Linear Network Coding Capacity Region of The Smart Repeater with Broadcast Erasure Channels

Jaemin Han and Chih-Chun Wang
School of Electrical and Computer Engineering, Purdue University, USA
{han83,chihw}@purdue.edu

***Abstract*—This work considers the smart repeater network where a single source s wants to send two independent packet streams to destinations $\{d_1, d_2\}$ with the help of relay r. The transmission from s or r is modeled by packet erasure channels: For each time slot, a packet transmitted by s may be received, with some probabilities, by a random subset of $\{d_1, d_2, r\}$; and those transmitted by r will be received by a random subset of $\{d_1, d_2\}$. Interference is avoided by allowing at most one of $\{s, r\}$ to transmit in each time slot. One example of this model is any cellular network that supports two cell-edge users when a relay in the middle uses the same downlink resources for throughput/safety enhancement.**

In this setting, we study the capacity region of (R_1, R_2) when allowing linear network coding (LNC). The proposed LNC inner bound introduces more advanced packing-mixing operations other than the previously well-known butterfly-style XOR operation on overheard packets of two co-existing flows. A new LNC outer bound is derived by exploring the inherent algebraic structure of the LNC problem. Numerical results show that, with more than 85% of the experiments, the relative sum-rate gap between the proposed outer and inner bounds is smaller than 0.08%, thus effectively bracketing the LNC capacity of the smart repeater problem.

I. Introduction

Increasing throughput/connectivity within scarce resources has been the main motivation for modern wireless communications. Among the various proposed techniques, the concept of *relaying* has attracted much attention as a cost-effective enabler to extend the network coverage and capacity. In recent 5G discussions, relaying became one of the core parts for the future cellular architecture including techniques of small cell managements and device-to-device communications [1].

In network information theory, many intelligent and cooperative relaying strategies have been devised such as decode-and-forward/compress-and-forward for relay networks [2], [3], network coding for noiseless networks [4], and general noisy network coding for discrete memoryless networks [5]. Among them, network coding has emerged as a promising technique for a practical wireless networking solution, which models the underlying wireless channels by a simple but non-trivial random packet erasure network. That is, each node is associated with its own broadcast packet erasure channel (PEC). Namely, each node can choose a symbol $X \in \mathbb{F}_q$ from some finite field $\mathbb{F}_q$, transmits X, and a random subset of receivers will receive the packet. In this setting, [6] proved that the *linear network coding* (LNC), operating only by "linear" packet-mixings, suffices to achieve the single-multicast capacity. Moreover, recent wireless testbeds have also demonstrated substantial LNC throughput gain for multiple-unicasts over the traditional store-and-forward 802.11 routing protocols [7], [8].

This work was supported in part by NSF grants: ECCS-1407603 and CCF-1422997.

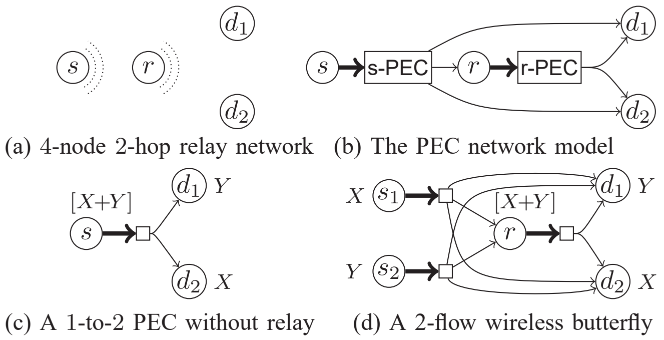

Fig. 1: The 2-flow Smart Repeater Network and its subset scenarios

Motivated by these results, we are interested in finding an optimal or near-optimal LNC strategy for wireless relaying networks. To simplify the analysis, we consider a 4-node 2-hop network with one source s, two destinations $\{d_1, d_2\}$, and a common relay r inter-connected by two broadcast PECs. See Fig. 1(a-b) for details. We assume time-sharing between s and r so that interference is fully avoided, and assume the causal packet ACKnowledgment feedback [7]–[10]. In this way, we can concentrate on how the relay r and source s can jointly exploit the broadcast channel diversity within the network.

When relay r is not present, Fig. 1(b) collapses to Fig. 1(c), the 2-receiver broadcast PEC, where a simple LNC scheme is capacity-achieving [9]. The idea is to exploit the wireless diversity created by random packet erasures, i.e., overhearing packets of other flows. Whenever a packet X intended for d_1 is received only by d_2 and a packet Y intended for d_2 is received only by d_1, s can transmit their linear mixture $[X + Y]$ to benefit both receivers simultaneously. This simple but elegant "butterfly-style" LNC operation achieves the Shannon capacity of Fig. 1(c) [9]. Another related scenario is a 2-flow wireless butterfly network in Fig. 1(d) that contains two separate sources s_1 and s_2 instead of a single source s as in our setting. In this butterfly scenario, each source can only mix packets of their own flow. [10] showed that the same butterfly-style LNC is no longer optimal but very close to optimal. In contrast, in our setting of Fig. 1(b), the two flows are originating from the same source s. Therefore, s can perform "inter-flow NC" to further improve the performance. As we will see, relay r should not just "forward" the packets it has received and need to actively perform coding in order to approach the capacity.

This is why we call such a scenario the smart repeater problem.

Contributions: This work investigates the LNC capacity region (R_1, R_2) of the smart repeater network. The outer bound is proposed by leveraging upon the algebraic structure of the underlying LNC problem. For the achievability scheme, we show that the classic butterfly-style is far from optimality and propose new LNC operations that lead to close-to-optimal performance. By numerical simulations, we demonstrate that the proposed outer/inner bounds are very close, thus effectively bracketing the LNC capacity of the smart repeater problem.

II. Problem Definition And Useful Notations

A. Problem Formation for The Smart Repeater Network

The 2-flow wireless smart repeater network with broadcast PECs, see Fig. 1(b), can be modeled as follows. Consider two traffic rates (R_1, R_2) and assume slotted transmissions. Within a total budget of n time slots, source s would like to send nR_k packets, denoted by a row vector $\mathbf{W}_k$, to destination d_k for all $k\in\{1,2\}$ with the help of relay r. Each packet is chosen uniformly randomly from a finite field $\mathbb{F}_q$ with size $q>0$. To that end, we denote $\mathbf{W} \triangleq (\mathbf{W}_1, \mathbf{W}_2)$ as an $n(R_1+R_2)$-dimensional row vector of all the packets, and define the linear space $\Omega \triangleq (\mathbb{F}_q)^{n(R_1+R_2)}$ as the *overall message/coding space*.

To represent the reception status, for any time slot $t \in \{1, \cdots, n\}$, we define two *channel reception status vectors*:

$$\mathbf{Z}_s(t) = (Z_{s\to d_1}(t), Z_{s\to d_2}(t), Z_{s\to r}(t)) \in \{1, *\}^3,$$
$$\mathbf{Z}_r(t) = (Z_{r\to d_1}(t), Z_{r\to d_2}(t)) \in \{1, *\}^2,$$

where "1" and "$*$" represent successful reception and erasure, respectively. For example, $Z_{s\to d_1}(t) = 1$ and $*$ represents whether d_1 can receive the transmission from source s or not at time slot t. We then use $\mathbf{Z}(t) \triangleq (\mathbf{Z}_s(t), \mathbf{Z}_r(t))$ to describe the 5-dimensional channel reception status vector of the entire network. We also assume that $\mathbf{Z}(t)$ is memoryless and stationary, i.e., $\mathbf{Z}(t)$ is independently and identically distributed over the time axis t.

We assume that either source s or relay r can transmit at each time slot, and express the *scheduling decision* by $\sigma(t)\in \{s, r\}$. For example, if $\sigma(t) = s$, then source s transmits a packet $X_s(t) \in \mathbb{F}_q$; and only when $Z_{s\to h}(t) = 1$, node h (one of $\{d_1, d_2, r\}$) will receive $Y_{s\to h}(t) = X_s(t)$. In all other cases, node h receives an erasure $Y_{s\to h}(t) = *$. The reception $Y_{r\to h}(t)$ of relay r's transmission is defined similarly.

Assuming that the 5-bit $\mathbf{Z}(t)$ vector is broadcast to both s and r after each packet transmission through a separate control channel, a *linear network code* contains n scheduling functions

$$\forall\, t \in \{1, \cdots, n\}, \;\; \sigma(t) = f_{\sigma,t}([\mathbf{Z}]_1^{t-1}), \tag{1}$$

where we use brackets $[\,\cdot\,]_1^\tau$ to denote the collection from time 1 to τ. Namely, at every time t, scheduling is decided based on the network-wide channel state information (CSI) up to time $(t-1)$. If source s is scheduled, then it can send a linear combination of any packets. That is,

$$\text{If } \sigma(t) = s, \text{ then } X_s(t) = \mathbf{c}_t\mathbf{W}^\top \text{ for some } \mathbf{c}_t \in \Omega, \tag{2}$$

where $\mathbf{c}_t$ is a row coding vector in Ω. The choice of $\mathbf{c}_t$ depends on the past CSI vectors $[\mathbf{Z}]_1^{t-1}$, and we assume that $\mathbf{c}_t$ is known causally to the entire network.[1] Therefore, decoding can be performed by simple Gaussian elimination.

We now define two important linear space concepts: The *individual message subspace* and the *knowledge subspace.* To that end, we first define $\mathbf{e}_l$ as an $n(R_1 + R_2)$-dimensional elementary row vector with its l-th coordinate being one and all the other coordinates being zero. Recall that the $n(R_1+R_2)$ coordinates of a vector in Ω can be divided into 2 consecutive "intervals", each of them corresponds to the information packets $\mathbf{W}_k$ for each flow from source to destination d_k. We then define the *individual message subspace* Ω_k:

$$\Omega_k \triangleq \mathsf{span}\{\mathbf{e}_l : l \in \text{"interval" associated to } \mathbf{W}_k\}, \tag{3}$$

That is, Ω_k is a linear space corresponding to any linear combination of $\mathbf{W}_k$ packets. By (3), each Ω_k is a linear subspace of the overall message space Ω and $\mathsf{rank}(\Omega_k) = nR_k$.

We define the knowledge space for $\{d_1, d_2, r\}$. The knowledge space $S_h(t)$ in the end of time t is defined by

$$S_h(t) \triangleq \mathsf{span}\{\mathbf{c}_\tau : \forall\tau \le t \text{ s.t. node } h \text{ receives the linear combination } (\mathbf{c}_\tau\cdot\mathbf{W}^\top) \text{ successfully in time } \tau\} \tag{4}$$

where $h\in\{d_1, d_2, r\}$. For example, $S_r(t)$ is the linear space spanned by the packets successfully delivered from source to relay up to time t. $S_{d_1}(t)$ is the linear space spanned by the packets received at destination d_1 up to time t, either transmitted by source or by relay.

For shorthand, we use $S_1(t)$ and $S_2(t)$ instead of $S_{d_1}(t)$ and $S_{d_2}(t)$, respectively. Then, by the above definitions, we quickly have that destination d_k can decode the desired packets $\mathbf{W}_k$ as long as $S_k(n) \supseteq \Omega_k$. That is, when the knowledge space in the end of time n contains the desired message space.

With the above linear space concepts, we now can describe the packet transmission from relay. Recall that, unlike the source where the packets are originated, relay can only send a linear mixture of *the packets that it has known.* Therefore, the encoder description from relay can be expressed by

$$\text{If } \sigma(t) = r, \text{ then } X_r(t) = \mathbf{c}_t\mathbf{W}^\top \text{ for some } \mathbf{c}_t \in S_r(t-1). \tag{5}$$

For comparison, in (2), the source s chooses $\mathbf{c}_t$ from Ω. We can now define the LNC capacity region.

Definition 1. Fix the distribution of $\mathbf{Z}(t)$ and finite field $\mathbb{F}_q$. A rate vector (R_1, R_2) is achievable by LNC if for any $\epsilon > 0$ there exists a joint scheduling and LNC scheme with sufficiently large n such that $\mathsf{Prob}(S_k(n) \supseteq \Omega_k) > 1 - \epsilon$ for all $k \in \{1, 2\}$. The LNC capacity region is the closure of all LNC-achievable (R_1, R_2).

B. A Useful Notation

In our network model, there are two broadcast PECs associated with s and r. For shorthand, we call those PECs the s-PEC and the r-PEC, respectively. The distribution of the network-wide channel status vector $\mathbf{Z}(t) = (\mathbf{Z}_s(t), \mathbf{Z}_r(t))$ can be described by the probabilities $p_{s\to T\overline{\{d_1,d_2,r\}\backslash T}}$ for all

[1] Coding vector $\mathbf{c}_t$ can either be appended in the header or be computed by the network-wide causal CSI feedback $[\mathbf{Z}]_1^{t-1}$.

$T \subseteq \{d_1, d_2, r\}$, and $p_{r\to U\overline{\{d_1,d_2\}\backslash U}}$ for all $U \subseteq \{d_1, d_2\}$. In total, there are $8 + 4 = 12$ channel parameters.[2]

For notational simplicity, we also define the following two probability functions $p_s(\cdot)$ and $p_r(\cdot)$, one for each PEC. The input argument of p_s is a collection of the elements in $\{d_1, d_2, r, \overline{d_1}, \overline{d_2}, \overline{r}\}$. The function $p_s(\cdot)$ outputs the probability that the reception event is compatible to the specified collection of $\{d_1, d_2, r, \overline{d_1}, \overline{d_2}, \overline{r}\}$. For example, $p_s(d_2\overline{r}) = p_{s\to\overline{d_1}d_2\overline{r}} + p_{s\to d_1d_2\overline{r}}$ is the probability that the input of the s-PEC is successfully received by d_2 but not by r. Herein, d_1 is a don't-care receiver and $p_s(d_2\overline{r})$ thus sums two joint probabilities together (whether d_1 receives it or not). Another example is $p_r(d_2) = p_{r\to d_1d_2} + p_{r\to\overline{d_1}d_2}$, which is the marginal probability from r to d_2. To slightly abuse the notation, we further allow $p_s(\cdot)$ and $p_r(\cdot)$ to take multiple input arguments separated by commas. With this new notation, they can represent the probability that the reception event is compatible to at least one of the input arguments. E.g., $p_s(d_1, d_2, r)$ represents the probability that a packet sent by s is received by at least one of the three nodes d_1, d_2, and r. The indicator function and taking expectation is denoted by $1_{\{\cdot\}}$ and $\mathbb{E}[\cdot]$, respectively.

III. Main Results

A. LNC Capacity Outer Bound

Since the coding vector $\mathbf{c}_t$ has $n(R_1+R_2)$ number of coordinates, there are exponentially many ways of jointly designing the scheduling $\sigma(t)$ and the coding vector $\mathbf{c}_t$ choices over time when sufficiently large n and $\mathbb{F}_q$ are used. Therefore, we will first simplify the aforementioned design choices by comparing $\mathbf{c}_t$ to the knowledge spaces $S_h(t-1)$, $h \in \{d_1, d_2, r\}$. Such a simplification allows us to derive Proposition 1, which uses a linear programming (LP) solver to exhaustively search over the entire coding and scheduling choices and thus computes an LNC capacity outer bound.

To that end, we use S_k as shorthand for $S_k(t-1)$, the knowledge space of destination d_k in the end of time $t-1$. We first define the following 7 linear subspaces of Ω.

$$A_1(t) \triangleq S_1, \qquad A_2(t) \triangleq S_2, \tag{6}$$
$$A_3(t) \triangleq S_1 \oplus \Omega_1, \qquad A_4(t) \triangleq S_2 \oplus \Omega_2, \tag{7}$$
$$A_5(t) \triangleq S_1 \oplus S_2, \tag{8}$$
$$A_6(t) \triangleq S_1 \oplus S_2 \oplus \Omega_1, \qquad A_7(t) \triangleq S_1 \oplus S_2 \oplus \Omega_2, \tag{9}$$

where $A \oplus B \triangleq \mathsf{span}\{\mathbf{v} : \mathbf{v} \in A \cup B\}$ is the *sum space* of any $A, B \subseteq \Omega$. In addition, we also define the following eight additional subspaces involving $S_r(t-1)$:

$$A_{i+7}(t) \triangleq A_i(t) \oplus S_r \quad \text{for all } i = 1, \cdots, 7, \tag{10}$$
$$A_{15}(t) \triangleq S_r, \tag{11}$$

where S_r is a shorthand notation for $S_r(t-1)$, the knowledge space of relay r in the end of time $t-1$.

[2]By allowing some coordinates of $\mathbf{Z}(t)$ to be correlated (i.e., spatially correlated as it is between coordinates, not over the time axis), our setting can also model the scenario in which d_1 and d_2 are situated in the same physical node and thus have perfectly correlated channel success events.

In total, there are $7+8 = 15$ linear subspaces of Ω. We then partition the overall message space Ω into 2^{15} disjoint subsets by the *Venn diagram* generated by these 15 subspaces. That is, at any time t, we can place any coding vector $\mathbf{c}_t$ in exactly one of the 2^{15} disjoint subsets by testing whether it belongs to which A-subspaces. We will often drop the input argument "(t)" when the time instant of interest is clear in the context.

We then use 15 bits to represent each disjoint subset in Ω: For any 15-bit string $\mathbf{b} = b_1b_2\cdots b_{15}$, we define the "coding type-$\mathbf{b}$" by

$$\mathsf{TYPE}^{(s)}_{\mathbf{b}} \triangleq \Big(\bigcap_{l:b_l=1} A_l\Big) \backslash \Big(\bigcup_{l:b_l=0} A_l\Big), \tag{12}$$

where the regions of these 2^{15} disjoint coding types may vary at every time instant as the 15 A-subspaces defined in (6) to (11) will evolve over the course of time. The superscript "(s)" indicates the source, meaning that s can send $\mathbf{c}_t$ in any coding type since source s knows all $\mathbf{W}_1$ and $\mathbf{W}_2$ packets to begin with. Note that not all 2^{15} disjoint subsets are feasible. For example, any $\mathsf{TYPE}^{(s)}_{\mathbf{b}}$ with $b_7 = 1$ but $b_{14} = 0$ is always empty because any coding vector that lies in $A_7 = S_1 \oplus S_2 \oplus \Omega_2$ cannot lie outside the larger $A_{14} = S_1 \oplus S_2 \oplus S_r \oplus \Omega_2$, see (9) and (10), respectively. We say those always empty subsets *infeasible coding types* and the rest is called *feasible coding types* (FTs). By exhaustive computer search, we can prove that out of $2^{15}{=}32768$ subsets, only 154 of them are feasible. Namely, the entire coding space Ω can be viewed as a union of 154 disjoint coding types. Source s can choose a coding vector $\mathbf{c}_t$ from one of these 154 types. See (2).

By (5), we know that when $\sigma(t) = r$, the $\mathbf{c}_t$ sent by relay must belong to its knowledge space $S_r(t-1)$. Hence, such $\mathbf{c}_t$ must always lie in $S_r(t-1)$, which is $A_{15}(t)$, see (11). As a result, any coding vector $\mathbf{c}_t$ sent by relay r must lie in those 154 subsets FTs that satisfy:

$$\mathsf{TYPE}^{(r)}_{\mathbf{b}} \triangleq \{\mathsf{TYPE}^{(s)}_{\mathbf{b}} : \mathbf{b} \in \mathsf{FTs} \text{ such that } b_{15} = 1\}. \tag{13}$$

Again by computer search, there are 18 such coding types out of 154 subsets FTs. We call those 18 subsets as *relay's feasible coding types* (rFTs). Obviously, $r\mathsf{FTs} \subseteq \mathsf{FTs}$. See [?] for the detailed enumeration of those FTs and rFTs.

We can then derive the following upper bound.

Proposition 1. A rate vector (R_1, R_2) is in the LNC capacity region only if there exists 154 non-negative variables $x^{(s)}_{\mathbf{b}}$ for all $\mathbf{b} \in \mathsf{FTs}$, 18 non-negative variables $x^{(r)}_{\mathbf{b}}$ for all $\mathbf{b} \in r\mathsf{FTs}$, and 14 non-negative y-variables, y_1 to y_{14}, such that jointly they satisfy the following three groups of linear conditions:

- Group 1, termed the *time-sharing condition*, has 1 inequality:

$$\Big(\sum_{\forall \mathbf{b}\in\mathsf{FTs}} x^{(s)}_{\mathbf{b}}\Big) + \Big(\sum_{\forall \mathbf{b}\in r\mathsf{FTs}} x^{(r)}_{\mathbf{b}}\Big) \leq 1. \tag{14}$$

- Group 2, the *rank-conversion conditions*, has 14 equalities:

$$y_{0+k+2l} = \Big(\sum_{\forall \mathbf{b}\in\mathsf{FTs} \text{ s.t. } b_{0+k+2l}=0} x^{(s)}_{\mathbf{b}} \cdot p_s(d_k)\Big) + \Big(\sum_{\forall \mathbf{b}\in r\mathsf{FTs} \text{ s.t. } b_{0+k+2l}=0} x^{(r)}_{\mathbf{b}} \cdot p_r(d_k)\Big) + R_k \cdot 1_{\{l=1\}}, \text{ for all } k \in \{1,2\} \text{ and } l \in \{0,1\}, \tag{15}$$

$$y_{5+k} = \left(\sum_{\forall \mathbf{b} \in \mathsf{FTs} \text{ s.t. } b_{5+k}=0} x_{\mathbf{b}}^{(s)} \cdot p_s(d_1, d_2) \right) + \left(\sum_{\forall \mathbf{b} \in r\mathsf{FTs} \text{ s.t. } b_{5+k}=0} x_{\mathbf{b}}^{(r)} \cdot p_r(d_1, d_2) \right)$$
$$+ R_k \cdot 1_{\{k \neq 0\}}, \text{ for all } k \in \{0,1,2\}, \tag{16}$$

$$y_{7+k+2l} = \left(\sum_{\forall \mathbf{b} \in \mathsf{FTs} \text{ s.t. } b_{7+k+2l}=0} x_{\mathbf{b}}^{(s)} \cdot p_s(d_k, r) \right) + R_k \cdot 1_{\{l=1\}},$$
$$\text{for all } k \in \{1,2\} \text{ and } l \in \{0,1\}, \tag{17}$$

$$y_{12+k} = \left(\sum_{\forall \mathbf{b} \in \mathsf{FTs} \text{ s.t. } b_{12+k}=0} x_{\mathbf{b}}^{(s)} \cdot p_s(d_1, d_2, r) \right) + R_k \cdot 1_{\{k \neq 0\}},$$
$$\text{for all } k \in \{0,1,2\}, \tag{18}$$

• Group 3, termed the *decodability conditions*, has 5 equalities:

$$y_1 = y_3, \quad y_2 = y_4, \quad y_8 = y_{11}, \quad y_9 = y_{11}, \tag{19}$$
$$y_5 = y_6 = y_7 = y_{12} = y_{13} = y_{14} = (R_1 + R_2). \tag{20}$$

The high-level intuition is as follows. Since we are partitioning Ω (the entire coding space) and S_r (the knowledge space of r) into 154 feasible coding types FTs and 18 subsets $r\mathsf{FTs}$, any LNC scheme achieving the given (R_1, R_2) can be classified as either s or r sending $\mathbf{c}_t$ in certain coding type at each time instant. Given one scheme, we use variables $x_{\mathbf{b}}^{(h)} \triangleq \frac{1}{n}\mathbb{E}\left[\sum_{t=1}^{n} 1_{\{\mathbf{c}_t \in \mathsf{TYPE}_{\mathbf{b}}^{(h)}\}}\right]$, $h \in \{s, r\}$, to compute the fraction of time that s (resp. r) spends on sending $\mathbf{c}_t$ in coding type-$\mathbf{b}$. Hence, jointly they must satisfy Group 1, the time-sharing condition. Group 2 conditions quantify the impact of sending $\mathbf{c}_t$ in different types on each subspace $A_l(t)$ until the end of time n, by the normalized rank $y_l \triangleq \frac{1}{n}\mathbb{E}\left[\mathsf{rank}\big(A_l(n)\big)\right]$, $l=1$ to 14. Finally, since the given scheme is "decodable", we must have that, for each $k \in \{1,2\}$, $\mathsf{rank}(A_k(n)) = \mathsf{rank}(S_k(n)) = \mathsf{rank}(S_k(n) \oplus \Omega_k) = \mathsf{rank}(A_{k+2}(n))$, implying $y_1 = y_3$ and $y_2 = y_4$ in (19). The detailed proof can be found in [11].

B. LNC Capacity Inner Bound

In the smart repeater problem, s can always take over relay's operations, and thus r becomes useless when the r-PEC is weaker than the s-PEC. To fully fetch the coding and diversity benefits using relay, we focus on the following assumption.

Definition 2. The smart repeater network with $\{d_1, d_2\}$ is *strong-relaying* if $p_r(T\overline{\{d_1, d_2\}\backslash T}) > p_s(T\overline{\{d_1, d_2\}\backslash T})$ for all $T \subseteq \{d_1, d_2\} \backslash \emptyset$. That is, the given r-PEC is stronger than the given s-PEC for all non-empty subsets of $\{d_1, d_2\}$.

We describe our capacity-approaching achievability scheme.

Proposition 2. (R_1, R_2) is LNC-achievable if there exist 2 non-negative variables t_s and t_r, $(6 \times 2 + 8)$ non-negative s-variables: $\{s_{\mathsf{UC}}^k, s_{\mathsf{PM1}}^k, s_{\mathsf{PM2}}^k, s_{\mathsf{RC}}^k, s_{\mathsf{DX}}^k, s_{\mathsf{DX}}^{\textcircled{k}} : \forall k \in \{1,2\}\}$; and $\{s_{\mathsf{CX};l}\ (l = 1, \cdots, 8)\}$, and $(3 \times 2 + 3)$ non-negative r-variables: $\{r_{\mathsf{UC}}^k, r_{\mathsf{DT}}^{\textcircled{k}}, r_{\mathsf{DT}}^{[k]} : \forall k \in \{1,2\}\}$; and $\{r_{\mathsf{RC}}, r_{\mathsf{XT}}, r_{\mathsf{CX}}\}$, such that jointly satisfy the following six groups of linear conditions:

• Group 1, the *time-sharing conditions*, has 3 inequalities:

$$1 > t_s + t_r, \tag{21}$$

$$t_s \geq \sum_{k \in \{1,2\}} \left(s_{\mathsf{UC}}^k + s_{\mathsf{PM1}}^k + s_{\mathsf{PM2}}^k + s_{\mathsf{RC}}^k + s_{\mathsf{DX}}^k + s_{\mathsf{DX}}^{\textcircled{k}} \right) + \sum_{l=1}^{8} s_{\mathsf{CX};l}, \tag{22}$$

$$t_r \geq \sum_{k \in \{1,2\}} \left(r_{\mathsf{UC}}^k + r_{\mathsf{DT}}^{\textcircled{k}} + r_{\mathsf{DT}}^{[k]} \right) + r_{\mathsf{RC}} + r_{\mathsf{XT}} + r_{\mathsf{CX}}. \tag{23}$$

• Group 2, termed the *packets-originating condition*, has 2 inequalities: Consider any $i, j \in \{1,2\}$ satisfying $i \neq j$. For each (i,j) pair (out of the two choices $(1,2)$ and $(2,1)$),

$$R_i \geq \left(s_{\mathsf{UC}}^i + s_{\mathsf{PM1}}^i \right) \cdot p_s(d_i, d_j, r), \tag{E}$$

• Group 3, termed the *packets-mixing condition*, has 4 inequalities: For each (i,j) pair,

$$\begin{aligned} \left(s_{\mathsf{UC}}^i + s_{\mathsf{PM1}}^i \right) \cdot p_{s \to \overline{d_i d_j} r} \geq \left(s_{\mathsf{PM1}}^j + s_{\mathsf{PM2}}^i \right) \cdot p_s(d_i, d_j) \\ + r_{\mathsf{UC}}^i \cdot p_r(d_i, d_j), \end{aligned} \tag{A}$$

$$s_{\mathsf{PM1}}^i \cdot p_{s \to \overline{d_i d_j r}} \geq s_{\mathsf{RC}}^i \cdot p_s(d_i, d_j, r), \tag{B}$$

and the following one inequality:

$$\begin{aligned} s_{\mathsf{PM1}}^1 \cdot p_s(d_1, d_2 r) + s_{\mathsf{PM1}}^2 \cdot p_s(d_2, d_1 r) + s_{\mathsf{PM2}}^1 \cdot p_s(\overline{d_1} d_2) + \\ s_{\mathsf{PM2}}^2 \cdot p_s(d_1 \overline{d_2}) + \left(s_{\mathsf{RC}}^1 + s_{\mathsf{RC}}^2 \right) \cdot p_{s \to \overline{d_1 d_2} r} \geq r_{\mathsf{RC}} \cdot p_r(d_1, d_2). \end{aligned} \tag{M}$$

• Group 4, termed the *classic XOR condition by source only*, has 4 inequalities: For each (i,j) pair,

$$\begin{aligned} \left(s_{\mathsf{UC}}^i + s_{\mathsf{RC}}^i \right) p_{s \to \overline{d_i d_j r}} \geq \left(s_{\mathsf{PM2}}^j + s_{\mathsf{DX}}^i \right) \cdot p_s(d_i, r) + \\ (s_{\mathsf{CX};1} + s_{\mathsf{CX};1+i}) \cdot p_s(d_i, r) + s_{\mathsf{CX};4+i} \cdot p_s(d_i, r), \end{aligned} \tag{S}$$

$$\begin{aligned} s_{\mathsf{RC}}^j \cdot p_{s \to \overline{d_i} d_j \overline{r}} \geq s_{\mathsf{DX}}^{\textcircled{i}} \cdot p_s(d_i, r) + r_{\mathsf{DT}}^{\textcircled{i}} \cdot p_r(d_i, d_j) + \\ (s_{\mathsf{CX};1+j} + s_{\mathsf{CX};4}) \cdot p_s(d_i, r) + s_{\mathsf{CX};6+i} \cdot p_s(d_i, r). \end{aligned} \tag{T}$$

• Group 5, termed the *XOR condition*, has 3 inequalities:

$$\sum_{l=1}^{4} s_{\mathsf{CX};l} \cdot p_{s \to \overline{d_1 d_2} r} \geq r_{\mathsf{XT}} \cdot p_r(d_1, d_2), \tag{X0}$$

and for each (i,j) pair,

$$\begin{aligned} & s_{\mathsf{PM2}}^j \cdot p_s(d_i d_j, \overline{d_i} r) + \left(s_{\mathsf{UC}}^i + s_{\mathsf{RC}}^i + s_{\mathsf{RC}}^j + \sum_{l=1}^{4} s_{\mathsf{CX};l} \right) \cdot p_{s \to \overline{d_i} d_j r} \\ & + \left(s_{\mathsf{CX};4+i} + s_{\mathsf{CX};6+i} + s_{\mathsf{DX}}^i + s_{\mathsf{DX}}^{\textcircled{i}} \right) \cdot p_s(\overline{d_i} r) \\ & + \left(r_{\mathsf{UC}}^i + r_{\mathsf{RC}} + r_{\mathsf{DT}}^{\textcircled{i}} + r_{\mathsf{XT}} \right) \cdot p_{r \to \overline{d_i} d_j} \\ & \geq (s_{\mathsf{CX};7-i} + s_{\mathsf{CX};9-i}) \cdot p_s(d_i) + \left(r_{\mathsf{CX}} + r_{\mathsf{DT}}^{[i]} \right) \cdot p_r(d_i). \end{aligned} \tag{X}$$

• Group 6, termed the *decodability condition*, has 2 inequalities: For each (i,j) pair,

$$\begin{aligned} & \left(s_{\mathsf{UC}}^i + s_{\mathsf{PM2}}^j + \sum_{k \in \{1,2\}} s_{\mathsf{RC}}^k + \sum_{l=1}^{8} s_{\mathsf{CX};l} + s_{\mathsf{DX}}^i + s_{\mathsf{DX}}^{\textcircled{i}} \right) \cdot p_s(d_i) + \\ & \left(r_{\mathsf{UC}}^i + r_{\mathsf{RC}} + r_{\mathsf{XT}} + r_{\mathsf{CX}} + r_{\mathsf{DT}}^{\textcircled{i}} + r_{\mathsf{DT}}^{[i]} \right) \cdot p_r(d_i) \geq R_i. \end{aligned} \tag{D}$$

This inner bound can be described based on packet movements in a queueing network, governed by the proposed LNC operations. Each s- and r-variable (except t-variables for time-sharing) is associated with a specific LNC operation performed by the source s and the relay r, respectively. The inequalities (E) to (D) then describe the queueing process, where LHS and RHS of each inequality implies the packet insertion and removal condition of a queue. E.g., suppose that $\mathbf{W}_1 = (X_1, \cdots, X_{nR_1})$ packets and $\mathbf{W}_2 = (Y_1, \cdots, Y_{nR_2})$ packets are

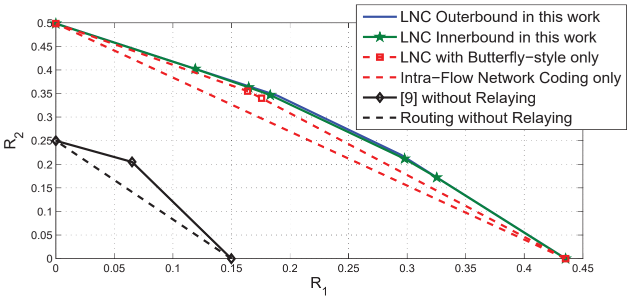

Fig. 2: Comparison of LNC regions with different achievable rates

initially stored in queues at source, denoted by Q^1_ϕ and Q^2_ϕ, respectively. The LNC operation corresponding to the variable s^1_{UC} (resp. s^2_{UC}) is that source s sends a packet $X_l \in Q^1_\phi$ for d_1 (resp. a packet $Y_l \in Q^2_\phi$ for d_2) uncodedly. Then the inequality (E) when $(i,j) = (1,2)$ (resp. $(i,j) = (2,1)$) implies that whenever it is received by at least one of $\{d_1, d_2, r\}$, this packet is removed from the queue of Q^1_ϕ (resp. Q^2_ϕ).

Depending on the reception status, a packet will either be moved to another queue or remain in the same queue. E.g., the use of the s^1_{UC}-operation will take one packet $X_l \in Q^1_\phi$ and insert it into the queue associated with the inequality (D) when $(i,j) = (1,2)$, as long as $Z_{s\to d_1}(t) = 1$, i.e., when d_1 correctly receives it. The details of the proposed queues and LNC operations, and their proofs are omitted due to space but can be found in [11].

At a high-level, variables $s_{\mathsf{CX};l}\,(l{=}1,\cdots,8)$, r_{CX}, and r_{XT} correspond to the coding operations that are direct generalization of the classic butterfly-style LNC, which when used alone are far from optimal. New LNC operations corresponding to s^k_{PM1}, s^k_{PM2}, s^k_{RC}, and r_{RC} are introduced in this work to significantly narrow the gap to the outer bound in Proposition 1.

C. Numerical Evaluation

Consider a smart repeater network with marginal channel success probabilities: (a) s-PEC: $p_s(d_1) = 0.15$, $p_s(d_2) = 0.25$, and $p_s(r) = 0.8$; and (b) r-PEC: $p_r(d_1) = 0.75$ and $p_r(d_2) = 0.85$. We assume that all the erasure events are independent. We will use the results in Propositions 1 and 2 to find the largest (R_1, R_2) value for this example scenario.

Fig. 2 compares the LNC capacity outer bound (Proposition 1) and the LNC inner bound (Proposition 2) with different achievability schemes. The smallest rate region is achieved by simply performing uncoded direct transmission without using the relay r. The second achievability scheme is the 2-receiver broadcast channel LNC from the source s in [9] while still not exploiting r at all. The third scheme performs the time-shared transmission between s and r, while allowing only intra-flow network coding. The forth scheme is derived from using only the classic butterfly-style LNCs corresponding to $s_{\mathsf{CX};l}\,(l{=}1,\cdots,8)$, r_{CX}, and r_{XT}. One can see that the result is strictly suboptimal. This shows that the newly-identified LNC operations other than the classic butterfly-style LNCs are critical in approaching the LNC capacity.

Fig. 3 examines the relative gaps between the outer and inner bounds by choosing the channel parameters $p_s(\cdot)$ and $p_r(\cdot)$ uniformly randomly while obeying the strong-relaying condition in Definition 2. For any chosen parameter instance, we use a linear programming solver to find the largest sum rate $(R_1 + R_2)$ of the LNC outer and inner bounds of Propositions 1 and 2, which are denoted by $R_{\mathsf{sum.outer}}$ and $R_{\mathsf{sum.inner}}$, respectively. We then compute the relative gap per each experiment, $(R_{\mathsf{sum.outer}} - R_{\mathsf{sum.inner}})/R_{\mathsf{sum.outer}}$, and then repeat the experiment 10000 times, and plot the cumulative distribution function (cdf) in unit of percentage. We can see that with more than 85% of the experiments, the relative gap between the outer and inner bound is smaller than 0.08%.

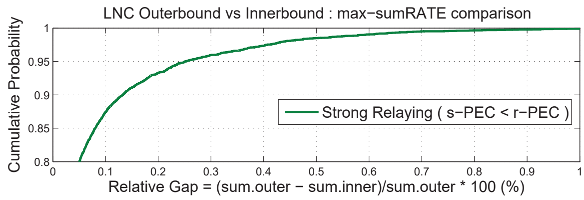

Fig. 3: The cumulative distribution of the relative gap between the outer and inner bounds.

IV. Conclusion

This work studies the LNC capacity of the smart repeater packet erasure network for two unicast flows. The capacity region has been effectively characterized by the proposed linear-subspace-based outer bound, and the capacity-approaching LNC scheme with newly identified LNC operations other than the previously well-known classic butterfly-style operations.

References

[1] S. Chen and J. Zhao, "The requirements, challenges, and technologies for 5g of terrestrial mobile telecommunication," *IEEE Commun. Mag.*, vol. 52, no. 5, pp. 36–43, May 2014.

[2] T. M. Cover and A. E. Gamal, "Capacity theorems for the relay channel," *IEEE Trans. Inform. Theory*, vol. 25, no. 5, pp. 572–584, Sept. 1979.

[3] G. Kramer, M. Gastpar, and P. Gupta, "Cooperative strategies and capacity theorems for relay networks," *IEEE Trans. Inform. Theory*, vol. 51, no. 9, pp. 3037–3063, September 2005.

[4] R. Ahlswede, N. Cai, S.-Y. Li, and R. Yeung, "Network information flow," *IEEE Trans. Inform. Theory*, vol. 46, no. 4, July 2000.

[5] S. H. Lim, Y.-H. Kim, A. E. Gamal, and S.-Y. Chung, "Noisy network coding," *IEEE Trans. Inform. Theory*, vol. 57, no. 5, May 2011.

[6] A. Dana, R. Gowaikar, R. Palanki, B. Hassibi, and M. Effros, "Capacity of wireless erasure networks," *IEEE Trans. Inform. Theory*, vol. 52, no. 3, pp. 789–804, March 2006.

[7] S. Katti, H. Rahul, W. Hu, D. Katabi, M. Medard, and J. Crowcroft, "Xors in the air: Practical wireless network coding," in *Proc. ACM SIGCOMM*, Pisa, Italy, September 2006, pp. 243–254.

[8] D. Koutsonikolas, C.-C. Wang, and Y. Hu, "Efficient network coding based opportunistic routing through cumulative coded acknowledgment," *IEEE/ACM Trans. Networking*, vol. 19, no. 5, pp. 1368–1381, 2011.

[9] L. Georgiadis and L. Tassiulas, "Broadcast erasure channel with feedback - capacity and algorithms," in *Proc. 5th Workshop on Network Coding, Theory, and Applications (NetCod)*, Lausanne, Switzerland, June 2009, pp. 54–61.

[10] W.-C. Kuo and C.-C. Wang, "Two-flow capacity region of the cope principle for two-hop wireless erasure networks," *IEEE Trans. Inform. Theory*, vol. 59, no. 11, pp. 7553–7575, November 2013.

[11] J. Han and C.-C. Wang, "Linear Network Coding Capacity Region of The Smart Repeater with Broadcast Erasure Channels," submitted to *IEEE Trans. Inform. Theory*, June 2016 [Online]. Available: http://arxiv.org/abs/1605.01331