, respectively. To illustrate the 9-dimensional capacity region, we assume that 3 flows are of the same rate $R_{1\to 2} = R_{1\to 3} = R_{1\to 23} = R_a$ and the other 6 flows are of rate $R_{2\to 1} = R_{2\to 3} = R_{3\to 1} = R_{3\to 2} = R_{2\to 31} = R_{3\to 12} = R_b$. Fig. 2 compares the Shannon capacity region of (R_a, R_b) with different achievability schemes.

The smallest achievable rate region is by simply performing uncoded direct transmission. The second achievability scheme combines the broadcast channel LNC in [8] with time-sharing among all three nodes. The third scheme performs two-way relay channel (TWRC) coding in node 1 for those $3 \to 2$ and $2 \to 3$ flows while allowing node 2 to relay the node 1's packets destined for node 3 and vice versa. The fourth scheme is the largest that can be derived from the existing LNC operations. Namely, we allow all three nodes to perform the broadcast-based LNC and/or TWRC-based LNC operations (coding choices $[\mathrm{c}, 1]$ and $[\mathrm{c}, 2]$ in Stage 2) but not the hybrid operations (coding choices $[\mathrm{c}, 3]$ and $[\mathrm{c}, 4]$) proposed in this work. One can see that the result is strictly suboptimal. It shows that the proposed hybrid operations are critical for achieving the Shannon capacity in Propositions 1 and 3.

[8]Those $\mathbf{W}^{(k)}_{i\to jk}$ packets are the common-information packets that are intended for both nodes j and k. However, since our definition of $\mathbf{W}^{(k)}_{i\to jk}$ counts only those that have already been received by node k only, we say herein their new intended destination is node j instead.

IV. Conclusion

This work studies the most general 9-dim. capacity region of the 3-node packet erasure network. The Shannon capacity has been exactly quantified when the casual ACK is available for free through a separate control channel. For the practical setting where control messages has to be sent through the regular forward channels, the proposed LNC scheme approaches the capacity as the packet size becomes large enough.

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