On finite-length analysis and channel dispersion for broadcast packet erasure channels with feedback

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Abstract*—Motivated by the applications for low-delay communication networks, the finite-length analysis, or channel dispersion identification, of the multi-user channel is very important. Recent studies also incorporate the effects of feedback in point-to-point and common-message broadcast channels (BCs). However, with private messages and feedback, finite-length results for BCs are much more scarce. Though it is known that feedback can strictly enlarge the capacity, the ultimate feedback capacity regions remain unknown for even some classical channels including Gaussian BCs. In this work, we study the two-user broadcast packet erasure channel (PEC) with causal feedback, which is one of the cleanest feedback capacity results and the capacity region can be achieved by elegant linear network coding (LNC). We first derive a new finite-length outer bound for any LNCs and then accompanying inner bound by analyzing a three-phase LNC. For the outer-bound, we adopt a linear-space-based framework, which can successfully find the LNC capacity. However, naively applying this method in finite-length regime will result in a loose outer bound. Thus a new bounding technique based on carefully labelling each time slot according to the type of LNC transmitted is proposed. Simulation results show that the sum-rate gap between our inner and outer bounds is within 0.02 bits/channel use. Asymptotic analysis also shows that our bounds bracket the channel dispersion of LNC feedback capacity for broadcast PEC to within a factor of $Q^{-1}(\epsilon/2)/Q^{-1}(\epsilon)$.

I. Introduction

In the future network, low-delay traffic will become more important due to many new Internet of Things (IoT) applications such as smart grid and future manufacturing. Indeed, 3GPP has already introduced new use cases such as ultra-reliable and low latency communications (URLLC) to accommodate the low-delay traffic. For URLLC, a hard deadline constraint such that the delay is no longer than 1ms is imposed. It is natural that only a short length channel coding is needed. However, the classical Shannon capacity is derived without any constraint on the blocklength and can be treated as a result from the first-order asymptotic rate analysis. A refined finite-length analysis is performed in [1], which results in matched second-order maximum rate bounds in a point-to-point channel. The backoff from channel capacity using this analysis is named as the channel dispersion.

It is well known that causal feedback can be helpful for enlarging the Shannon capacity regions for some multi-user channels. However, the ultimate feedback capacity regions remain open for even some classical scenarios including Gaussian broadcast channels (BC)s [2]. Among all known feedback capacity results, those for broadcast packet erasure channel (PEC) are prominent [3] [4] since even the capacity with one-sided channel state feedback is found [5]. The broadcast PEC with feedback belongs to the non-degraded BC and thus studying its capacity is theoretically important. Moreover, the corresponding optimal scheme also motivates many practical wireless network coding [4].

With a fixed blocklength and feedback, the finite-length analysis for point-to-point PEC was studied in [6]. Unfortunately, extensions from single-user results in [1] [6] to more complicated networks are hard, even for channels without feedback. Due to a close resemblance to the single-user channel, the channel dispersion of a BC with only a common message requested by all receivers was identified in [7] under a fixed blocklength. From [7], output feedback can improve the channel dispersion. Without feedback, finite-length achievable rate analysis was studied in the BC with a degraded message set and multiple access channel (MAC) [8] [9] [10].

In this paper, we focus on the finite-length analysis for a two-user broadcast PEC with feedback, where two private messages must be sent from the source. Unlike [7], the private messages will make each destination suffer from an additional interference. The single-user outer bound in [1] [6], which is based on the detection probability of a hypothesis testing problem, is hard to be generalized due to the inference at each destination. To make the problem traceable, we adopt the linear network coding (LNC) setting where the channel encoder at the source is linear. Sophisticated LNC solutions are proved to achieve capacity regions in [3]–[5]. Though LNCs were also considered in [11] [12], the channels are quite different to ours. Multi-hop network is considered in [11], however, not only single user is allowed but also the finite-length outer bound is absent. Also to identify the channel dispersion, our error probability is evaluated over a single coding block while that in [12] is allowed to be averaged over many blocks.

To find a finite-length outer bound for all LNCs, we adopt a linear-space-based framework [4] which partitions all possible coding vectors into different types, according to whether a vector is in a certain received space. The outer bound in [4] is calculated via comparing the long-term expectation of random ranks for these linear spaces. However, this method will result in a loose bound due to our short-term error

This work was supported in parts by NSF under Grant CCF-1422997, CCF-1618475, CCF-1816013 and CCF-2008527, and by MOST Taiwan under Grant 107-2628-E-011-003-MY3 and 110-2636-E-009-016.

probability is evaluated during only a finite length. We propose a new technique by labelling each time slot according to the coding type transmitted and identifying that only two labels matter to obtain a tight outer bound. A three-phase LNC is proposed for the inner bound, and the asymptotic analysis shows that we bracket the channel dispersion to within a factor of $Q^{-1}(\epsilon/2)/Q^{-1}(\epsilon)$. Simulation results verify that the sum-rate gap between our bounds is within 0.02 bits/channel use.

II. System model

Given a finite field $\mathsf{GF}(q)$, a 2-destination broadcast PEC is defined as follows. At time index t, the source sends a symbol $X_t \in \mathsf{GF}(q)$, and each destination either perfectly receives X_t or an erasure. In other words, let the binary erasure state at time t of destination d_i be $S_{i,\,t}, i = 1, 2$, d_i receives $Z_{i,\,t} = X_t$ if $S_{i,\,t} = 1$ and an erasure $Z_{i,\,t} = *$ if $S_{i,\,t} = 0, i = 1, 2$. Here $\{S_{1,\,t}\}$ and $\{S_{2,\,t}\}$ are independent Bernoulli p processes respectively, and i.i.d. over time. Within D time slots, the source would like to send two independent messages (packet streams) $\mathbf{w}_i$, each being a L_i-dimensional column vector in $\mathsf{GF}(q)$, to destination $d_i, i = 1, 2$ respectively. The messages are uniformly distributed. At the end of each time slot, each d_i feeds back to the source whether the transmitted symbol has received through the use of ACK or NACK.

A linear network encoder at the source can be described by D linear encoding functions: for all $t = 1, 2, \ldots, D$,

$$X_t = [\mathbf{w}_1^T\ \mathbf{w}_2^T]\ \mathbf{c}_t, \tag{1}$$

where $\mathbf{c}_t$ is a $(L_1{+}L_2){\times}1$ vector in $\mathsf{GF}(q)$. Let $(S_{1,\,t}, S_{2,\,t}) \triangleq S_t$, the choices of coding vector $\mathbf{c}_t$ depends on the past erasure state sequence $\{S_1, S_2, \ldots, S_{t-1}\} \triangleq S^{t-1}$ known via feedback, but not on $(\mathbf{w}_1, \mathbf{w}_2)$. Also the decoding function g_i at destination d_i is $\hat{\mathbf{w}}_i = g_i(Z_i^D, S^D), i = 1, 2.$ where the received sequence Z_i^D is defined similarly as S^D. Note that with S^D, coding vectors $\mathbf{c}_1, \ldots \mathbf{c}_D$ are known at each destination for decoding.

Now we have the following definitions. With a fixed length D and error probability ϵ, a rate vector $(L_1/D, L_2/D)$ is achievable if there exists a LNC such that

$$P(\mathbf{w}_1 \neq \hat{\mathbf{w}}_1 \text{ or } \mathbf{w}_2 \neq \hat{\mathbf{w}}_2) \leq \epsilon. \tag{2}$$

The maximum LNC rate region is defined as the closure of all $(L_1/D, L_2/D)$ that are achievable. As [4], the unit of our rate is packets per time slot and can be converted to the traditional unit bits per time slot by multiplying a factor of $\log_2(q)$.

III. Main results

The first result is a novel non-asymptotic outer bound.

Theorem 3.1: In a two-user broadcast PEC with a fixed length D and erasure probability $1{-}p$, for delivering L_i private packets to user $i, i = 1, 2$, the error probability (2) of any feedback-aided LNC must satisfy

$$\epsilon \geq P\left(\sum_{k=1}^{L_1} \tilde{t}_{1,k} + \sum_{k=1}^{L_2} \tilde{t}_{2,k} > D\right) \tag{3}$$

$$\epsilon \geq P\left(\sum_{k=1}^{L_2} \tilde{t}_{1,k} + \sum_{k=1}^{L_1} \tilde{t}_{2,k} > D\right) \tag{4}$$

where $\tilde{t}_{1,k}$ and $\tilde{t}_{2,k}$ respectively are i.i.d geometric distributed random variables with parameter p and $p(2-p)$; moreover, we have the asymptotic rate outer bounds for any LNCs

$$\frac{L_1}{Dp}+\frac{L_2}{Dp(2-p)} \leq 1-\sqrt{\frac{1-p}{Dp^2}\left(\frac{L_1}{D}+\frac{L_2(1-p)}{D(2-p)^2}\right)}Q^{-1}(\epsilon)+O\left(\frac{1}{D}\right) \tag{5}$$

$$\frac{L_2}{Dp}+\frac{L_1}{Dp(2-p)} \leq 1-\sqrt{\frac{1-p}{Dp^2}\left(\frac{L_2}{D}+\frac{L_1(1-p)}{D(2-p)^2}\right)}Q^{-1}(\epsilon)+O\left(\frac{1}{D}\right) \tag{6}$$

Proof: The key idea to reach (3) is labelling each time slot t according to the type of its transmitted coding vector $\mathbf{c}_t$ in (1), such that one can monitor the ranks for decoding desired messages. Relationship between the required rank and the number of successfully received $\mathbf{c}_t$ for a certain type can be established. Then to meet (2) we show that only two labels, say labels A and B, matter. That is, the source must transmit L_1 time slots with label A and at least L_2 time slots with label B before the deadline D. Details for proving (3) and other inequalities are referred to Sec. IV. ∎

Next, we propose a three-phase LNC for the achievability. Compared with the LNCs in [3] [4], the simplicity of our LNC facilitates much more complicated second-order asymptotic analysis in upcoming Theorem 3.2. Our LNC is outlined with binary input $q = 2$ as follows. Consider virtual queues $(Q^{[i]}, Q^{[i]}_{o.h.}), i = 1, 2$. In the beginning the L_i bits of message $\mathbf{w}_i$ for user i are first stored in queue $Q^{[i]}$, and then

Phase 1 With aids of state feedback, the source sends each bit in queue $Q^{[1]}$ until it is delivered to d_1 or over-heard at d_2. In both cases, the transmitted bit is removed from $Q^{[1]}$ but it will further be stored in the overheard queue $Q^{[1]}_{o.h.}$ for the latter case. Phase 1 will end if length $q^{[1]} = 0$ for queue $Q^{[1]}$.

Phase 2 The same coding operation is performed by swapping the role of two users.

Phase 3 The source picks the first recycled bit b_1 from $Q^{[1]}_{o.h.}$ and XOR it with the first bit b_2 from $Q^{[2]}_{o.h.}$. The XOR is sent and if it is delivered to destination d_i, b_i is removed from $Q^{[i]}_{o.h.}, i = 1, 2$. It a bit is removed in the over-heard queue, the rest bits will be shifted forward such that the original second bit will be the first one. The source then picks the first bit from each over-heard queue and sends their XOR, and performs bit-removal as aforementioned. This procedure is repeated until both $Q^{[1]}_{o.h.}$ and $Q^{[2]}_{o.h.}$ are empty. When one of the queues, say $Q^{[1]}_{o.h.}$, is empty first, degenerated XOR [12] is applied to send all remaining bits in $Q^{[2]}_{o.h.}$. That is, each of the remaining bits in $Q^{[2]}_{o.h.}$ is sent until it is delivered to d_2 as standard ARQ.

Theorem 3.2: For considered two-user broadcast PEC, we have the asymptotic inner bounds from the three-phase LNC

$$\frac{L_1}{Dp}+\frac{L_2}{Dp(2-p)} \leq 1-\sqrt{\frac{1-p}{Dp^2}\left(\frac{L_1}{D}+\frac{L_2(1-p)}{D(2-p)^2}\right)}Q^{-1}\left(\frac{\epsilon}{2}\right)+O\left(\frac{1}{D}\right) \tag{7}$$

$$\frac{L_2}{Dp}+\frac{L_1}{Dp(2-p)} \leq 1-\sqrt{\frac{1-p}{Dp^2}\left(\frac{L_2}{D}+\frac{L_1(1-p)}{D(2-p)^2}\right)}Q^{-1}\left(\frac{\epsilon}{2}\right)+O\left(\frac{1}{D}\right) \tag{8}$$

Proof: The proof is presented in Sec. V. ∎

For comparison, we also consider the a simple routing protocol which consists of only two phases. That is, after delivering L_1 bits to d_1 the source sends L_2 bits to d_2. Each bit is repeated until being delivered to its target destination as ARQ. Similar asymptotic analysis as that for Theorem 3.2 will result in

$$\frac{L_1+L_2}{D} \le p - \sqrt{\frac{(L_1+L_2)}{D}(1-p)}Q^{-1}(\epsilon) + O\left(\frac{1}{D}\right) \quad (9)$$

for this routing protocol, and the rate region is smaller than that in Theorem 3.2.

Under symmetric rate $L_1 = L_2$, our outer and inner bounds from in Theorem 3.1 and 3.2 bracket the channel dispersion of LNC sum-rate capacity to within a factor of $Q^{-1}(\epsilon/2)/Q^{-1}(\epsilon/2)$. Note that the converse for the sum-capacity is not previous reported even for MAC [9]. We further plot these bounds under binary inputs in Fig. 1. From Fig. 1, our sum-rate outer bound is quite tight and the maximum gap to the LNC inner bound is 0.02 bits. Also the sum rate of our three-phase LNC has significant gain over that of routing. The maximum outage probability ϵ in (2) is 0.01 while the erasure probability is 0.5. For the rate outer-bound, given a blocklength D, we search the largest L_1 meeting (3) (which equals to (4) when $L_1 = L_2$) by using the L_1 from (5) as the starting point (ignore the $O(1/D)$ term). The LNC inner bound is obtained by exhaustively searching the largest L_1 such that the resulting error probability from 10^5 trials meets (2).

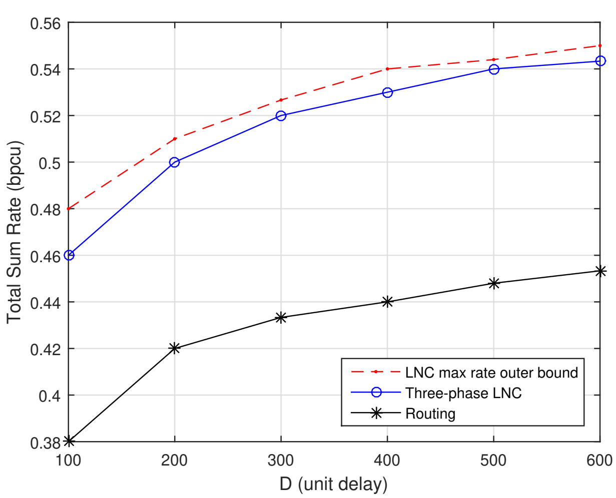

Fig. 1. Outer and inner bounds for LNC maximum sum-rate of broadcast erasure channels with feedback and a finite blocklength D.

IV. Proof Sketch of Theorem 3.1

With only a single user, the outer bound with a fixed length was given in [6, Theorem 14] (or [1, Theorem 53]), which is based on the detection probability of a hypothesis testing problem [1, Theorem 38]. This method is hard to be generalized to our two-private-message broadcast PEC not only because we will simultaneously face two hypothesis testing problems but also the interference for one destination is a desired codeword for the other. Furthermore, unlike no-feedback settings [1] [9] [10], the transmitter can change the interference and message at each destination according to the feedback and have much more coding choices to boost the rate. Thus we adopt the linear-space-based framework in [4] [12], which partitions all possible coding vector $\mathbf{c}_t$s in (1) according to whether $\mathbf{c}_t$ is in a certain received space. These knowledge spaces are also known at the source via feedback S^{t-1}. However, both [4] and [12] consider long-term error probabilities, that is, the error probability in [4] is evaluated under $D \to \infty$ while that in [12] is averaged over many blocks. On the contrary, our short-term error probability in (2) is evaluated over finite D channel uses within a single coding block. Naive outer bound from [4] [12] is not tight due to this fundamental difference and new challenge arises.

To solve the aforementioned difficulty, besides partitioning all possible LNC coding vector $\mathbf{c}_t$s into knowledge-space-based coding types as [4] the new ingredient is further labelling each time slot t according to the coding type transmitted. Then one can monitor the random ranks for decoding desired messages. To proceed, we first recall some definitions in [4]. For $j = 1$ to $L_1 + L_2$, let δ_j denote an (L_1+L_2)-dimensional elementary delta (row) vector with its j-th coordinate being one and all the other coordinates being zero. Define $\Omega_i \triangleq \text{span}\{\delta_j : j = 1, \ldots, L_i\}$ as the individual message spaces for user i. The knowledge space $S_i(t)$ in the end of time t is

$$\mathcal{S}_i(t) \triangleq \text{span}\{\mathbf{c}_\tau : \forall \tau \le t \ \text{ s.t. } S_{i,\tau} = 1(d_i \ \text{ is on at time } \ \tau)\}.$$

That is, $\mathcal{S}_i(t)$ is the linear span of the vectors of those coded packets that have successfully received at d_i. For any two linear subspaces $\mathcal{A}, \mathcal{B} \subseteq \Omega_1 \bigcup \Omega_2$, define $\mathcal{A} \oplus \mathcal{B} \triangleq \text{span}\{\mathbf{v} : \forall \mathbf{v} \in \mathcal{A} \bigcup \mathcal{B}\}$ as the linear sum space of $\mathcal{A}$ and $\mathcal{B}$. Destination d_i can decode its desired message $\mathbf{w}_i$ if and only if in the end of deadline D we have $\Omega_i \subseteq \mathcal{S}_i(D)$, or equivalently

$$\mathcal{S}_i(D) \oplus \Omega_i = \mathcal{S}_i(D). \quad (10)$$

At time t, the source knows $\mathcal{S}_1$ and $\mathcal{S}_2$, shorthand respectively for $\mathcal{S}_1(t-1)$ and $\mathcal{S}_2(t-1)$, according to the feedback. There are seven linear coding spaces from [4]

$$\begin{aligned} &\mathcal{A}_1 = \mathcal{S}_1, \mathcal{A}_2 = \mathcal{S}_2, \mathcal{A}_3 = \mathcal{S}_1 \oplus \Omega_1, \mathcal{A}_4 = \mathcal{S}_2 \oplus \Omega_2 \\ &\mathcal{A}_5 = \mathcal{S}_1 \oplus \mathcal{S}_2, \mathcal{A}_6 = \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \Omega_1, \mathcal{A}_7 = \mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \Omega_2. \end{aligned} \quad (11)$$

These linear coding spaces can help to track the interference and message spaces known at destinations by partitioning the overall coding vector space $\Omega_1 \bigcup \Omega_2$ into disjoint subsets. For example, we call transmitted $\mathbf{c}_t$ is with coding type-27 if it belongs to the set

$$(\mathcal{A}_7 \cap \mathcal{A}_6 \cap \mathcal{A}_4 \cap \mathcal{A}_3) \setminus (\mathcal{A}_5 \cup \mathcal{A}_2 \cup \mathcal{A}_1), \quad (12)$$

where the binary expression of 27 is 0011011 and the i-th bit of this type index (from the leftmost) denotes whether $\mathbf{c}_t \in \mathcal{A}_i$

Now we show how to label the coding type at each time slot and trace the random rank to ensure (10) under the short-term error probability (2), which leads to (3). Though different coding types can have the same label, to simplify the number of labels, we form a physically degraded BC by giving the observations of d_1 to d_2. Upcoming Proposition 4.1 shows that only two labels matter for the decodability (10). Note that even the maximum LNC rate with finite D and feedback of this physically degraded BC is open, since the proof of [4] is based on [13] which assumes $D \to \infty$. For this BC, compared with [4], there are only nine coding types

$$\{0, 2, 9, 11, 18, 27, 47, 63, 127\}. \tag{13}$$

since $\mathcal{A}_5 = \mathcal{A}_2$ and $\mathcal{A}_7 = \mathcal{A}_4$ in (11). Now one can *causally* label each slot t as either A, or B, or C such that

- A-slot : $\mathbf{c}_t$ is with coding types 18, 27, or 63
- B-slot : $\mathbf{c}_t$ is with coding types 9, 11, or 0
- C-slot : $\mathbf{c}_t$ is with the rest coding types 2, 47, or 127

It is clear that we can label all time slots causally and no single slot t will receive two labels. Now we have

Proposition 4.1: For any LNC (1) in the aforementioned physically degraded broadcast BC, the successful decoding probability

$$1 - P(\mathbf{w}_1 \neq \hat{\mathbf{w}}_1 \text{ or } \mathbf{w}_2 \neq \hat{\mathbf{w}}_2)$$

is upper-bounded by the probability of the following event : before the deadline D, there are exactly L_1 coded symbols delivered to destination d_1 among the A-slots and there are at least L_2 coded symbols delivered to destination d_2 among the B-slots.

Proof: Here we focus on d_1. It is clear that only those coding vector $\mathbf{c}_t$s with types 18, 27, and 63, once received by d_1, can close the gap between $\operatorname{rank}(\mathcal{S}_1(t) \oplus \Omega_1)$ and $\operatorname{rank}(\mathcal{S}_1(t))$ in (10) by 1. This can be done by checking the first and third bits of the binary expressions for type indices in (13), which corresponds to whether $\mathbf{c}_t$ belongs to $\mathcal{A}_1$ and $\mathcal{A}_3$ defined in (11). As a result, successfully decoding of $\mathbf{w}_1$ is equivalent to that that there are exactly L_1 symbols delivered to d_1 among the A-slots. The proof for the result for d_2 is more involved and relegated to Appendix A ■

This physically degraded BC always has larger successful decoding probability than that of the original channel. Also the erasure probability at d_2 reduces to $1 - p(2-p)$. Then if (2) is met before enhancement, we must have

$$P\left(\sum_{k=1}^{L_1} \tilde{t}_{1,k} + \sum_{k=1}^{L_2} \tilde{t}_{2,k} \leq D\right) \geq 1 - \epsilon$$

where the total length of A-slots is $\sum_{k=1}^{L_1} \tilde{t}_{1,k}$ and that of B-slots is at least $\sum_{k=1}^{L_2} \tilde{t}_{2,k}$. Then (3) is valid. By swapping the role of d_1 and d_2, we also have (4).

Now $\sum_{k=1}^{L_1} \tilde{t}_{1,k} + \sum_{k=1}^{L_2} \tilde{t}_{2,k}$ is sum of independent random variables, though $\tilde{t}_{1,k}$ and $\tilde{t}_{2,k}$ have different distributions. The asymptotic analysis (5)(6) then comes from applying the Berry-Esseen Theorem [14] to (3)(4). The details are given in Appendix B. Finally, for the single user case $L_2 = 0$, outer-bounds (5)(6) implies the second-order term of L_1 versus D is $\sqrt{Dp(1-p)}Q^{-1}(\epsilon)$, which matches [6, Theorem 14].

V. Proof sketch of Theorem 3.2

Without loss of generality, we consider binary input. At the beginning of Phase 3 of the three-phase LNC, all coded bits in the queue $Q^{[1]}_{o.h.}$ are known at destination d_2, and verse versa. By using such a receiver side-information, the transmission in Phase 3 is equivalent to simultaneously unicast bits in $Q^{[i]}_{o.h.}$ from the source to destination $d_i, i = 1, 2$ through two parallel erasure links. Let the length of $Q^{[1]}_{o.h.}$ be $q^{[1]}_{o.h.}$. Assume bit k in $Q^{[1]}_{o.h.}$ is repeated $t^1_{o.h.}[k]$ times, then all over-heard bits in Phase 1 can be recovered at d_1 after $\sum_{k=1}^{q^{[1]}_{o.h.}} t^1_{o.h.}[k]$ time slots. Then the successful decoding at d_1 before deadline D is ensured by $T_1 + T_2 + \sum_{k=1}^{q^{[1]}_{o.h.}} t^1_{o.h.}[k] \leq D$, where T_1 and T_2 are lengthes of Phase 1 and 2 respectively, and the overall successful probability from (2) is

$$P\left(T_1 + T_2 + \sum_{k=1}^{q^{[1]}_{o.h.}} t^1_{o.h.}[k] \leq D, T_1 + T_2 + \sum_{k=1}^{q^{[2]}_{o.h.}} t^2_{o.h.}[k] \leq D\right) \tag{14}$$

Now we perform asymptotic large D analysis based on (14). By union bound, the error probability is upper-bounded by

$$\sum_{i=1}^{2} P\left(T_1 + T_2 + \sum_{k=1}^{q^{[i]}_{o.h.}} t^1_{o.h.}[k] > D\right) \tag{15}$$

If both summands are smaller than $\epsilon/2$, then (2) is met. For the first summand in (15) (i=1), we first show that it is the same as the RHS of (3) in Theorem 3.1. Note the following two terms have same distributions

$$T_1 + \sum_{k=1}^{q^{[1]}_{o.h.}} t^1_{o.h.}[k] \stackrel{d}{=} \sum_{k=1}^{L_1} \tilde{t}_{1,k} \tag{16}$$

For the LHS, assume a certain bit of user 1 is delivered to d_1 using t time slots. This event may happens in Phase 1 with probability $(1-p)^{t-1}(1-p)^{t-1}p$ or in Phase 3 with probability $(1-p)^{t-1}(1-(1-p)^{t-1})p$ (not all states of d_2 are erased before t since this bit is delivered to d_2 first in Phase 1). The probability of this event is $(1-p)^{t-1}p$, and the same as $P(\tilde{t}_{1,k} = t)$ since $\tilde{t}_{1,k}$ is geometric distributed with parameter p. Also the length T_2 of phase 2 is sum of L_2 i.i.d. geometric distributed random variables, each with success probability $1-(1-p)^2 = p(2-p)$ as $\tilde{t}_{2,k}$. Then by replacing the LHS of (3) by $\epsilon/2$, the asymptotic analysis in Appendix B results in (7). Similarly, one can show that the second summand in (15) (i=2) is the same as the RHS of (4), which results in (8).

VI. Discussions

Besides the asymptotic analysis in Section V, the successful probability (14) of the three-phase LNC can be obtained in closed-form for any finite $D > L_1 + L_2$. This closed-form expression can significantly reduce the time to search the maximum rate under (2) in Figure 1. To see this, given $T_1 + T_2, q^{[1]}_{o.h.}, q^{[2]}_{o.h.}$, the conditional successful probability can be re-written from (14) as

$$\prod_{i=1}^{2} P\left(\sum_{k=1}^{q^{[i]}_{o.h.}} t^i_{o.h.}[k] \leq D - T_1 + T_2\right), \tag{18}$$

$$D \geq \frac{L_1}{p} + \frac{L_2}{p(2-p)} + \sqrt{\frac{L_1(1-p)}{p^2} + \frac{L_2\left(1-p(2-p)\right)}{p^2(2-p)^2}}\, Q^{-1}\left(\epsilon - \frac{B}{\sqrt{\frac{L_1(1-p)}{p^2} + \frac{L_2(1-p(2-p))}{p^2(2-p)^2}}}\right) \tag{17}$$

which can be expressed in closed-form since $\sum_{k=1}^{q^{[i]}_{o.h.}} t^i_{o.h.}[k]$ is negative binomial distributed with $q^{[i]}_{o.h.}$ successes and probability of success p. Moreover, T_1+T_2 is also negative binomial distributed ; while the queue length $q^{[1]}_{o.h.}$ is independent of $T_1 + T_2$, $q^{[2]}_{o.h.}$ and binomial distributed with L_1 trials and success probability $\frac{(1-p)p}{1-(1-p)^2}$. The joint probability of $T_1 + T_2, q^{[1]}_{o.h.}, q^{[2]}_{o.h.}$, and thus (14), can both be obtained in closed-form.

Appendix

A. Proof of the result for user d_2 in Proposition 4.1

Similar to the proof for A-slots, by looking at the second and the fourth bits of the type indices, which corresponds to whether $\mathbf{c}_t$ belongs to $\mathcal{A}_2$ and $\mathcal{A}_4$, it is clear that only types 9, 11, 27, once received by d_2, can close the gap between

$$\text{rank}(\mathcal{S}_2(t) \oplus \Omega_2) \text{ and } \text{rank}(\mathcal{S}_2(t))$$

by 1. As a result, successfully decoding of $\mathbf{w}_2$ is equivalent to that there are exactly L_2 symbols delivered to d_2 among the slots sending types 9, 11, 27. Note that the current coding vector $\mathbf{c}_t$ is known at both destinations since they depends only on the global state sequence S^{t-1}.

Next, we will prove that the total number of successful type-27 delivery to d_2 must be upper bounded by the total number of successful type-0 delivery to d_2. Then the successfully decoding of $\mathbf{w}_2$ happens only if there are at least L_2 symbols delivered to d_2 among the B-slots. It is due to that whenever a type-27 coding vector is delivered to d_2, there must be a type-0 coding vector delivered to d_2 in a previous time slot. To see this, from (11)(12) and degradedness, the type-27 $\mathbf{c}_t$ belongs to the set

$$(\mathcal{S}_2 \oplus \Omega_2 \cap \mathcal{S}_2 \oplus \Omega_1 \cap \mathcal{S}_1 \oplus \Omega_1) \setminus (\mathcal{S}_2 \cup \mathcal{S}_1), \tag{19}$$

here we recall that $\mathcal{S}_i$ is a shorthand of $\mathcal{S}_i(t-1)$. We notice that type-27 $\mathbf{c}_t$ is feasible only if

$$\text{rank}(\mathcal{S}_2 \oplus \Omega_1 \cap \mathcal{S}_2 \oplus \Omega_2) > \text{rank}(\mathcal{S}_2)$$

from definition (19). Now from [4, Lemma 6],

$$\begin{aligned}&\text{rank}(\mathcal{S}_2 \oplus \Omega_1 \cap \mathcal{S}_2 \oplus \Omega_2) + \text{rank}(\mathcal{S}_2 \oplus \Omega_1 \oplus \mathcal{S}_2 \oplus \Omega_2),\\ =&\text{rank}(\mathcal{S}_2 \oplus \Omega_1) + \text{rank}(\mathcal{S}_2 \oplus \Omega_2),\end{aligned}$$

we know type-27 is feasible only if

$$\begin{aligned}&\text{rank}(\mathcal{S}_2 \oplus \Omega_1) + \text{rank}(\mathcal{S}_2 \oplus \Omega_2)\\ &-\text{rank}(\mathcal{S}_2) - \text{rank}(\Omega_1 \oplus \Omega_2) > 0.\end{aligned} \tag{20}$$

Initially the gap in the LHS of (20) is zero. Then we notice that only type-0 $\mathbf{c}_t$ can increase the gap by 1 once being received by d_2 since all ranks related to $\mathcal{S}_2(t)$ in (20) increase by 1. No other types can increase the gap no matter what is the reception state status. Next, whenever a type-27 $\mathbf{c}_t$ is received by d_2, the gap will decrease by 1 since only $\text{rank}(\mathcal{S}_2(t))$ increases by 1. Then before a type-27 $\mathbf{c}_t$, another type-0 one must be received by d_2. Indeed, the so-called "reverse XOR" transmission [4] belongs to such a pair of type-0 and the type-27 coding selections.

B. Proof of asymptotic results in Theorem 3.1

To perform asymptotic analysis on the RHS of (3), from Berry-Esseen Theorem [14]

$$\begin{aligned}&\Bigg| P\Big(\sum_{k=1}^{L_1} \tilde{t}_{1,k} + \sum_{k=1}^{L_2} \tilde{t}_{2,k} - \big(L_1\mathbb{E}[\tilde{t}_{1,k}] + L_2\mathbb{E}[\tilde{t}_{2,k}]\big)\\ &\geq \lambda\sqrt{L_1\text{Var}[\tilde{t}_{1,k}] + L_2\text{Var}[\tilde{t}_{2,k}]}\,\Big) - Q(\lambda) \Bigg|\\ &\leq \frac{6\left(L_1\mathbb{E}\left[\left|\tilde{t}_{1,k} - \mathbb{E}[\tilde{t}_{1,k}]\right|^3\right] + L_2\mathbb{E}\left[\left|\tilde{t}_{2,k} - \mathbb{E}[\tilde{t}_{2,k}]\right|^3\right]\right)}{(L_1\text{Var}[\tilde{t}_{1,k}] + L_2\text{Var}[\tilde{t}_{2,k}])^{\frac{3}{2}}}\end{aligned} \tag{21}$$

for any $-\infty < \lambda < \infty$. By choosing λ such that

$$D \geq \big(L_1\mathbb{E}[\tilde{t}_{1,k}] + L_2\mathbb{E}[\tilde{t}_{2,k}]\big) + \lambda\sqrt{L_1\text{Var}[\tilde{t}_{1,k}] + L_2\text{Var}[\tilde{t}_{2,k}]} \tag{22}$$

The RHS of (3) is upper-bounded by

$$Q(\lambda) + \frac{6\left(L_1\mathbb{E}\left[\left|\tilde{t}_{1,k} - \mathbb{E}[\tilde{t}_{1,k}]\right|^3\right] + L_2\mathbb{E}\left[\left|\tilde{t}_{2,k} - \mathbb{E}[\tilde{t}_{2,k}]\right|^3\right]\right)}{(L_1\text{Var}[\tilde{t}_{1,k}] + L_2\text{Var}[\tilde{t}_{2,k}])^{\frac{3}{2}}}. \tag{23}$$

Note that since $\tilde{t}_{1,k}$ and $\tilde{t}_{2,k}$ are both geometric distributed

$$\begin{aligned}\mathbb{E}[\tilde{t}_{1,k}] &= \frac{1}{p}, \mathbb{E}[\tilde{t}_{2,k}] = \frac{1}{p(2-p)},\\ \text{Var}[\tilde{t}_{1,k}] &= \frac{1-p}{p^2}, \text{Var}[\tilde{t}_{2,k}] = \frac{1-p(2-p)}{p^2(2-p)^2}.\end{aligned} \tag{24}$$

To meet (3), from (23), we select

$$\lambda = Q^{-1}\left(\epsilon - \frac{B}{\sqrt{L_1\text{Var}[\tilde{t}_{1,k}] + L_2\text{Var}[\tilde{t}_{2,k}]}}\right),$$

where

$$B = \frac{6\left(L_1\mathbb{E}\left[\left|\tilde{t}_{1,k} - \mathbb{E}[\tilde{t}_{1,k}]\right|^3\right] + L_2\mathbb{E}\left[\left|\tilde{t}_{2,k} - \mathbb{E}[\tilde{t}_{2,k}]\right|^3\right]\right)}{L_1\text{Var}[\tilde{t}_{1,k}] + L_2\text{Var}[\tilde{t}_{2,k}]}$$

is a constant since L_2/L_1 is a constant. Then from (22)(24), we have constraint (17) at the top of this page. Finally, from Taylor's series on infinitely differentiable $Q^{-1}(.)$, we have the LNC outer-bound (5). The other outer-bound (6) can be obtained similarly from (4).

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