Concatenated Coding Using Linear Schemes for Gaussian Broadcast Channels with Noisy Channel Output Feedback

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Abstract

Linear coding schemes for the additive white Gaussian noise broadcast channel (AWGN-BC) with noiseless feedback have been the main choice of coding for such channels in the literature. The achievable rate regions of these schemes go well beyond the capacity region of the AWGN-BC without feedback. In this paper, a concatenated coding design for the K-user AWGN-BC with noisy feedback is proposed that relies on linear feedback schemes to achieve rate tuples outside the no-feedback capacity region. Specifically, a linear feedback code for the AWGN-BC with noisy feedback is used as an inner code that creates an effective single-user channel from the transmitter to each of the receivers, and then open-loop coding is used for coding over these single-user channels. An achievable rate region of linear feedback schemes for noiseless feedback is shown to be achievable by the concatenated coding scheme for sufficiently small feedback noise level. Then, a linear feedback coding scheme for the K-user symmetric AWGN-BC with noisy feedback is presented and optimized for use in the concatenated coding scheme.

Index Terms

Broadcast channel, noisy feedback, linear feedback, concatenated coding, network information theory.

I. Introduction

The demand for higher data rates in wireless communication systems continues to increase. However, there is concern that many of the popular approaches to physical layer design are only capable of minimal further enhancements [1]. The use of feedback in wireless communication systems has been mainly focused

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on channel state feedback and automatic repeat request (ARQ) feedback [2]. Both of the aforementioned feedback types can enhance the rate and reliability of a practical wireless communication system. However, channel output feedback, where the channel output is fed back from the receiver to the transmitter, has not been fully explored, especially for imperfect feedback.

The use of channel output feedback in Gaussian channels dates back to the seminal paper by Schakwijk and Kailath (S-K) [3]. Assuming a noiseless feedback link available from the receiver to the transmitter, the paper presented a simple linear scheme that achieves the capacity of the single-user additive white Gaussian noise (AWGN) channel. More importantly, the scheme has a probability of error that decays double exponentially with the blocklength as compared to at most linearly exponential decay for no feedback techniques[4]. The scheme was then extended by Ozarow to the AWGN multiple-access channel (AWGN-MAC) with noiseless feedback to achieve the feedback capacity region of the channel [5]. Ozarow also extended the scheme to the AWGN broadcast channel (AWGN-BC) using noiseless feedback, which is the focus of this paper, yielding an improvement on the no-feedback capacity region [6]. There has been a revived interest in channel output feedback for the AWGN-BC. Assuming noiseless feedback, the works in [7], [8], [9] show further improvements. In fact, the recent interest in channel output feedback extends to other Gaussian channel models such as the Gaussian MAC channel [5], [7], [10], Guassian interference channel [11], [12], [13], and the Guassian relay channel [14], [15], [16].

The AWGN-BC models an important setting in practical wireless systems, the downlink setting, where the base station is transmitting information streams to different users. Hence, any improvement in rates over the AWGN-BC model would have a significant impact on practical systems. However, except for a few exceptions [17], [18], the literature on the AWGN-BC with feedback is limited to noiseless feedback, which is an assumption that is far from practical. The schemes considered in the literature for noiseless feedback are all linear schemes, which, as we show in this paper by extension from the single-user result, fail to achieve any positive rates when feedback channel are subject to noise. The works in [17], [18] present achievable rate regions for the broadcast channel with general (including noisy) feedback that are larger than the no-feedback capacity region. Both these regions where derived using schemes inspired by the example in [19]. In [18], it is shown for two types of discrete memoryless channels that noisy feedback, specifically with sufficiently small feedback noise level, improves on the no-feedback capacity region. In [17], the achievable rate region is evaluated for the symmetric two-user AWGN-BC with a single feedback link from one of the receivers. In the high forward channel signal-to-noise ratio (SNR) regime, the scheme improves on the no-feedback sum-capacity for a feedback noise level as high as the forward noise level. However, for low SNR (but still within practical values), the scheme's improvement over the no-feedback sum-capacity is negligible even for noiseless feedback.

In this paper, we consider the AWGN-BC with either noiseless or noisy feedback. Noiseless feedback will mean the transmitter has perfect access to the channel outputs in a causal fashion. Noisy feedback will mean the transmitter has causal access to the channel outputs corrupted by AWGN in the feedback link from each receiver. Concatenated coding, using a linear feedback encoding scheme as the inner code, has been previously considered for the AWGN-MAC with noisy feedback [10], Gaussian broadcast channel with noiseless feedback [20], and point-to-point AWGN channel with noisy feedback [21]. However, concatenated coding for the AWGN-BC with noisy feedback has not been explored previously. In this work, we extend the concatenated coding scheme that was presented in [21] for the point-to-point AWGN channel with noisy feedback to the K-user AWGN-BC with noisy feedback. Specifically, a linear feedback code for the AWGN-BC with noisy feedback is used as an inner code that creates an effective single-user channel from the transmitter to each of the receivers, and then open-loop (i.e., without feedback) coding is used for coding over these single-user channels.

For the single-user case, the scheme in [21] improves upon the no-feedback error-exponents. For the AWGN-BC with noisy feedback, we use the extended concatenated coding scheme to show improvements on the no-feedback capacity region. The contributions and improvements on previous works will be stated towards the end of this section. Before that, we would like to comment on the practicality of the concatenated coding scheme presented in this paper. In fact, the concatenated coding scheme presented in this paper has the following attractive properties for practical systems:

- Feedback information is utilized using simple linear processing.
- Open-loop coding is only used over single-user channels. Furthermore, when interference from the message points of other users is canceled out by the linear feedback code (as in the scheme of Section IV), the effective single-user channels are pure AWGN channels for which open-loop codes are well developed in practice.
- No broadcast channel coding techniques, like dirty paper coding or superposition coding, are required.

The results of Theorem 1 and Theorem 2 are for sufficiently small feedback noise levels (compared to forward noise levels). However, many broadcast communication systems can have high SNR. This is especially true for systems where the receivers have a larger power available at their disposable than the transmitter. One example of such a system is found in satellite communications. In a satellite communication system, the transmitter which is at the satellite would be broadcasting (possibly independent) data streams to different gateways present on earth. Satellites have much less power available than the gateways on earth. Another important application that possesses the same distribution of power is communication with implantable chips. In such an application, the chip implanted in the body of a human would like to broadcast different measurements to different devices that are located outside the body. Since the implantable chip powers itself from energy harvesting systems that convert ambient energy to electrical energy, the transmitter would have a very small power available as compared to the receivers that are located outside the body. Therefore, assuming a low feedback noise level as compared to the forward noise level still captures many important applications that starve for improvement in rates or lower transmitter power consumption.

The contributions of the paper can be summarized by the following:

- We show that if the feedback noise level for a receiver is strictly larger than zero, no matter how low the level is, linear feedback schemes can only achieve the zero rate to that receiver. This is an extension of the result derived in [22], [21] for the single-user case.
- We extend the concatenated coding scheme presented in [21] to the K-user AWGN-BC with noisy feedback and show an achievable rate region for a sufficiently small feedback noise level. From this result, it is deduced that any achievable rate tuple by Ozarow's scheme [6] for noiseless feedback can be achieved by the concatenated coding scheme for some, but non-zero, feedback noise level. The same result can be extended to the two-user AWGN with single feedback channel from one of the receivers by using the scheme in [23], with some modifications, as an inner code. The region achieved by the scheme in [23] for noiseless feedback can be achieved by concatenated coding but with sufficiently small feedback noise level which shows achievable rate tuples outside what is presented in [17], especially for low forward channel SNR.
- We present a linear feedback scheme for the symmetric K-user AWGN-BC channel with noisy feedback that is optimized and used as an inner code in the concatenated coding scheme. For noiseless feedback, it is shown that the scheme achieves the same sum-rate as in [9] but over the real channel, unlike the scheme presented in [9] requiring a complex channel. We show that the latter sum-rate is also achievable for sufficiently small feedback noise level. We also present achievable sum-rates versus feedback noise level obtained using the same linear scheme in the design of the concatenated

coding scheme.

The paper is organized as follows: In Section II, we describe the channel setup and give a general framework for linear feedback coding. In Section III, we present the concatenated coding scheme and its achievable rate region. In Section IV, we present a linear feedback coding scheme for the symmetric AWGN-BC with noisy feedback that is utilized in the concatenated coding scheme in Section V for the same channel. The paper is concluded in Section VI.

II. General Framework for Linear Feedback Coding

In this section, we formulate a general framework for linear feedback coding schemes for the K-user AWGN-BC with noisy feedback.

A. Channel Setup

We start by describing the channel setup that is depicted in Fig. 1. The channel at hand has one transmitter and K receivers. Before every block of transmission, the transmitter will have K independent messages W_1, W_2, $\ldots$, W_K, each to be conveyed reliably to the respective receiver.

After channel use ℓ, the channel output at receiver k, for $k \in \mathbb{K} = \{1, 2, \ldots, K\}$, is given by

$$y_k[\ell] = x[\ell] + z_k[\ell], \tag{1}$$

where $x[\ell] \in \mathbb{R}$ is the transmitted symbol at time l and $\{z_k[\ell]\}$ are i.i.d. and such that $z_k[l] \sim \mathcal{N}(0, \sigma_{z_k}^2)$. $z_k[\ell]$ is assumed independent of $x[\ell]$ for $k \in \mathbb{K}$. An average transmit power constraint, P, is imposed so that

$$E\left[\sum_{\ell=1}^{L} x^2[\ell]\right] \leq LP, \tag{2}$$

where L is the length of the transmission block.

Through the presence of feedback links from each receiver to the transmitter, the transmitter will have access to noisy versions of the channel outputs of all receivers in a causal fashion. In particular, to form $x[\ell]$, the transmitter can use $\{y_1[1]+n_1[1], \ldots, y_K[1]+n_K[1], \ldots, y_1[\ell-1]+n_1[\ell-1], \ldots, y_K[\ell-1]+n_K[\ell-1]\}$, where $\{n_k[\ell]\}$ are i.i.d. and $n_k[\ell] \sim \mathcal{N}(0, \sigma_{n_k}^2)$. Since the transmitter knows what it had transmitted in the previous transmissions, it can subtract it and equivalently use $\{z_1[1]+n_1[1], \ldots, z_K[1]+n_K[1], \ldots, z_1[\ell-1]+n_1[\ell-1], \ldots, z_K[\ell-1]+n_K[\ell-1]\}$ for encoding. It is assumed that $n_k[\ell]$ is independent of a $x[\ell]$ for $k \in \mathbb{K}$, and $n_i[t]$ is independent of $z_j[s]$ for any $t, s \in \mathbb{N}$ and $i, j \in \mathbb{K}$.

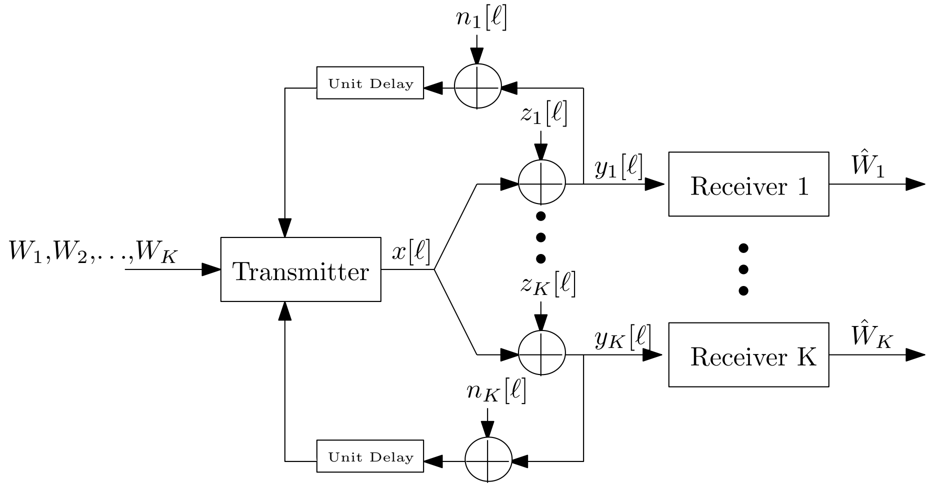

Fig. 1. AWGN-BC with feedback.

At the end of the transmission block, receiver k will have an estimate of its message denoted by $\hat{W}_k$ for $k \in \mathbb{K}$.

B. Linear Feedback Coding Framework

A general linear coding framework for the channel setup just described is presented next. Before each block of transmission, the transmitter maps each of the K messages to a point in $\mathbb{R}$, which is termed a *message point*. Specifically the message for the k-th receiver is mapped to $\theta_k \in \Theta_k \subseteq \mathbb{R}$ such that $|\Theta_k| = \lceil 2^{LR_k} \rceil$ and $E[\theta_k] = 0$, where L is the length of the transmission block and R_k is the rate of transmission for receiver k.

Let $\mathbf{x} = [x[1], x[2], \ldots, x[L]]^T$, $\mathbf{z}_k = [z_k[1], z_k[2], \ldots, z_k[L]]^T$, $\mathbf{n}_k = [n_k[1], n_k[2], \ldots, n_k[L]]^T$, and $\mathbf{y}_k = [y_k[1], y_k[2], \ldots, y_k[L]]^T$, where the superscript T denotes matrix transposition. Then we can write

$$\mathbf{x} = \sum_{k=1}^{K} \left[\mathbf{g}_k \theta_k + \mathbf{F}_k(\mathbf{z}_k + \mathbf{n}_k)\right],$$

where $\mathbf{g}_k \in \mathbb{R}^{L\times 1}$ and $\mathbf{F}_k \in \mathbb{R}^{L\times L}$ such that $\{\mathbf{F}_k\}$ are lower triangular matrices with zeros on the main diagonal so that casuality is ensured. In this work, we focus exclusively on linear encoding schemes and focus mainly on optimizing $\mathbf{F}_k$.

The average transmit power constraint (2) can be written as

$$E[\mathbf{x}^T\mathbf{x}] = \sum_{k=1}^{K} \mathbf{g}_k^T \mathbf{g}_k E[\theta_k^2] + \sum_{k=1}^{K} (\sigma_{z_k}^2 + \sigma_{n_k}^2) \|\mathbf{F}_k\|_F^2 \leq LP, \tag{3}$$

where $\|\mathbf{F}_k\|_F$ is the Frobenius norm of $\mathbf{F}_k$.

The received sequence at the k-th receiver can be written as

$$\mathbf{y}_k = \mathbf{x} + \mathbf{z}_k. \tag{4}$$

Each receiver will form an estimate of its message as a linear combination of its observed channel output sequence. Specifically, receiver k will form an estimate $\hat{\theta}_k$ of θ_k as

$$\hat{\theta}_k = \mathbf{q}_k^T \mathbf{y}_k, \tag{5}$$

where $\mathbf{q}_k \in \mathbb{R}^{L\times 1}$.

C. An Achievable Rate Region For Linear Feedback Coding

Breaking down $\hat{\theta}_k$ we have

$$\hat{\theta}_k = \mathbf{q}_k^T \mathbf{g}_k \theta_k + \sum_{\substack{i=1\\ i\neq k}}^{K} \mathbf{q}_k^T \mathbf{g}_i \theta_i + \sum_{j=1}^{K} \mathbf{q}_k^T \mathbf{F}_j (\mathbf{z}_j + \mathbf{n}_j) + \mathbf{q}_k^T \mathbf{z}_k. \tag{6}$$

A rate tuple $(R_1, \ldots, R_K)$ is achievable by a linear feedback scheme if the probability of error $Pr\{\hat{\theta}_k \neq \theta_k\}$ goes to zero as $L \to \infty$ for all $k \in \mathbb{K}$ while the power constraint in (3) is satisfied for $L \to \infty$. The third term plus the fourth term in the right-hand side of (6) is a zero mean additive Gaussian noise. Now, if the second term, whose power is constrained, is dealt with as additive noise, the probability distributions of $\theta_1, \ldots, \theta_{k-1}, \theta_{k+1}, \ldots, \theta_K$ that minimizes the mutual information between θ_k and $\hat{\theta}_k$ is the Gaussian distribution [24].

Hence, any rate tuple $(R_1, \ldots, R_K)$ that satisfies for all $k \in \mathbb{K}$

$$R_k < \lim_{L\to\infty} \frac{1}{2L} \log\left(1 + SNR_k(L)\right), \tag{7}$$

where

$$SNR_k(L) = \frac{(\mathbf{q}_k^T \mathbf{g}_k)^2 E[\theta_k^2]}{\sum\limits_{\substack{i=1\\ i\neq k}}^{K} \mathbf{q}_k^T \mathbf{g}_i E[\theta_i^2] + \sum\limits_{\substack{j=1\\ j\neq k}}^{K} (\sigma_{z_j}^2 + \sigma_{n_j}^2) \|\mathbf{q}_k^T \mathbf{F}_j\|^2 + \sigma_{z_k}^2 \|\mathbf{q}_k^T (\mathbf{I} + \mathbf{F}_k)\|^2 + \sigma_{n_k}^2 \|\mathbf{q}_k^T \mathbf{F}_k\|^2}, \tag{8}$$

and $\mathbf{I}$ is the identity matrix, is achievable by the linear feedback scheme given that the power constraint in (3) is satisfied for $L \to \infty$.

Before closing this section, we show that for any linear feedback scheme, if the feedback noise variance of receiver k is strictly greater than zero, i.e., if $\sigma^2_{n_k} > 0$, then the only achievable rate for receiver k is zero. This result is a direct extension of that of the single-user case shown in [22],[21].

Lemma 1. *For any linear feedback scheme for the AWGN-BC with noisy feedback, if the feedback noise of receiver k is strictly larger than zero, i.e., $\sigma^2_{n_k} > 0$, then the only achievable rate R_k for receiver k is zero.*

Proof: The proof is a direct extension from the single-user case in [22],[21]. Assume a positive rate to receiver k. Remove interference from other users. Then, the rate to receiver k will the stay the same or become larger. But, the channel from the transmitter to receiver k is a single-user AWGN channel. Hence, a contradiction with single-user result in [22],[21].

■

III. Concatenated Coding Scheme

From Lemma 1, we see that linear processing alone can only achieve zero rate with noisy feedback. Therefore, we need to do more than linear processing of noisy feedback in order to achieve positive rates and possibly achieve rate tuples that are outside the no-feedback capacity region. We describe such a scheme in this section that uses open-loop coding on top of linear processing to achieve rate tuples outside the no-feedback capacity region.

For any linear feedback code, we observe from (6) that for receiver k, the stochastic relation between θ_k and $\hat{\theta}_k$ can be modeled as a single-user channel without feedback, as in Fig. 2. This channel will be termed the k-th user *superchannel*. Since we can perform open-loop coding over the superchannel for each user, we have converted the problem to single-user coding without feedback. This will be the main idea behind the concatenated coding scheme to be described in this section. We call the scheme a concatenated coding scheme because of the use of open-loop codes in concatenation with a linear feedback code that creates the superchannels, which shares many similarities to the definition in [25] but here for a multi-user channel. Note that the time index m in Fig. 2 is shown to indicate that the superchannel will be used more than once for open-loop coding. The time index m will be defined later as we describe open-loop coding over the superchannels.

Fig. 3 shows the overall concatenated coding scheme that will be described next. The transmitter will use an open-loop code to encode each of the messages (i.e., will use K open-loop encoders). All open-loop encoders use codebooks of equal blocklength M. Let the chosen codeword of the k-th open-loop encoder

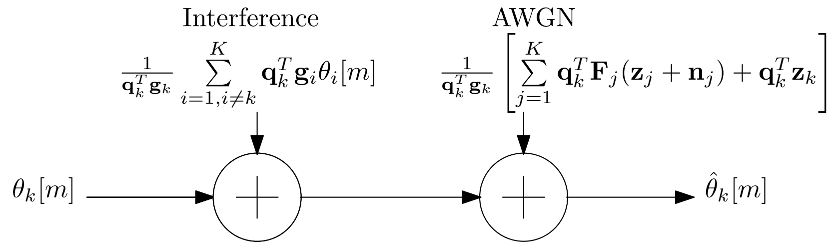

Fig. 2. Superchannel model.

be $[\theta_k[1], \theta_k[2], \ldots, \theta_k[M]]$. Similar to [25] but for the AWGN-BC, we will term the block consisting of the K open-loop encoders, which takes the K messages as input and gives as an output K codewords each of length M, the *outer code encoder*. At each time $m \in \{1, 2, \ldots, M\}$, the outer code encoder will have as an output, $\theta_1[m], \theta_2[m], \ldots, \theta_K[m]$.

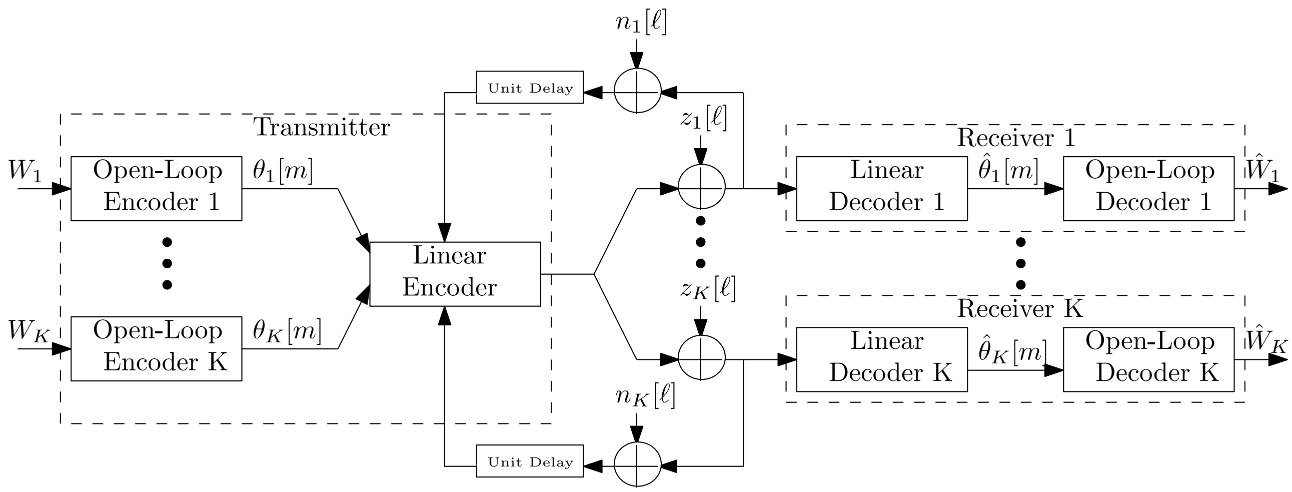

Fig. 3. Concatenated coding scheme.

At each time m, the output of the outer code encoder, particularly $\theta_1[m], \theta_2[m], \ldots, \theta_K[m]$, will be fed to a linear feedback encoder at the transmitter. The linear feedback encoder will use the channel for L times, after which each receiver will form an estimate, by linear processing of its intended symbol. Specifically, receiver k will form $\hat{\theta}_k[m]$ as the estimate of $\theta_k[m]$. The linear feedback code, formed of the linear encoder at the transmitter and a linear decoder at each receiver will be called the $innercode$. Receiver k will use an open-loop decoder, termed the *k-th outer code decoder*, that corresponds to its open-loop encoder, to decode its message by observing the sequence $\hat{\theta}_k[1], \hat{\theta}_k[2], \ldots, \hat{\theta}_k[M]$.

The overall code for the AWGN-BC with feedback is of blocklength ML. Since for each m, the inner code encoder transmits with at most LP of power, then the overall code uses a transmit power of at most MLP, and hence satisfies the codeword average power constraint. At receiver k, the SNR is the same for all $\hat{\theta}_k[1], \hat{\theta}_k[2], \ldots, \hat{\theta}_k[M]$ and is given by (8). Note the time index is dropped (i.e., if $\theta_k[m]$ is simply written as θ_k for all m). Thus, if a linear code has blocklength L, the concatenated coding scheme described above can be designed to achieve any rate tuple $(R_1, R_2, \ldots, R_k)$ that satisfies

$$R_k < \frac{1}{2L} \log\left(1 + SNR_k(L)\right) \tag{9}$$

for all $k \in \mathbb{K}$. It is important to note that in (9), the interference term from the message points of other users is dealt with as additive noise with bounded power. Since, the "worst" noise distribution for an additive noise with bounded power is the Gaussian distribution [24], the use of the AWGN channel capacity formula in the right-hand side of (9) is justified.

Theorem 1. *Given a linear feedback scheme over an AWGN-BC with noiseless feedback, i.e.,* $\sigma_{n_k} = 0$, *for any rate tuple* $(R_1, R_2, \ldots, R_K)$ *that satisfies* (7) *for* $k \in \mathbb{K}$, *there exist* $\epsilon_1 > 0, \ldots, \epsilon_K > 0$ *such that the same rate tuple* $(R_1, R_2, \ldots, R_K)$ *can be achieved by the concatenated coding scheme (scheme of Fig. 3) over the same AWGN-BC but with* $\sigma^2_{n_k}$ *as large as* ϵ_k *for* $k \in \mathbb{K}$.

Proof: For the given linear feedback coding scheme the SNR at receiver k for blocklength L is given by $SNR_k(L)$ of (8). In this proof, we will make the dependence of the SNR on the blocklength and the feedback noise variances explicit, e.g., for a linear feedback code with blocklength L that works according to the given linear feedback coding scheme over AWGN-BC with feedback noise variance for receiver k of $\sigma^2_{n_k}$ will be written as $SNR_k(L, \sigma^2_{n_1}, \ldots, \sigma^2_{n_K})$. Note here the dependence on $\sigma^2_{n_1}, \ldots, \sigma^2_{n_K}$ is just for the explicit values (i.e., if $\mathbf{g}_1, \ldots, \mathbf{g}_K$, $\mathbf{F}_1, \ldots, \mathbf{F}_K$, or $\mathbf{q}_1, \ldots, \mathbf{q}_K$ depend on $\sigma^2_{n_1}, \ldots, \sigma^2_{n_K}$, it is not captured by the arguments of SNR_k).

For the given rate tuple $(R_1, R_2, \ldots, R_K)$, assume $R_k > 0$ for $k \in \mathbb{K}$; for the case of $R_k = 0$ for some k, the proof below works the same but with trivially achieving the zero rates. Then,

$$R_k < \lim_{L \to \infty} \frac{1}{2L} \log\left(1 + SNR_k(L, 0, \ldots, 0)\right),$$

for $k \in \mathbb{K}$. Hence, there exists L_0 such that

$$R_k < \frac{1}{2L_0} \log\left(1 + SNR_k(L_0, 0, \ldots, 0)\right)$$

for all $k \in \mathbb{K}$. Let the matrices of the given linear scheme for blocklength L_0 be $\mathbf{g}_1, \ldots, \mathbf{g}_K$, $\mathbf{F}_1, \ldots, \mathbf{F}_K$, and $\mathbf{q}_1, \ldots, \mathbf{q}_K$ with the power constraint

$$\sum_{k=1}^{K} \mathbf{g}_k^T \mathbf{g}_k E[\theta_k^2] + \sum_{k=1}^{K} \sigma_{z_k}^2 \|\mathbf{F}_k\|_F^2 \leq L_0 P.$$

Let g_{k1} be the first entry of $\mathbf{g}_k$ for $k \in \mathbb{K}$. Since $R_k > 0$, then at least one entry of $\mathbf{g}_k$ is non-zero. Assume without loss of generality that g_{k1} is non-zero. Also, assume that $g_{k1} > 0$ (the proof still works in a similar way if g_{k1} is assumed negative). For $k \in \mathbb{K}$, let $\mathbf{g}'_k$ be such that

$$\mathbf{g}'_k = \mathbf{g}_k - \epsilon'_k \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{10}$$

where $g_{k1} - \epsilon'_k > 0$ and $\epsilon'_1 > 0$, $\epsilon'_2 > 0$, $\ldots$, $\epsilon'_K > 0$ are to be chosen next.

Choose $\epsilon'_1 > 0$,$\epsilon'_2 > 0$,$\ldots$,$\epsilon'_K > 0$ such that

$$R_k < \frac{1}{2L_0} \log\left(1 + SNR'_k(L_0, 0, \ldots, 0)\right)$$

for all $k \in \mathbb{K}$, where SNR'_k is the same function as SNR_k but uses $\mathbf{g}'_1, \ldots, \mathbf{g}'_K$ in place of $\mathbf{g}_1$,$\ldots$,$\mathbf{g}_K$. This is possible by the continuity of SNR_k at $\mathbf{g}_1$, $\mathbf{g}_2$, $\ldots$, $\mathbf{g}_K$.

Now, choose $\epsilon''_1 > 0$, $\epsilon''_2 > 0$,$\ldots$, $\epsilon''_K > 0$ such that

$$\epsilon''_k \leq \frac{(\mathbf{g}_k^T \mathbf{g}_k - \mathbf{g'_k}^T \mathbf{g}'_k) E[\theta_k^2]}{\|\mathbf{F}_k\|_F^2}$$

for $k \in \mathbb{K}$. Also, choose $\epsilon'''_1 > 0$, $\epsilon'''_2 > 0$, $\ldots$, $\epsilon'''_K > 0$ such that

$$R_k < \frac{1}{2L_0} \log\left(1 + SNR'_k(L_0, \epsilon'''_1, \ldots, \epsilon'''_K)\right)$$

for $k \in \mathbb{K}$.

Let $\epsilon_k = \min\{\epsilon''_k, \epsilon'''_k\}$ for $k \in \mathbb{K}$. Then, we have

$$\sum_{k=1}^{K} \mathbf{g'_k}^T \mathbf{g}'_k E[\theta_k^2] + \sum_{k=1}^{K} (\sigma_{z_k}^2 + \epsilon_k) \|\mathbf{F}_k\|_F^2 \leq L_0 P,$$

and

$$R_k < \frac{1}{2L_0} \log\left(1 + SNR'_k(L_0, \epsilon_1, \ldots, \epsilon_K)\right)$$

for $k \in \mathbb{K}$. Hence, for the same foward AWGN-BC but with feedback noise variances $\epsilon_1 > 0, \ldots, \epsilon_K > 0$, we have found a linear feedback code of blocklength L_0 defined by the matrices $\mathbf{g}'_1, \ldots, \mathbf{g}'_K, \mathbf{F}_1, \ldots, \mathbf{F}_K$, and $\mathbf{q}_1, \ldots, \mathbf{q}_K$, that satisfies the power constraint, and that attains an SNR at receiver k of $SNR'_k(L_0, \epsilon_1, \ldots, \epsilon_K)$ that is such that

$$R_k < \frac{1}{2L_0} \log \left(1 + SNR'_k(L_0, \epsilon_1, \ldots, \epsilon_K)\right).$$

Using this linear code as an inner code, and by (9), the concatenated coding scheme achieves the rate tuple $(R_1, \ldots, R_K)$. ■

Remark 1. *The result of Theorem 1 can be directly extended to the complex AWGN-BC with complex AWGN feedback channels.*

Remark 2. *In [6], the scheme is linear, and hence for any fixed block length can be described using vectors as in the framework described in Section II-B. Using the vector construction, the achievable rate region presented in [6] can be similarly described by the set of rate tuples that satisfy* (7) *for* $k \in \mathbb{K}$. *Hence, the achievable rate region for noiseless feedback in [6] can be achieved by the concatenated coding scheme of Fig. 3 for sufficiently small feedback noise level. The only difference in [6] is the addition of an auxiliary Gaussian random variable* w *to the first two transmissions, and only minor steps are needed to accommodate that in the proof of Theorem 1.*

Remark 3.

A similar result to Theorem 1 can be shown for the scheme in [23] for the two-user AWGN-BC with a single feedback channel from one of the receivers. Specifically, the scheme in [23] can be used, with some modifications, as the inner code to show that any rate tuple that is achievable by the scheme in [23] for noiseless feedback can be achieved by concatenated coding for sufficiently small feedback noise level. Hence, by concatenated coding we can achieve rate tuples outside what is presented in [17], especially for low forward channel SNR.

IV. A Linear Coding Scheme For The Symmetric AWGN-BC with Feedback

For designing the inner code of the concatenated coding scheme presented in Section III, we would ultimately like to find a linear coding scheme that maximizes the SNR at all receivers. However, to make the problem more tractable, we focus our attention on the symmetric case and impose some constraints on the scheme.

Using these constraints and the same channel setup of Section II, we present a linear coding scheme for the symmetric K-user AWGN-BC with feedback. Symmetric here means that all forward noises are of equal variance and all feedback noises are of equal variance. Denote by σ_z^2 the forward noise variance and by σ_n^2 the feedback noise variance. We will set $\sigma_z^2 := 1$ so that σ_n^2 will represent the ratio $\frac{\sigma_n^2}{\sigma_z^2}$ and P will represent the channel SNR $\frac{P}{\sigma_z^2}$. The scheme we develop will rely on techniques similar to code division multiple access (CDMA) for nulling cross user interference [7]. In this section, the total blocklength will be $L = \tilde{L} + K - 1$, where $\tilde{L} \in \mathbb{N}$. The reason behind introducing a new parameter $\tilde{L}$ will be clearer as we describe the scheme. We assume that K is an integer power of 2, specifically $K \in \{2, 4, 8, 16, \dots\}$.

Similar to the general formulation of Section II, the transmitter will map each of the independent K messages to a message point in $\mathbb{R}$. Specifically, the transmitter maps the message intended to receiver k to a point $\theta_k \in \Theta_k$ where $\Theta_k \subseteq \mathbb{R}$, $|\Theta_k| = \lceil 2^{LR_k} \rceil$ and $E[\theta_k] = 0$. R_k denotes the rate for receiver k.

Similar to Ozarow's scheme [6], the first K transmissions are used to send the message points in an orthogonal fashion. We will assume that time division is used for achieving that and let $x[k] = \theta_k$ for $k \in \mathbb{K}$ (note that the traditional CDMA could be used too). The remaining $L - K$ transmissions will be used for sending feedback information in a CDMA-like manner where interference nulling is imposed similar to the techniques used in [7].

Receiver k will ignore the channel uses where other users send their message points, i.e., channels uses $1, \dots, k-1, k+1, \dots, K$. Now, if we let $\tilde{\mathbf{z}}_k = [z_k[k], z_k[K+1], \dots, z_k[L]]^T$, $\tilde{\mathbf{n}}_k = [n_k[k], n_k[K+1], \dots, n_k[L]]^T$, and $\tilde{\mathbf{y}}_k = [y_k[k], y_k[K+1], \dots, y_k[L]]^T$, then the observed sequence at receiver k could be written as

$$\tilde{\mathbf{y}}_k = \mathbf{e_1}\theta_k + \sum_{k=1}^{K} \tilde{\mathbf{F}}_k(\tilde{\mathbf{z}}_k + \tilde{\mathbf{n}}_k) + \tilde{\mathbf{z}}_k, \tag{11}$$

where $\tilde{\mathbf{F}}_k \in \mathbb{R}^{\tilde{L}\times\tilde{L}}$ is a lower triangular matrix with zeros on the main diagonal to ensure causality and $\mathbf{e}_1$ is the first column of the $\tilde{L} \times \tilde{L}$ identity matrix.

With $\mathbf{x}$ defined as in Section II, the average transmit power is bounded by

$$E[\mathbf{x}^T\mathbf{x}] \leq LP = (\tilde{L} + K - 1)P. \tag{12}$$

From (3), we have

$$E[\mathbf{x}^T\mathbf{x}] = \sum_{k=1}^{K} \left[E[\theta_k^2] + (1 + \sigma_n^2)\|\tilde{\mathbf{F}}_k\|_F^2 \right]. \tag{13}$$

The first quantity on the right hand side, $\sum_{k=1}^{K} E[\theta_k^2]$, can be seen as the power used for transmitting the messages while the second term, $\sum_{k=1}^{K}(1+\sigma_n^2)\|\tilde{\mathbf{F}}_k\|_F^2$, is interpreted as the power utilized for transmitting feedback information. Due to this trade-off, a new parameter $\gamma \in [0,1]$ is introduced such that

$$\sum_{k=1}^{K} E[\theta_k^2] = (1-\gamma)(\tilde{L}+K-1)P, \tag{14}$$

and

$$\sum_{k=1}^{K}(1+\sigma_n^2)\|\tilde{\mathbf{F}}_k\|_F^2 \leq \gamma(\tilde{L}+K-1)P. \tag{15}$$

Thus, γ can be thought of as the normalized ratio of power spent on encoding feedback information. Since the channel is symmetric, we will assume that

$$E[\theta_k^2] = \frac{1}{K}(1-\gamma)(\tilde{L}+K-1)P$$

for all users.

The receiver creates its estimate, $\hat{\theta}_k$ as

$$\hat{\theta}_k = \tilde{\mathbf{q}}_k^T \tilde{\mathbf{y}}_{\mathbf{k}}, \tag{16}$$

where $\tilde{\mathbf{q}}_k \in \mathbb{R}^{\tilde{L}}$.

Then the received SNR for the k-th receiver is given by (17).

$$SNR_k = \frac{(\tilde{q}_k[1])^2 \frac{1}{K}(1-\gamma)(\tilde{L}+K-1)\rho}{\|\tilde{\mathbf{q}}_k^T(\mathbf{I}+\tilde{\mathbf{F}}_k)\|^2 + \sigma_n^2\|\tilde{\mathbf{q}}_k^T\tilde{\mathbf{F}}_k\|^2 + (1+\sigma_n^2)\sum_{\substack{i=1\\i\neq k}}^{K}\|\tilde{\mathbf{q}}_k^T\mathbf{C}_k\mathbf{C}_i\tilde{\mathbf{F}}_k\|^2}. \tag{17}$$

Definition 1. *A sum-rate R is said to be achievable if there exists a rate tuple $(R_1, R_2, \ldots, R_k)$ that is achievable and satisfying*

$$R = \sum_{i=1}^{K} R_k. \tag{18}$$

Hence, any sum-rate R that satisfies

$$R < \lim_{\tilde{L}\to\infty} \sum_{i=1}^{K} \frac{1}{2(\tilde{L}+K-1)} \log\left(1 + SNR_k(\tilde{L})\right), \tag{19}$$

is achievable where $SNR_k(\tilde{L})$ is written to show the dependence of the received SNR on the blocklength.

A. Linear Scheme Construction Summary for the Symmetric Case

In this section, we summarize the construction of our linear scheme for the symmetric AWGN-BC channel with noisy feedback in six simple steps. The underlying design logic behind the construction will be described in the following two sections. What we are really after are the vectors $\tilde{\mathbf{F}}_1, \ldots, \tilde{\mathbf{F}}_k$ and $\tilde{\mathbf{q}}_1, \ldots, \tilde{\mathbf{q}}_k$, which fully describe the scheme.

Assume a channel with a fixed number of users K, a fixed P, and a fixed σ_n^2. Let L be the blocklength of interest.

1) Fix $\gamma, \beta \in (0, 1)$
2) Construct $\mathbf{C}_1, \ldots, \mathbf{C}_k$ as in (21).
3) Construct $\mathbf{q}, \mathbf{f}$, and $\mathbf{v_i}$ for $i = 1, \ldots, \tilde{L} - K$ as in (25), (26), and (30), respectively.
4) Let $\boldsymbol{\mu}_i$ be constructed as in (31) for $i = 1, \ldots, \tilde{L} - K$, where λ is chosen such that (32) is satisfied.
5) Construct $\mathbf{F}$ as such:

   The i^{th} column of the $\mathbf{F}$ matrix is built by $\lfloor \frac{\tilde{L}-i}{K} \rfloor$ scaled copies of $\mathbf{f}$ below the main diagonal and the remaining entries are set to zero. The scaling coefficient for the i^{th} column and the j^{th} copy of $\mathbf{f}$ will be $\boldsymbol{\mu}_i[j]$, the j^{th} entry of $\boldsymbol{\mu}_i$. Specifically, the i^{th} column of the $\mathbf{F}$ matrix is given by

$$[\underbrace{0 \ \ldots \ 0}_{i} \ \ \boldsymbol{\mu}_i[1]\mathbf{f}^T \ \ \boldsymbol{\mu}_i[2]\mathbf{f}^T \ \ \ldots \ \ \boldsymbol{\mu}_i[\lfloor \frac{\tilde{L}-i}{K} \rfloor]\mathbf{f}^T \ \ \underbrace{0 \ \ldots \ 0}_{\tilde{L}-i-K\lfloor \frac{\tilde{L}-i}{K} \rfloor} \ ]^T. \tag{20}$$

6) Finally, $\tilde{\mathbf{F}}_k = \mathbf{C}_k \mathbf{F}$ and $\tilde{\mathbf{q}}_k = \mathbf{C}_k \mathbf{q}$ for $k \in \{1, 2, \ldots, K\}$.

Example 1. *Assume a channel with parameters $K = 2$, $P = 10$ and $\sigma_n^2 = 0.01$. Also, assume the block length of interest is $L = 9$. Then $\tilde{L} = L - 1 = 8$. Now, we follow the steps presented above:*

1) *Fix $\gamma = \frac{1}{3}$ and $\beta = \frac{1}{4}$.*
2) $\mathbf{C}_1 = diag([1, -1, 1, -1, 1, -1, 1, -1])$*, where $diag(\mathbf{x})$ is a diagonal matrix with the vector $\mathbf{x}$ as its main diagonal. For receiver 2, $\mathbf{C}_2 = -\mathbf{C}_1$.*
3)

$$\mathbf{q} = [\ 1 \ \ \tfrac{1}{16} \ \ \tfrac{1}{256} \ \ \tfrac{1}{4096} \ \ \tfrac{1}{65536} \ \ \tfrac{1}{1048576} \ \ \tfrac{1}{16777210} \ \ \tfrac{1}{268435456} \ ]^T, \qquad \mathbf{f} = [\ 1 \ \ 16\ ]^T,$$

$$\mathbf{v}_1 = 2[\ 1 \ \ \tfrac{1}{16} \ \ \tfrac{1}{256}\ ]^T, \qquad \mathbf{v}_2 = \frac{1}{2}[\ 1 \ \ \tfrac{1}{16} \ \ \tfrac{1}{256}\ ]^T,$$

$$\mathbf{v}_3 = \frac{1}{8}[\ 1 \ \ \tfrac{1}{16}\ ]^T, \qquad \mathbf{v}_4 = \frac{1}{32}[\ 1 \ \ \tfrac{1}{16}\ ]^T, \qquad \mathbf{v}_5 = \frac{1}{128}, \qquad \mathbf{v}_6 = \frac{1}{512}.$$

4) *Using numerical tools to solve for* λ *in* (32)*, we find* $\lambda = 3.7589$*. Using this value of* λ *we have*

$$\boldsymbol{\mu}_1 = [\ -0.2095 \quad -0.0131 \quad -0.0008\ ]^T, \qquad \boldsymbol{\mu}_2 = 10^{-2}[\ -0.7788 \quad -0.0487 \quad -0.0030\ ]^T,$$

$$\boldsymbol{\mu}_3 = 10^{-3}[\ -0.1294 \quad -0.0081\ ]^T, \qquad \boldsymbol{\mu}_4 = 10^{-5}[\ -0.2029 \quad -0.0127\ ]^T,$$

$$\boldsymbol{\mu}_5 = -3.1714 \times 10^{-8}, \qquad \boldsymbol{\mu}_6 = -4.9552 \times 10^{-10}.$$

5) *Using* $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_{\tilde{L}-K}$ *and* $\mathbf{f}$*, following step 4 we can construct* $\mathbf{F}$*,*

$$\mathbf{F} = 10^{-2} \times \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -20.9500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -335.2000 & -0.7788 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.3100 & -12.4608 & -0.0129 & 0 & 0 & 0 & 0 & 0 \\ -20.9600 & -0.0487 & -0.2070 & -0.0002 & 0 & 0 & 0 & 0 \\ -0.0800 & -0.7792 & -0.0008 & -0.0032 & -0.0000 & 0 & 0 & 0 \\ -1.2800 & -0.0030 & -0.0130 & -0.0000 & -0.0001 & -0.0000 & 0 & 0 \\ 0 & -0.0480 & 0 & -0.0002 & 0 & -0.0000 & 0 & 0 \end{bmatrix}.$$

6) *Finally,* $\tilde{\mathbf{F}}_1 = \mathbf{C}_1\mathbf{F}$ *and* $\tilde{\mathbf{q}}_1 = \mathbf{C}_1\mathbf{q}$*. For receiver 2,* $\tilde{\mathbf{F}}_2 = -\tilde{\mathbf{F}}_1$ *and* $\tilde{\mathbf{q}}_2 = -\tilde{\mathbf{q}}_1$*.*

B. Design Logic Behind Steps 2, 3, and 5: Interference Nulling

In this section, we describe the reasoning behind steps 2, 3 and 5 of the construction summary presented in the previous section. The main idea is based on interference nulling between users that will be described mathematically later in this section. Specifically, receiver k, we will introduce a diagonal matrix $\mathbf{C}_k \in \mathbb{R}^{\tilde{L}\times\tilde{L}}$ that is such that

$$[\mathbf{C}_k]_{ij} = \begin{cases} \mathbf{c}_k[i \bmod K], & i = j, \\ 0, & i \neq j, \end{cases} \tag{21}$$

where for $k = 1, 2 \ldots, K$, $\mathbf{c}_k \in \mathbb{R}^{1\times K}$ is of entries in $\{-1, 1\}$ and is such that

$$\mathbf{c}_i^T \mathbf{c}_j = \begin{cases} K, & i = j, \\ 0, & i \neq j. \end{cases}$$

Remark 4. *The vectors* $\mathbf{c}_1, \ldots, \mathbf{c}_K$ *can be chosen as the columns of a* $K \times K$ *Hadamard matrix. For this reason, we have constrained* K *to be an integer power of 2. Note, however, that if the channel at hand was complex, this constraint on* K *can be alleviated by using complex Hadamard matrices, and all sum-rates derived for the real channel can be similarly achieved per real dimension over the complex channel for any* $K \geq 2$.

We will restrict $\tilde{\mathbf{F}}_k$ to be such that

$$\tilde{\mathbf{F}}_k = \mathbf{C}_k \mathbf{F}, \tag{22}$$

where $\mathbf{F} \in \mathbb{R}^{\tilde{L} \times \tilde{L}}$ is a lower triangular matrix with zeros on the main diagonal to ensure causality and whose construction will be described later. We chose $\mathbf{F}$ to be the same for all users because of the symmetry of the channel.

The receiver creates its estimate, $\hat{\theta}_k$ as

$$\hat{\theta}_k = \mathbf{q}^T \mathbf{C}_k \tilde{\mathbf{y}}_\mathbf{k}, \tag{23}$$

where $\mathbf{q} \in \mathbb{R}^{\tilde{L}}$. Again, we chose $\mathbf{q}$ to be the same for all users because of the symmetry of the channel.

We will constrain our scheme to satisfy

$$\sum_{\substack{i=1 \\ i \neq k}}^{K} \|\mathbf{q}^T \mathbf{C}_k \mathbf{C}_i \mathbf{F}\|^2 = 0, \tag{24}$$

so that cross-user interference is nulled to zero.

In the following lemma, constraints on the transmission scheme are given to satisfy requirement (24).

Lemma 2. *Let* $\mathbf{C}_k$ *be defined as in* (21) *for* $k = 1, 2, \ldots, K$. *Then, the following forms of* $\mathbf{q}$ *and* $\mathbf{F}$ *satisfy* (24):

- *For a real number* $\beta \in (0, 1)$

$$\mathbf{q} = \left[1, \beta^2, \beta^4, \ldots, \beta^{2(\tilde{L}-1)}\right]^T. \tag{25}$$

- *Let*

$$\mathbf{f} = \left[1, \beta^{-2}, \beta^{-4}, \ldots, \beta^{-2(\tilde{K}-1)}\right]^T. \tag{26}$$

The i^{th} *column of the* $\mathbf{F}$ *matrix is built by* $\lfloor \frac{\tilde{L}-i}{K} \rfloor$ *scaled copies of* $\mathbf{f}$ *below the main diagonal and the remaining entries are set to zero. The scaling coefficient for the* i^{th} *column and the* j^{th} *copy of*

$\mathbf{f}$ *will be called* $\mu_{i,j} \in \mathbb{R}$. *Specifically, the* i^{th} *column of the* $\mathbf{F}$ *matrix is given by*

$$[\underbrace{0 \ \ldots \ 0}_{i} \ \mu_{i,1}\mathbf{f}^T \ \ \mu_{i,2}\mathbf{f}^T \ \ \ldots \ \ \mu_{i,\lfloor \frac{\tilde{L}-i}{K} \rfloor}\mathbf{f}^T \ \ \underbrace{0 \ \ldots \ 0}_{\tilde{L}-i-K\lfloor \frac{\tilde{L}-i}{K} \rfloor} \ ]^T. \tag{27}$$

Proof: The form of $\mathbf{F}$ stems from the following observation. For $\mathbf{v}_1 \in \mathbb{R}^{\tilde{L}}$ and $\mathbf{v}_2 \in \mathbb{R}^{\tilde{L}}$ to satisfy

$$\mathbf{v}_1^T \mathbf{C}_i \mathbf{C}_j \mathbf{v}_2 = \begin{cases} \mathbf{v}_1^T \mathbf{v}_2, & i = j \\ 0, & i \neq j \end{cases}$$

the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ can be constructed as $v_2[i] = \frac{1}{v_1[i]}$ for all $i = 1, 2, \ldots, \tilde{L}$. Using this fact and the condition that it must hold between $\mathbf{q}$ and K shifts of $\mathbf{f}$, the lemma is constructed. The further choice that $\beta \in (0, 1)$ is to keep the norm of $\mathbf{q}$ bounded as $\tilde{L} \to \infty$. Note that $\mathbf{F}$ is all zeros for $\tilde{L} \leq K$. ■

Define

$$\boldsymbol{\mu}_i = \left[\mu_{i,1}, \mu_{i,2}, \ldots, \mu_{i,\left\lfloor \frac{\tilde{L}-i}{K} \right\rfloor} \right]^T, \tag{28}$$

With $\mathbf{q}$ and $\mathbf{F}$ having forms as in Lemma 2, the SNR at any of the receivers can be written as

$$SNR(\sigma_n^2, \tilde{L}, \gamma, \beta, \boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_{\tilde{L}-K}) = \frac{\frac{1}{K}(1-\gamma)(\tilde{L}+K-1)P}{\|\mathbf{q}^T(\mathbf{I}+\mathbf{F})\|^2 + \sigma_n^2 \|\mathbf{q}^T \mathbf{F}\|^2}. \tag{29}$$

C. Design Logic Behind Step 4: SNR Optimization

In the following lemma, given γ and β, we optimize SNR in (29) over $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_{\tilde{L}-K}$.

Lemma 3. *Assume* $\tilde{L} > K$. *Given* $\gamma, \beta \in (0, 1)$ *and following the forms of* $\mathbf{q}$ *and* $\mathbf{F}$ *as in Lemma 2, the* $\mu_{i,j}$ *values of* $\mathbf{F}$ *that maximize the received SNR* (29) *given the power constraint* (12) *can be obtained as follows:*

1) *For* $i = 1, 2, ..., \tilde{L} - K$, *define*

$$\mathbf{v}_i = K\beta^{i-1} \left[1, \beta^K, \ldots, \beta^{K\left(\left\lfloor \frac{\tilde{L}-i}{K} \right\rfloor - 1\right)} \right]^T, \tag{30}$$

2) *Then, the* $\boldsymbol{\mu}_i$ *that maximize the received SNR are constructed as*

$$\boldsymbol{\mu}_i = -\frac{q_i}{(1+\sigma_n^2)\|\mathbf{v}_i\|^2 + \lambda} \mathbf{v}_i, \tag{31}$$

where $\lambda \geq 0$ is chosen to satisfy

$$\sum_{i=1}^{\tilde{L}-K} \|\boldsymbol{\mu}_i\|^2 = \frac{\gamma(\tilde{L}+K-1)P}{K(1+\sigma_n^2)\|\mathbf{f}\|^2}. \tag{32}$$

Proof: With the definitions in Lemma 3, the denominator of the received SNR in (29) can be rewritten as

$$\sum_{i=\tilde{L}-K+1}^{\tilde{L}} q_i^2 + \sum_{i=1}^{\tilde{L}-K} \left(q_i + \mathbf{v}_i^T \boldsymbol{\mu}_i\right)^2 + \sigma_n^2 \sum_{i=1}^{\tilde{L}-K} \left(\mathbf{v}_i^T \boldsymbol{\mu}_i\right)^2. \tag{33}$$

Then, it can be shown that to minimize (33), one should let $\boldsymbol{\mu}_i = -b_i \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|}$ for some scalars b_i for $i = 1, 2, \ldots, \tilde{L} - K$. The sum of the second and third terms of (33) can now be rewritten as

$$\|\mathbf{A}\mathbf{b} - \mathbf{q}\|^2 + \sigma_n^2 \|\mathbf{A}\mathbf{b}\|^2, \tag{34}$$

where $\mathbf{A} \in \mathbb{R}^{\tilde{L} \times \tilde{L}-K}$ is

$$\mathbf{A} = \begin{bmatrix} \|\mathbf{v}_1\| & 0 & 0 & \cdots & & 0 \\ 0 & \|\mathbf{v}_2\| & 0 & \cdots & & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & \cdots & & 0 & \|\mathbf{v}_{\tilde{L}-K}\| \\ 0 & 0 & \cdots & & & 0 \\ \vdots & & \vdots & & & \vdots \\ 0 & 0 & \cdots & & & 0 \end{bmatrix}$$

and $\mathbf{b} = [b_1, b_2, \ldots, b_{\tilde{L}-K}]^T$. To minimize (34) and abide by the average power constraint, we use Lagrange multipliers to obtain the $\mathbf{b}$ that minimizes (34) is

$$\mathbf{b}_{min} = \left[(1+\sigma_n^2)\mathbf{A}^T\mathbf{A} + \lambda \mathbf{I}\right]^{-1} \mathbf{A}^T \mathbf{q}, \tag{35}$$

where λ is chosen to satisfy the power constraint. Thus, using $\mathbf{b}_{min}$ to build $\boldsymbol{\mu}_i$, we produce the lemma. ■

Using the optimal values of $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_{\tilde{L}-K}$, call them $\boldsymbol{\mu}_1^*, \ldots, \boldsymbol{\mu}_{\tilde{L}-K}^*$, the optimal received SNR will be

$$SNR_{optimal}(\sigma_n^2, \tilde{L}, \gamma, \beta) = SNR(\sigma_n^2, \tilde{L}, \gamma, \beta, \boldsymbol{\mu}_1^*, \ldots, \boldsymbol{\mu}_{\tilde{L}-K}^*). \tag{36}$$

Note, however, that the optimal form of $\boldsymbol{\mu}_i$ in Lemma 3 depends on λ for which a closed form is generally hard to obtain.

1) Special case at $L = \infty$: Notice that $\lambda \to 0$ as $L \to \infty$ in which case it can be shown that

$$\mu_{i,j} = -\frac{1-\beta^{2K}}{(1+\sigma_n^2)K}\beta^{K(j-1)}. \tag{37}$$

Furthermore, as $\sigma_n^2 \to 0$, we have

$$\mu_{i,j} = -\frac{1-\beta^{2K}}{K}\beta^{K(j-1)}. \tag{38}$$

D. Suboptimal SNR

The form in (38) can be used as a suboptimal solution instead of the optimal solution in Lemma 3 that does not have a closed form. With $\mathbf{q}$ and $\mathbf{F}$ following the form in Lemma 2, and using (38) as a suboptimal solution, the SNR for $\tilde{L} > K$ at any of the receivers can be written as

$$SNR_{suboptimal}(\sigma_n^2, \tilde{L}, \gamma, \beta) = \frac{\frac{1}{K}(1-\gamma)(\tilde{L}+K-1)P}{g(\tilde{L},\beta)+\sigma_n^2 h(\tilde{L},\beta)}, \tag{39}$$

where

$$g(\tilde{L},\beta) = \sum_{i=\tilde{N}-K+1}^{\tilde{L}} \beta^{2(i-1)} + \sum_{i=1}^{\tilde{L}-K} \beta^{\left[2(i-1)+4K\left\lfloor\frac{\tilde{L}-i}{K}\right\rfloor\right]}$$

and

$$h(\tilde{L},\beta) = \sum_{i=1}^{\tilde{L}-K} \beta^{2(i-1)}\left(1-\beta^{2K\left\lfloor\frac{\tilde{L}-i}{K}\right\rfloor}\right)^2,$$

The power constraint (15) can be written as

$$e(\tilde{L},\beta) \le \frac{\gamma(\tilde{L}+K-1)P}{K(1+\sigma_n^2)}, \tag{40}$$

where

$$e(\tilde{L},\beta) = \frac{(1-\beta^{2K})^2}{K^2(1-\beta^2)\beta^{2K}}\left(\tilde{L}-K-\sum_{i=1}^{\tilde{L}-K}\beta^{2K\left\lfloor\frac{\tilde{L}-i}{K}\right\rfloor}\right).$$

Note that suboptimal solution coincides with the optimal solution for $L = \infty$ and $\sigma_n^2 = 0$, i.e., $SNR_{optimal}(\sigma_n^2 = 0, \tilde{L} = \infty, \gamma, \beta) = SNR_{suboptimal}(\sigma_n^2 = 0, \tilde{L} = \infty, \gamma, \beta)$.

1) Upper and Lower Bounds on $SNR_{suboptimal}$: In the next lemma, we find upper and lower bounds on $SNR_{suboptimal}(\sigma_n^2 = 0, \tilde{L}, \gamma, \beta)$.

Lemma 4. *$SNR_{suboptimal}(\sigma_n^2 = 0, \tilde{L}, \gamma, \beta)$ can be bounded as*

$$SNR_{suboptimal,lb}(\tilde{L}, \gamma, \beta) \leq SNR_{suboptimal}(\sigma_n^2 = 0, \tilde{L}, \gamma, \beta) \leq SNR_{suboptimal,ub}(\tilde{L}, \gamma, \beta),$$

where

$$SNR_{suboptimal,lb}(\tilde{L}, \gamma, \beta) = \frac{a_{lb}(1-\gamma)(\tilde{L}+K-1)\frac{P}{K}}{\beta^{2\tilde{L}}},$$

$$SNR_{suboptimal,ub}(\tilde{L}, \gamma, \beta) = \frac{(1-\beta^2)(1-\gamma)(\tilde{L}+K-1)\frac{P}{K}}{\beta^{2(\tilde{L}-K)} - \beta^{2\tilde{L}} + \beta^{2(\tilde{L}-K-1)}\left(1-\beta^{2(\tilde{L}-K)}\right)},$$

and

$$a_{lb} = \frac{(1-\beta^2)}{\beta^{-2K}(1+\beta^2)-1}.$$

Proof: The second term of $g(\tilde{L}, \beta)$ can be upper bounded as

$$\begin{aligned}
\sum_{i=1}^{\tilde{L}-K} \beta^{\left[2(i-1)+4K\left\lfloor\frac{\tilde{L}-i}{K}\right\rfloor\right]} &\leq \sum_{i=1}^{\tilde{L}-K} \beta^{\left[2(i-1)+4(\tilde{L}-i-K+1)\right]} \\
&= \beta^{2(\tilde{L}-K+1)} \frac{1-\beta^{2(\tilde{L}-K)}}{1-\beta^2} \\
&\leq \beta^{2\tilde{L}} \frac{\beta^{-2(K-1)}}{1-\beta^2},
\end{aligned}$$

where the first inequality is due to the fact that $\left\lfloor\frac{\tilde{L}-i}{K}\right\rfloor \geq \frac{\tilde{L}-i-K+1}{K}$. Using this bound, $SNR_{suboptimal,lb}(\tilde{L}, \gamma, \beta)$ can be reached.

On the other hand, the second term of $g(\tilde{L}, \beta)$ can be lower bounded as

$$\begin{aligned}
\sum_{i=1}^{\tilde{L}-K} \beta^{\left[2(i-1)+4K\left\lfloor\frac{\tilde{L}-i}{K}\right\rfloor\right]} &\geq \sum_{i=1}^{\tilde{L}-K} \beta^{\left[2(i-1)+4(\tilde{L}-i+K)\right]} \\
&= \beta^{2(\tilde{L}-K-1)} \sum_{i=1}^{\tilde{L}-K} \beta^{2(\tilde{L}-K-i)} \\
&= \beta^{2(\tilde{L}-K-1)} \frac{1-\beta^{2(\tilde{L}-K)}}{1-\beta^2}
\end{aligned}$$

where the first inequality is due to the fact that $\left\lfloor\frac{\tilde{L}-i}{K}\right\rfloor \leq \frac{\tilde{L}-i}{K}+1$. Using this bound, $SNR_{suboptimal,ub}(\tilde{L}, \gamma, \beta)$ can be reached.

■

E. Achievable Sum-Rate For Noiseless Feedback

For the noiseless feedback case (i.e., for $\sigma_n^2 = 0$), from Lemma 4, we see that

$$\lim_{\tilde{L}\to\infty} \frac{K}{2(\tilde{L}+K-1)} \log\left(1 + SNR_{suboptimal,lb}(\tilde{L},\gamma,\beta)\right) = \lim_{\tilde{L}\to\infty} \frac{K}{2(\tilde{L}+K-1)} \log\left(1 + SNR_{suboptimal,ub}(\tilde{L},\gamma,\beta)\right) = -K\log(\beta),$$

and hence

$$\lim_{\tilde{L}\to\infty} \frac{K}{2(\tilde{L}+K-1)} \log\left(1 + SNR_{suboptimal}(\sigma_n^2 = 0,\tilde{L},\gamma,\beta)\right) = -K\log(\beta).$$

Thus, any sum-rate R is achievable if

$$R < -K\log(\beta). \tag{41}$$

In the following lemma, we show that β and γ can in fact be chosen so that the right-hand side of (41) is equal to the linear-feedback sum-rate bound derived in [9].

Lemma 5. *Let $\phi \in [1, K]$ be the solution of*

$$(1+P\phi)^{K-1} - \left[1 + \frac{P}{K}\phi(K-\phi)\right]^K = 0. \tag{42}$$

The power constraint allows β to be chosen to statisfy $\beta^{-2K} = 1 + P\phi$ so that the scheme achieves any sum-rate R satisfying

$$R < \frac{1}{2}\log\left(1+P\phi\right). \tag{43}$$

Proof: Choose $\gamma = \frac{L-1}{L}$. We choose β such that all available power is consumed. Specifically, we choose β such that

$$\lim_{\tilde{L}\to\infty} \frac{e(\tilde{L},\beta)}{\gamma(\tilde{L}+K-1)} = \frac{P}{K}.$$

The left-hand side of the above equation is equal to $\frac{(1-\beta^{2K})^2}{K^2(1-\beta^2)\beta^{2K}}$. Let $\beta^{-2K} = 1 + P\phi$ and solve for ϕ instead of β. The resulting equation in ϕ can be reduced to (42). By (41), the proof is complete. ■

The sum-rate achieved here is the same as in [9]. However, in [9] the scheme requires a complex channel in order to achieve, per real dimension, the same sum-rate of Lemma 5. This is especially true for $K > 2$. Note, however, that the number of users K is constrained to be an integer power of 2 for the real channel case.

V. Concatenated Coding for the Symmetric AWGN-BC with Noisy Feedback

In this section, we consider the same concatenated scheme that was described in Section III, but that relies on the linear scheme of Section IV for coding over the symmetric AWGN-BC with noisy feedback. From Section III and by the symmetry of the channel and scheme, if we fix a linear code of blocklength L designed according to the scheme described in Section IV, any sum rate, R, can be achieved by the concatenated scheme just described if

$$R < \frac{K}{2(\tilde{L}+K-1)} \log\Big(1 + SNR_{optimal}(\sigma_n^2, \tilde{L}, \gamma, \beta)\Big). \tag{44}$$

A. Achievable Sum-Rates For Small Enough Feedback Noise Level

In this section, we discuss the achievable sum-rates for small enough feedback noise variance. From Theorem 1, we know that what is achieved for the noiseless feedback case in Lemma 5 can be achieved for small enough feedback noise level by the concatenated coding scheme. However, for sum-rates close to the bound in Lemma 5, the required inner code blocklength will be larger with small σ_n^2, this makes the choice of $\mu_{i,j}$ in (38) approximately optimal. For such case, and given a value for γ, Lemma 6 and Lemma 7 will be useful for choosing the value of β. We will also use those lemmas to rederive the result of Theorem 1 but using the specifics of the scheme of this section.

In the next lemma, we find a stricter power constraint than the power constraint (40) that does not depend on $\tilde{L}$.

Lemma 6. *For a fixed γ, if β satisfies*

$$\frac{(1-\beta^{2K})^2}{K(1-\beta^2)\beta^{2K}} \le \frac{\gamma P}{1+\sigma_n^2}, \tag{45}$$

the the power constraint (40) *is satisfied for any $\tilde{L}$.*

Proof: $e(\tilde{L}, \beta)$ of (40) can be upper bounded as follows

$$e(\tilde{L}, \beta) \le \frac{(1-\beta^{2K})^2}{K^2(1-\beta^2)\beta^{2K}}(\tilde{L}+K-1).$$

Hence, β that satisfies

$$\frac{(1-\beta^{2K})^2}{K^2(1-\beta^2)\beta^{2K}}(\tilde{L}+K-1) \le \frac{\gamma(\tilde{L}+K-1)P}{K(1+\sigma_n^2)}$$

satisfies (40). ■

Note that for large $\tilde{L}$, the power lost by assuming the power constraint (45) instead of (40) becomes negligible. In the next lemma, we show that the left-hand side of (45), as a function of β, is positive and decreasing, and bijective from $(0, 1)$ to $(0, \infty)$.

Lemma 7. *Let* $f(\beta) = \frac{(1-\beta^{2K})^2}{K(1-\beta^2)\beta^{2K}}$. *Then*

- f *is a decreasing positive function on* $(0, 1)$. *Specifically, if* $\beta_1, \beta_2 \in (0, 1)$ *are such that* $\beta_1 < \beta_2$, *then* $0 < f(\beta_2) < f(\beta_1)$.
- f *is a bijective function from* $(0, 1)$ *to* $(0, \infty)$.

Proof: Let f' denote the first derivative of f with respect to β. It can be shown that $f'(\beta) < 0$ for $\beta \in (0, 1)$ if and only if $p(x) > 0$ for $x \in (0, 1)$, where $p(x) = (1 - K)x^{K+1} + Kx^K - (K + 1)x + K$. Now, let p' and p'' denote the first and the second derivatives of p with respect to x, respectively. To show that $p(x) > 0$ for $x \in (0, 1)$, we will use the fact that $p(1) = 0$ and show that $p(x)$ is strictly decreasing on $(0, 1]$. We have

$$p'(x) = (1 - K)(K + 1)x^K + K^2 x^{K-1} - (K + 1)$$

and

$$p''(x) = x^{K-2} K(K - 1) \left[K - (K + 1)x\right].$$

From $p''(x)$, we notice that $p'(x)$ is strictly increasing for $x \in (0, \frac{K}{K+1})$ and is strictly decreasing for $x \in (\frac{K}{K+1}, 1]$, and hence its maximum value on $(0, 1]$ is at $x = \frac{K}{K+1}$. For $x \in (0, 1]$,

$$\begin{aligned} p'(x) &\leq p'\left(\frac{K}{K+1}\right) \\ &= K\left(\frac{K}{K+1}\right)^{K-1} - (K+1) < 0. \end{aligned}$$

Therefore, $p(x)$ is a strictly decreasing function on $(0, 1]$. But since $p(1) = 0$, then $p(x) > 0$ for $x \in (0, 1)$. So far, we have shown that f is a strictly deceasing function on $(0, 1)$. Now, since f is a continous function on $(0, 1)$ and since $\lim_{\beta \to 0} f(\beta) = \infty$ and $\lim_{\beta \to 1} f(\beta) = 0$, then $f((0, 1)) = (0, \infty)$. This completes the proof. ■

Theorem 2. *For any sum-rate* $R < \frac{1}{2}\log(1 + P\phi)$, *where* ϕ *is as defined in Lemma 5, there exists* $\epsilon > 0$ *such that the same sum-rate* R *can be achieved by the concatenated coding with* $\sigma_n^2 \leq \epsilon$.

Proof: For $R = 0$, the proof is trivial. For $R > 0$, choose γ large enough such that $\frac{1}{2}\log(1 + P\gamma\phi) >$

R, where $\phi \in [1, K]$ is the solution of

$$(1 + P\gamma\phi)^{K-1} - \left[1 + \frac{P\gamma}{K}\phi(K - \phi)\right]^K = 0.$$

Call this γ value γ_0.This allows us to choose $\beta \in [0, 1]$ such that $-K \log(\beta) > R$ and $f(\beta) \leq P\gamma$. Choose, $\beta_0 \in [0, 1] > \beta$ such that $-K \log(\beta) > -K \log(\beta_0) > R$. By Lemma 7, there exists $\epsilon_1 > 0$ such that

$$f(\beta_0) \leq \frac{P\gamma}{1 + \epsilon_1}.$$

Define

$$\tilde{R}(\sigma_n^2, \tilde{L}) = \frac{K}{2(\tilde{L} + K - 1)} \log\left(1 + SNR_{suboptimal}(\sigma_n^2, \tilde{L}, \gamma_0, \beta_0)\right).$$

Since $\lim_{\tilde{L}\to\infty} \tilde{R}(0, \tilde{L}) = -k \log(\beta_0) > R$, there exists $\tilde{L}_0$ such that $\tilde{R}(0, \tilde{L}_0) > R$. There also exists $\epsilon_2 > 0$ such that $\tilde{R}(\epsilon_2, \tilde{L}_0) > R$.

Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Since $\tilde{R}(\epsilon, \tilde{L}_0) \geq \max\{\tilde{R}(\epsilon_1, \tilde{L}_0), \tilde{R}(\epsilon_2, \tilde{L}_0)\}$ and since $f(\beta_0) \leq \frac{P\gamma}{1+\epsilon}$, by (44) and by Lemma 6, we have found γ_0, β_0, and $\tilde{L}_0$ such that the concatenated coding scheme achieves any sum-rate below $\tilde{R}(\tilde{L}_0, \epsilon) > R$ for feedback noise variance as large as ϵ. Hence, R is achieved. ■

B. Inner Code Blocklength

In this section, we find an upper bound on the inner code blocklength required for the concatenated coding scheme to achieve a certain sum-rate above the no-feedback sum-capacity. To do that, we assume noiseless feedback and make use of the SNR lower bound in Lemma 4 and Lemma 6. For sum-rates closer to the bound in Lemma 5, the inner code block length is larger. For larger inner code block length, $\mu_{i,j}$ in (38) becomes approximately optimal, the power lost in Lemma 6 becomes negligible, and the SNR lower bound, $SNR_{suboptimal,lb}(\tilde{L}, \gamma, \beta)$ in Lemma 4, becomes closer to $SNR_{suboptimal}(\sigma_n^2 = 0, \tilde{L}, \gamma, \beta)$. As a result, the upper bound becomes tighter for sum-rates closer to the bound in Lemma 5.

Lemma 8. *Fix $\gamma, \beta \in (0, 1)$ such that $-K \log(\beta) > \frac{1}{2} \log(1 + P)$. Let a_{lb} be defined as in Lemma 4, and assume noiseless feedback, i.e., $\sigma_n^2 = 0$. For any sum-rate R such that*

$$\frac{1}{2} \log(1 + P) < R < -K \log \beta,$$

let L_0 be the smallest integer $\tilde{L}$ such that

$$\frac{K}{2(\tilde{L}+K-1)}\log\left(1+SNR_{suboptimal,lb}(\tilde{L},\gamma,\beta)\right)\geq R,$$

where $SNR_{suboptimal,lb}(\tilde{L},\gamma,\beta)$ is defined as in Lemma 4. Then

$$L_0\leq\left\lceil\frac{-W(-\frac{a\ln 2}{b}2^{-\frac{ac}{b}})}{a\ln 2}-\frac{a}{b}\right\rceil, \tag{46}$$

where $a=2\left(\frac{R}{K}+\log\beta\right)$, $b=a_{lb}(1-\gamma)2^{-2\frac{R}{K}(K-1)}$, $c=\left[a_{lb}(1-\gamma)(K-1)+1\right]2^{-2\frac{R}{K}(K-1)}$, and W is the Lambert W function, i.e., $W(x)$ is the solution to $x=W(x)e^{W(x)}$.

Proof: Define

$$R_{lb}(\tilde{L},\gamma,\beta)=\frac{K}{2(\tilde{L}+K-1)}\log\left(1+SNR_{suboptimal,lb}(\tilde{L},\gamma,\beta)\right),$$

where $SNR_{suboptimal,lb}(\tilde{L},\gamma,\beta)$ is defined as in Lemma 4. To derive the upper bound on L_0, we solve for $\tilde{L}$ that satisfies $R_{lb}(\tilde{L},\gamma,\beta)=R$. After some manipulations, the preceding equation in $\tilde{L}$ reduces to

$$2^{a\tilde{L}}=b\tilde{L}+c, \tag{47}$$

which is known to have, by substitution, the term inside the ceil operator in (46) as a solution in $\tilde{L}$.

It can be easily shown that $R_{lb}(1,\gamma,\beta)\leq\frac{1}{2}\log(1+P)$ and that $\lim_{\tilde{L}\to\infty}R_{lb}(\tilde{L},\gamma,\beta)=-K\log(\beta)$. Hence, there exists at least one $\tilde{L}$ such that $R_{lb}(\tilde{L},\gamma,\beta)=R$. Now, let us analyze (47). The left-hand side of the equation is a decreasing exponential function in $\tilde{L}$ because a is negative. The right-hand side is a straight line in $\tilde{L}$ with a positive slope. Hence, (47) can have one real valued solution only, call it $\hat{L}$. Then, $R_{lb}(\tilde{L},\gamma,\beta)\geq R$ for all $\tilde{L}\geq\hat{L}$. This validates the use of the ceil operater in (46).

∎

For a fixed $\tilde{L}$, let γ^* and β^* be the $\gamma\in(0,1)$ and $\beta\in(0,1)$ values, respectively, that maximize

$$\frac{K}{2(\tilde{L}+K-1)}\log\left((1+SNR_{optimal}(\sigma_n^2=0,\tilde{L},\gamma,\beta\right).$$

For a sum-rate $R\in(\frac{1}{2}\log(1+P),\frac{1}{2}\log(1+P\phi))$, define $\tilde{L}_s(R)$ as the smallest $\tilde{L}>K$ such that

$$\frac{K}{2(\tilde{L}+K-1)}\log\left(1+SNR_{optimal}(\sigma_n^2=0,\tilde{L},\gamma^*,\beta^*)\right)\geq R. \tag{48}$$

In the following corollary, we find an upper bound on $\tilde{L}_s(R)$.

Corollary 1. *Let f be defined as in Lemma 7 and ϕ defined as in Lemma 5.*

1) *Fix a sum-rate R such that $\frac{1}{2}\log(1+P) < R < \frac{1}{2}\log(1+P\phi)$.*
2) *Choose $\gamma \in (0,1)$ such that*
$$\gamma > \gamma_{lb} = \frac{1}{P} f(2^{-\frac{R}{K}}). \tag{49}$$
3) *Choose β such that*
$$\beta = f^{-1}(\gamma P), \tag{50}$$
where f^{-1} is the inverse of f.
4) *Then $\tilde{L}_s(R)$ can be upper bounded as follows*
$$\tilde{L}_s(R) \leq \tilde{L}_{ub}(R) = \max\{K+1, \left\lceil \frac{-W(-\frac{a\ln 2}{b}2^{-\frac{ac}{b}})}{a\ln 2} - \frac{a}{b} \right\rceil\}, \tag{51}$$
where the parameters a, b, and c are as defined in Lemma 8.

Proof: First, we choose γ such that the linear coding scheme for the noiseless feedback case can achieve a sum-rate larger than R. To do so, we need $-K\log\beta > R$. This implies $\beta < 2^{-\frac{R}{K}}$ which also implies that $f(\beta) > f(2^{-\frac{R}{K}})$. But for $\tilde{L} \to \infty$, the power constraint of the linear scheme reduces to $f(\beta) = \gamma P$. Then $\gamma > \frac{1}{P} f(2^{-\frac{R}{K}})$, where the right-hand side is exactly γ_{lb}.

Now, for any $\gamma > \gamma_{lb}$, choosing $\beta = f^{-1}(\gamma P)$ satisfies the power constraint for any $\tilde{L} > K$ (Lemma 6). The proof then follows by Lemma 8. Note that the use of the $\max$ function in (51) is to ensure that the upper bound on L_0 is no smaller than $K+1$. This is because of the way the linear scheme is constructed that requires $\tilde{L} > K$ for $R > \frac{1}{2}\log(1+P)$. By the discussion in the proof of Lemma 8, a larger blocklength is still a valid upper bound on $\tilde{L}_s(R)$.

■

Following the steps in Corollary 1, we plot $\tilde{L}_{ub}(R)$ in Fig. 4 for sum-rates R between $C_{nf} + 0.01\Delta$ and $C_{nf} + 0.9\Delta$, where $C_{nf} = \frac{1}{2}\log(1+P)$ and $\Delta = \frac{1}{2}\log(1+\phi P) - \frac{1}{2}\log(1+P)$. We consider $P = 10$ and $K = 2$. For each sum-rate point R, the γ chosen in step 2 was $\gamma = \gamma_{lb} + 0.2(1-\gamma_{lb})$.

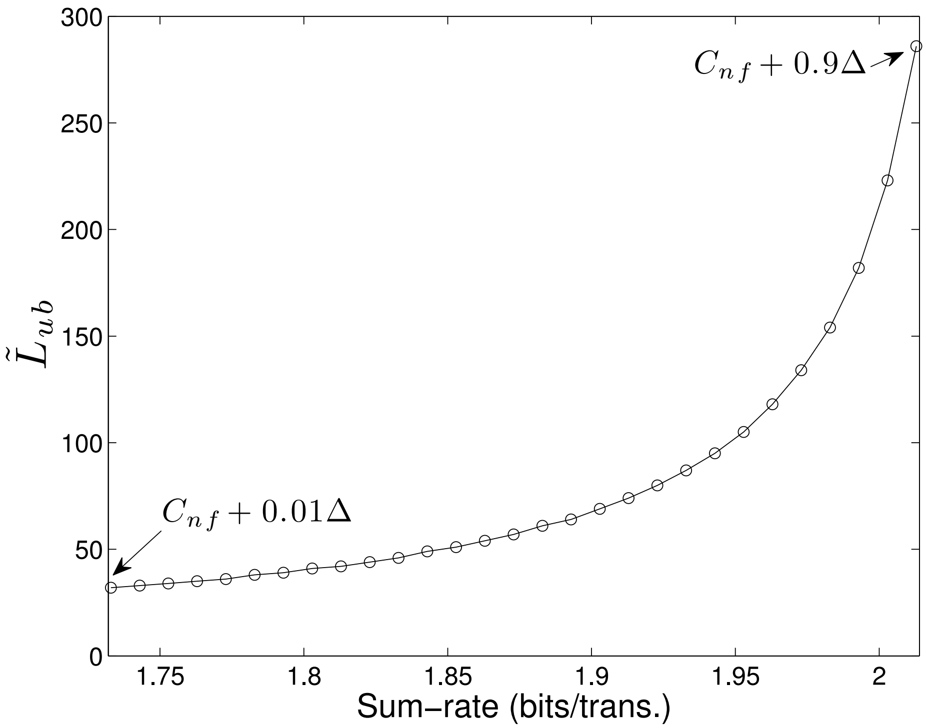

Fig. 4. Upper bound on the $\tilde{L}$ needed for the concatenated coding scheme to start to outperform a certain sum-rate for noiseless feedback. The values of the channel parameters are: $P = 10$ and $K = 2$.

C. Sum-Rate Versus Feedback Noise Level

In this section, we present, using computer experiments for numerical optimization, the achievable sum-rates given a certain feedback noise level. Specifically, we calculated the following

$$R^*(K, P, \sigma_n^2) = \sup_{\substack{\tilde{L} \in \mathbb{N} \\ \beta \in (0,1) \\ \gamma \in [0,1]}} \frac{K}{2(\tilde{L} + K - 1)} \log \left(1 + SNR_{optimal}(\sigma_n^2, \tilde{L}, \gamma, \beta)\right). \tag{52}$$

In Fig. 5, we compare the sum-rates achievable by the proposed concatenated coding scheme with the no-feedback sum-capacity for $P = 10$ and $K = 2$. The solid line is the plot of $R^*(2, 10, \sigma_n^2)$ as a function of σ_n^2. The dashed line is the plot of the no-feedback sum-capacity. The chosen points for σ_n^2 are $0.01 \times 10^{-4}, 0.015 \times 10^{-4}, 0.02 \times 10^{-4}, 0.025 \times 10^{-4}, \ldots, 0.995 \times 10^{-4}, 1 \times 10^{-4}$. The sharp edges of $R^*(2, 10, \sigma_n^2)$ curve occur with the change of the optimal value of $\tilde{L}$. The optimal $\tilde{L}$ is shown on the curve between any two of these edges. Although not exactly a piecewise linear curve, we can observe from the the plot of $R^*(2, 10, \sigma_n^2)$ that $R^*(., ., \sigma_n^2)$ is close to a piecewise linear curve and hence could be estimated, with small error, by a piecewise linear function. However, in this paper, we do not perform such an analysis. From the plot, we can see that for σ_n^2 values close to 1×10^{-4}, the optimal $\tilde{L}$ is 1, i.e., feedback

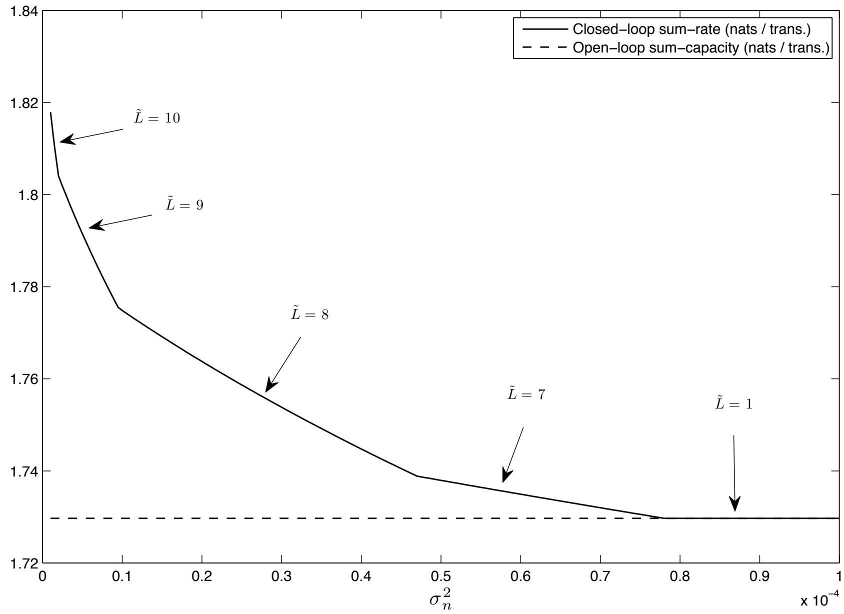

Fig. 5. Comparison between the sum-rates achievable by the proposed concatenated coding scheme with the no-feedback sum-capacity for $P = 10$ and $K = 2$. Specifically, $R^*(2, 10, \sigma_n^2)$ (solid line) and the no-feedback sum-capacity (dashed line) are plotted for σ_n^2 values $0.01 \times 10^{-4}, 0.015 \times 10^{-4}, 0.02 \times 10^{-4}, 0.025 \times 10^{-4}, \ldots, 0.995 \times 10^{-4}, 1 \times 10^{-4}$.

is not utilized. (It is important to note that for the symmetric AWGN-BC orthogonal signaling is optimal for open-loop coding, which is encompassed by our scheme by having $\tilde{L} = 1$ and $\gamma = 0$). However, as σ_n^2 becomes smaller, the concatenated coding scheme outperforms the no-feedback sum-capacity. As $\sigma_n^2 \to 0$, $R^*(., ., \sigma_n^2)$ should approach the bound in Lemma 5 with the optimal $\tilde{L} \to \infty$. On the other hand, for all values of σ_n^2 greater than 1×10^{-4}, the optimal $\tilde{L}$ should remain equal to 1 (with $\gamma = 0$), at which open-loop coding outperforms the use of feedback information.

VI. Conclusion

In this paper, we proposed a concatenated coding design that uses a linear feedback scheme as an inner code to achieve rate tuples for the K-user AWGN-BC with noisy feedback outside the no-feedback capacity region. We have shown an achievable rate region of linear feedback schemes for the noiseless feedback case to be achievable by the concatenated coding scheme for sufficiently small feedback noise level. We also presented a linear feedback scheme for the symmetric K-user AWGN-BC with noisy feedback that was used as an inner code in the concatenated coding scheme that was itself optimized to achieve sum-rates above the no-feedback sum-capacity. The concatenated coding design can also be applied to the two-user AWGN-BC with a single noisy feedback link from one of the receivers.

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