Sequentially Mixing Randomly Arriving Packets Improves Channel Dispersion Over Block-Based Designs

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***Abstract*—Channel dispersion quantifies the convergence speed of coding rate to channel capacity under different latency constraints. Under the setting of packet erasure channels (PECs) with Bernoulli packet arrivals, *this work characterizes the channel dispersions of random linear streaming codes (RLSCs) and MDS block codes, respectively.* New techniques are developed to quantify the channel dispersion of sequential (non-block-based) coding, the first in the literature. The channel dispersion expressions are then used to compare the levels of error protection between RLSCs and MDS block codes. The results show that *if and only if the target error probability p_e is smaller than a threshold (≈ 0.1774), RLSCs offer strictly stronger error protection than MDS block codes,* which is on top of the already significant 50% latency savings of RLSCs that eliminate the queueing delay completely.**

I. Introduction

This work considers an information source that generates message packets sequentially, which are then encoded and sent through a noisy packet erasure channel (PEC). In every time slot $t \geq 1$, one message packet $m(t) \in \mathsf{GF}(2^q)$ may arrive at the encoder with probability $R \in (0, 1)$. The message packet $m(t)$ is *causally* encoded to generate the coded packet $x(t) \in \mathsf{GF}(2^q)$, which is then transmitted through a PEC with the output being perfect $y(t) = x(t)$ or erased $y(t) = *$, with erasure probability ϵ. The goal is to decode the message packet $m(t)$ transmitted at time t by time $t+\Delta$. Decoding after $t+\Delta$ is considered useless and will be counted towards the error probability p_e.

Such a system model characterizes the interaction among the sequential arrival rate R, the channel capacity $C = 1 - \epsilon$, the hard decoding deadline Δ and the deadline-constrained error probability p_e, and is motivated by next-generation low latency communications [1] and some existing error protection techniques such as erasure network coding [2]–[12].

Following the seminal work of Polyanskiy *et al.* [13], this work studies the setting of fixed $p_e > 0$ and $C = 1 - \epsilon$ and characterizes the convergence speed of $R \nearrow C$ when $\Delta \to \infty$. Specifically, [13] shows that for any fixed $p_e < 0.5$, we have

$$R = C - \sqrt{V} \cdot \frac{1}{\sqrt{\Delta}} \cdot Q^{-1}(p_e) + o\left(\frac{1}{\sqrt{\Delta}}\right) \qquad (1)$$

for a wide variety of channel models[1], block coding schemes and converse results [11]–[17], where $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$ is the complimentary cumulative distribution function (CCDF) of the standard Gaussian distribution, and $Q^{-1}(\cdot)$ is the inverse of $Q(\cdot)$. The constant V is termed the *channel dispersion*. Given the system reliability requirement (fixed p_e), the channel dispersion equation (1) shows that the gap-to-optimality $C - R$ admits an important *product form* consisting of $\sqrt{V}$, determined by the underlying channel distribution, $\frac{1}{\sqrt{\Delta}}$, the convergence rate versus the deadline constraint Δ, and $Q^{-1}(p_e)$, the cost needed to achieve p_e.

All existing results of channel dispersion studied an infinite backlogged setting [11]–[17], for which all the messages are available to the encoder before transmission. In contrast, this work considers the sequential Bernoulli arrival setting and characterizes the channel dispersion of *random linear streaming codes* (RLSCs) over PECs. RLSCs belong to the class of sequential coding for which the encoder receives a string of message packets sequentially and outputs a string of coded packets in real time. With continuous arrival of new messages and continuous transmission of coded packets simultaneously, there is no concept of "blocks" in its operations. RLSCs thus do not experience any queueing delay typically associated to the block-based designs, a feature that is especially appealing for the next-generation low latency communications [18]. Meanwhile, the non-existence of "blocks" means that all the existing intuitions and techniques of block-based channel dispersion analysis no longer hold and we have to devise new ways of quantifying the convergence speed of $R \nearrow C$ when $\Delta \to \infty$ under fixed p_e.

The main contributions of this work are:

(i) We show that even with a non-block-based design, the channel dispersion equation of RLSCs, surprisingly, still has the same form as (1). Specifically, the gap-to-optimality is once again a product of $\sqrt{V}$ and $\frac{1}{\sqrt{\Delta}}$ but one has to replace the cost term $Q^{-1}(p_e)$ by a different function $S^{-1}(p_e)$ to precisely account for the non-block nature of RLSCs. Note that our results focus exclusively on achievability without any converse discussion. At this moment, it remains an open problem

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[1]The remainder term $o(\sqrt{1/\Delta})$ in (1) can be further improved for many models such as discrete memoryless channels and Gaussian channels [13].

whether this is a phenomenon limited to the RLSCs over PECs considered herein or a result with broader generality.

(ii) We also consider a simple and reasonable block-based scheme that first "queues" the incoming Bernoulli-arrival message packets and then protects them by the optimal maximum distance separable (MDS) block code. We analyze the channel dispersion of such a block-MDS-based scheme, termed BMDS. By comparing the dispersion equations of the two fundamentally different designs, we show that *if and only if the target error probability p_e is smaller than a threshold (≈ 0.1774), RLSCs offer strictly stronger error protection than BMDS*. This implies that, for any commonly used reliability requirement $p_e \in [10^{-6}, 10^{-2}]$, RLSCs not only save 50% of the latency by eliminating queueing delay, but also offer stronger error protection than traditional block-based designs.

(iii) If we ignore the remainder term $o(\sqrt{1/\Delta})$, the channel dispersion equation (1) can be used as an approximation of the finite-length performance. We conduct extensive numerical evaluation and compare the new channel-dispersion-based approximation versus the exact finite-length results in our previous work [19]–[21]. The gap-to-optimality $C - R$ predicted by the new approximation is within 7.1% of the actual $C - R$ from exact analysis even for Δ as small as 100, which shows numerically the power of the new channel dispersion result when used as a finite-length approximation.

II. Dispersion for Block-MDS-Based Design

This section describes a traditional way of protecting sequentially arriving packets by BMDS, and then characterizes its channel dispersion over PECs. While the result is new, the techniques follow directly from [13]. Our goal is to establish a baseline for our main results, the RLSC schemes in Sec. III.

For any integer $n_B \geq 1$, the BMDS encoder first queues the message packets $m(t)$ arrived in time slot $t \in [1, n_B]$. Denote $M_1^{n_B}$ the number of message packets arrived during $[1, n_B]$, which is binomially distributed with parameters (n_B, R). At the end of time n_B, those $M_1^{n_B}$ message packets are encoded into n_B coded packets by a non-systematic[2] MDS block code. The resulting packets are sent in time slots $[n_B + 1, 2n_B]$. At the end of time $2n_B$, the receiver attempts decoding.

The BMDS can be naturally *pipelined* for continuous operation. That is, during the transmission in $[n_B + 1, 2n_B]$, the encoder queues the next batch of incoming messages, encodes and transmits them in $[2n_B+1, 3n_B]$, and so forth. To accommodate the queueing delay n_B incurred in $[1, n_B]$ plus the transmission delay n_B incurred in $(n_B, 2n_B]$, we choose the block length n_B according to the deadline constraint Δ by

$$n_B = \lfloor 0.5\Delta \rfloor. \tag{2}$$

If we use $Y_{n_B+1}^{2n_B}$ to denote the number of successful deliveries in $[n_B + 1, 2n_B]$, the error probability becomes

$$p_e = \Pr\left(M_1^{n_B} > Y_{n_B+1}^{2n_B}\right). \tag{3}$$

[2]In both BMDS and the RLSCs in Sec. III, we use non-systematic construction for fair comparison. A systematic construction may further reduce the error probability but the analysis involves various code-dependent "partial observing/decoding" events, which is beyond the scope of this work.

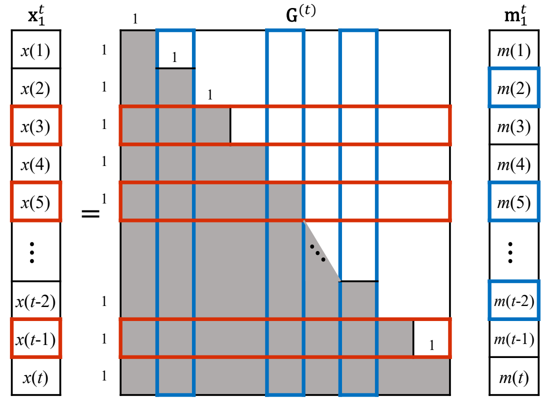

Fig. 1: The illustration of the cumulative generator matrix $\mathbf{G}^{(t)}$ in (5) and the random column and row removal to derive $\mathbf{H}^{(t)}$.

Proposition 1. *With the capacity being $C = 1 - \epsilon$, for any $p_e \in (0, 0.5)$, we have*

$$R = C - \sqrt{2\epsilon(1-\epsilon)}\frac{1}{\sqrt{n_B}}Q^{-1}(p_e) + o\left(\frac{1}{\sqrt{n_B}}\right). \tag{4}$$

Sketch of the proof: The error event can be rewritten as $\{\sum_{i=1}^{n_B} Z_i > 0\}$ where $Z_i = M_i - Y_{i+n_B}$, and $M_i \sim \text{Bern}(R)$ and $Y_i \sim \text{Bern}(1-\epsilon)$ denote the arrival of the message and the delivery of the coded packet at time i, respectively. Because Z_i is i.i.d. with $\mathbb{E}\{Z_i\} = R - C$ and $\mathsf{Var}(Z_i) = \mathsf{Var}(M_i) + \mathsf{Var}(Y_{i+n_B}) = R(1-R) + \epsilon(1-\epsilon)$, by the Berry-Essen Theorem and by noting that $R \nearrow C = 1 - \epsilon$ when n_B is large, we have (4).

Remark 1: Eq. (4) is written in terms of the n_B, which can be easily converted to an expression of the Δ by (2).

Remark 2: The channel dispersion of PECs for the infinite backlogged setting is $V = \epsilon(1 - \epsilon)$ [13, Theorem 53]. The new Bernoulli arrival setting effectively doubles the dispersion V in (4) to $2\epsilon(1 - \epsilon)$ since the dispersion V is additionally contributed by the random arrivals that are absent in an infinite-backlogged setting.

III. Dispersion for RLSCs

A. Construction of RLSCs

Following the sequential arrival setting in Sec. I, the RLSC encoder uses all the received packets until time t as input and outputs one coded packet $x(t) \in \mathsf{GF}(2^q)$ per time slot. We first describe the operations assuming no randomness. That is, a new message arrives in each time slot ($R = 1$), and there is no erasure ($\epsilon = 0$). Define $\mathbf{G}_t$ as the 1-by-t *generator matrix* and $\mathbf{G}^{(t)}$ as the t-by-t *cumulative generator matrix* for slot t. We have

$$x(t) = \mathbf{G}_t\mathbf{m}_1^t \qquad \text{and} \qquad \mathbf{x}_1^t = \mathbf{G}^{(t)}\mathbf{m}_1^t \tag{5}$$

where each entry of $\mathbf{G}_t$ is chosen uniformly and randomly from $\mathsf{GF}(2^q)$ excluding 0, $\mathbf{G}^{(t)}$ is obtained by vertically concatenating $\mathbf{G}_1$ to $\mathbf{G}_t$, and $\mathbf{m}_1^t$ (resp. $\mathbf{x}_1^t$) is the cumulative column vector consisted by vertically concatenating $m(1)$ to $m(t)$ (resp. $x(1)$ to $x(t)$). Because of the causal construction, $\mathbf{G}^{(t)}$ is a lower triangular matrix, see Fig. 1.

We now consider random message arrival ($R < 1$) and random erasure ($\epsilon > 0$). Essentially the effect of random message arrival is to delete each column of $\mathbf{G}^{(t)}$ with probability $1-R$ and the effect of random erasure is to delete each row of $\mathbf{G}^{(t)}$ with probability ϵ. After applying random column and row deletions to $\mathbf{G}^{(t)}$, we denote the final matrix by $\mathbf{H}^{(t)}$, which we term the *cumulative receiver matrix*. We then have

$$\mathbf{y}_1^t = \mathbf{H}^{(t)} \mathbf{m}_1^t \tag{6}$$

where we slightly abuse the notation and use $\mathbf{m}_1^t$ to denote the message packets that actually arrive during $[1, t]$ and $\mathbf{y}_1^t$ to denote the coded packets delivered successfully during $[1, t]$.

In Fig. 1, the columns boxed in blue signify no message arrival at time 2, 5 and $t-2$, and the rows boxed in red signify packet erasure at time 3, 5 and $t-1$. Removing the blue columns and red rows, the remaining matrix is the $\mathbf{H}^{(t)}$.

With sufficiently large finite field $\mathsf{GF}(2^q)$, we assume that Bernoulli arrival patterns are conveyed to the receiver with negligible overhead. As a result, the receiver has perfect knowledge of the random matrix $\mathbf{H}^{(t)}$ for all t. The goal is to decode $m(t)$ (if a message did arrive at time t) by the observation $\mathbf{y}_1^{t+\Delta}$ and the knowledge of $\mathbf{H}^{(t+\Delta)}$. We thus have

Definition 1. *A packet $m(t)$ is decodable by time $t + \Delta$ if the transpose of its* location *column vector $\vec{\delta}_{m(t)}$ is in the row space of $\mathbf{H}^{(t+\Delta)}$, where $\vec{\delta}_{m(t)}$ is of the same dimension as the $\mathbf{m}_1^{t+\Delta}$ and its entry corresponding to the location of $m(t)$ is one and all other $M_1^{t+\Delta} - 1$ entries are zero.*

B. Existing Results on RLSCs

In [22], the concept of *information debt* was proposed to describe the error events of (random) linear streaming codes:

Definition 2. *Initializing $I_d(0) \triangleq 0$, the information debt $I_d(t)$ at time $t \geq 1$ can be computed iteratively by*

$$I_d(t) \triangleq (M_t - Y_t + I_d(t-1))^+ \tag{7}$$

where M_t and Y_t are the numbers of the message arrivals and successful packet deliveries at time t and $(\cdot)^+ \triangleq \max(\cdot, 0)$.

[22, Lemma 6] states the following converse statement on decodability: If $I_d(t) > 0$, at least one message packet $m(\tau)$ with $\tau \in [1, t]$ cannot be decoded by time t. Our previous work [20] reused this definition of $I_d(t)$ but significantly strengthened the relationship between error events and $I_d(t)$.

Definition 3. *Define $t_0 \triangleq 0$ and define iteratively*

$$t_i \triangleq \inf\{\tau : \tau > t_{i-1}, I_d(\tau) = 0\} \tag{8}$$

as the i-th time that $I_d(t)$ hits 0.

Lemma 1 (Proposition 1 in [20])**.** *Assume the* **Generalized MDS** *condition* (**GMDS**) *in [20] (which is similar to assuming an infinite field $q \to \infty$). For any time $\tau \in (t_i, t_{i+1}]$ for some fixed $i \geq 0$, the earliest decoding time (EDT) of the message $m(\tau)$ is the next 0-hitting time t_{i+1}.*

Compared to the converse-only statement in [22, Lemma 6], Lemma 1 implies both (i) $m(\tau)$ can be decoded at the next time $I_d(t)$ hits 0, and (ii) $m(\tau)$ cannot be decoded at any earlier time slot. As a result, for any fixed τ the deadline-constrained error probability for $m(\tau)$ is equal to $\Pr(H_\tau(0) - \tau > \Delta)$, where $H_\tau(0) = \inf\{t \geq \tau : I_d(t) = 0\}$ is the next 0-hitting time after (and including) time τ.

Since $I_d(t)$ is a *random walk* bounded below by 0, by the standard *renewal theorem* we immediately have

Lemma 2 (Lemma 1 in [20])**.** *Assume $R < C = 1 - \epsilon$ and the* **GMDS***, the error probability of RLSCs (averaged over all $m(\tau)$) can be computed by*

$$p_e = \frac{\mathbb{E}\left\{(t_{i_0+1} - t_{i_0} - (\Delta+1))^+\right\}}{\mathbb{E}\left\{t_{i_0+1} - t_{i_0}\right\}} \tag{9}$$

for any arbitrary constant $i_0 \geq 0$.

C. Main Results on RLSCs

We define a new function before stating our main results.

Definition 4. *For any $x \in (0, \infty)$, define*

$$S(x) \triangleq \frac{4}{\pi} x \cdot e^{-\frac{x^2}{2}} \int_0^\infty \frac{u^2}{(x^2+u^2)^2} e^{-\frac{u^2}{2}} du. \tag{10}$$

Some properties of the function $S(x)$ are in order:

Proposition 2. *(i) $S(x)$ is non-increasing; (ii) $\lim_{x \searrow 0} S(x) = 1$; and (iii) $\lim_{x \to \infty} S(x) = 0$. Specifically, if we expand the domain of $S(x)$ by letting $S(0) \triangleq \lim_{x \searrow 0} S(x)$, then $S(x)$ is a CCDF of a positive finite random variable.*

The proof is omitted due to space constraints. It is worth noting that Property (iii) can be easily derived by examining (10). The proofs of Properties (i) and (ii) are quite involved and we need special techniques beyond simply taking the derivatives. For example, directly plugging $x = 0$ in (10) gives 0 rather 1. Comparing this fact to Property (ii), we can see that (10) is discontinuous at $x = 0$, while being continuous everywhere in $(0, \infty)$.

We also analyze the asymptotics of $S(x)$. For two arbitrary functions $f(x)$ and $g(x)$, we say $f(x) \sim g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. For example, it is known that $Q(x) \sim \frac{1}{\sqrt{2\pi}} x^{-1} e^{-\frac{x^2}{2}}$. We then have

Proposition 3.

$$S(x) \sim \frac{2\sqrt{2}}{\sqrt{\pi}} x^{-3} e^{-\frac{x^2}{2}} \tag{11}$$

Using the $S(x)$ function, we state our main results.

Proposition 4. *For the random packet arrival setting in this work, the channel dispersion equation of RLSCs becomes*

$$R = C - \sqrt{2\epsilon(1-\epsilon)} \frac{1}{\sqrt{\Delta}} S^{-1}(p_e) + o\left(\frac{1}{\sqrt{\Delta}}\right) \tag{12}$$

where $p_e \in (0, 1)$ is the target error probability and $S^{-1}(\cdot)$ is the inverse of $S(\cdot)$.

The similarity between (4) and (12) is striking, especially considering the fundamental differences between BMDS and

RLSCs. The former has a block structure that partitions the time-axis into disjoint segments. RLSCs have no block structure and the decodings of past and future messages $m(t)$ are inherently correlated across the entire time axis. Regardless, Proposition 4 shows that the channel dispersion of RLSCs still admits an identical form as that of BMDS (4). In both cases, the gap-to-optimality $C - R$ is a product involving the same dispersion term $(\sqrt{2\epsilon(1-\epsilon)})$, augmented by the additional randomness of packet arrivals. Both share the same $(\cdot)^{-.5}$ decrease term due to the n_B in (4) or the Δ in (12).

Even the cost of achieving the target probability p_e shares a similar form, both being the inverse of a CCDF with BMDS using the $Q^{-1}(\cdot)$ due to the underlying central limit theorem and the RLSCs using the new $S^{-1}(\cdot)$ function that is clearly related to Gaussian distributions (10) but has a much different expression due to the sequential nature of RLSCs.

Fig. 2 plots the CCDFs $Q(x)$ versus $S(x)$. One can see that when $p_e = 0.1774$, the inverse terms are identical $Q^{-1}(0.1774) = S^{-1}(0.1774)$. This implies that if and only if $p_e < 0.1774$, RLSCs will have a smaller gap-to-optimality $(C - R)$ due to the smaller cost term $S^{-1}(p_e) < Q^{-1}(p_e)$, and thus offer stronger error protection. Note that even though Fig. 2 plots $S(x)$ versus $Q(x)$ only for $x \in (0, 5]$, we have $S(x) < Q(x)$ for all $x \in (5, \infty)$ since the asymptotics of $S(x)$ in Proposition 3 has the x^{-3} term, which decays faster than the asymptotics of $Q(x)$ that only has the x^{-1} term. The error protection benefits of RLSCs thus persist for all $p_e < 0.1774$.

It is worth reiterating that there is no "queueing phase" in RLSC and it thus has no queueing delay. For comparison, because of the queueing delay in BMDS, the n_B values in (4) is roughly half of the Δ value in (12), see (2). *The aforementioned benefit of $S^{-1}(p_e) < Q^{-1}(p_e)$ is thus on top of the already significant savings of eliminating the queueing delay (i.e., $\frac{1}{\sqrt{\Delta}} \approx \frac{1}{\sqrt{2n_B}}$).*

The proof of Proposition 4 consists of the following results.

Lemma 3. *Assuming $R < C = 1 - \epsilon$, for any fixed $i_0 \geq 0$,*

$$\mathbb{E}\{t_{i_0+1} - t_{i_0}\} = \frac{(1-R)(1-\epsilon)}{1-R-\epsilon}. \tag{13}$$

Recall that $I_d(t)$ is updated in (7) via the random movement $(M_t - Y_t)$. For $z \in \{-1, 0, 1\}$, define $P_z = \Pr(M_t - Y_t = z)$. We have $P_{+1} = R\epsilon$, $P_{-1} = (1-R)(1-\epsilon)$, and

Lemma 4. *Assuming $R < C = 1 - \epsilon$, for any fixed $i_0 \geq 0$,*

$$\begin{aligned} &\mathbb{E}\left\{(t_{i_0+1} - t_{i_0} - (\Delta+1))^+\right\} \qquad (14)\\ &= \int_0^1 8P_{-1}P_{+1}\sin^2\left(\frac{\pi}{2}x\right)\left(1 - \sin^2\left(\frac{\pi}{2}x\right)\right)\\ &\cdot \frac{\left(1 - \left(\sqrt{P_{+1}} - \sqrt{P_{-1}}\right)^2 - 4\sqrt{P_{-1}P_{+1}}\sin^2\left(\frac{\pi}{2}x\right)\right)^\Delta}{\left(\left(\sqrt{P_{+1}} - \sqrt{P_{-1}}\right)^2 + 4\sqrt{P_{-1}P_{+1}}\sin^2\left(\frac{\pi}{2}x\right)\right)^2}dx. \end{aligned}$$

Sketch of the proof: The term $t_{i_0+1} - t_{i_0}$ is the classic definition of hitting time. We can thus easily use Doob's optional stopping theorem to derive its moment generating function (MGF). Lemma 3 is then a direct result of the MGF

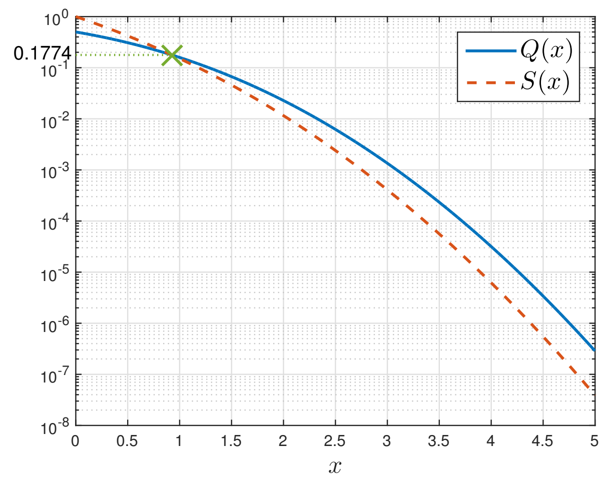

Fig. 2: Comparison between $Q(x)$ and $S(x)$.

expression. Nonetheless, how to evaluate the left-hand side of (14) using the MGF is an open problem. Instead, we take the following different approach.

Assuming $R = \mathbb{E}\{M_t\} < C = \mathbb{E}\{Y_t\}$, $I_d(t)$ is a random walk with negative drift. We thus have t_{i_0} and t_{i_0+1} be finite almost surely. For any fixed finite integers $n \in [0, \infty)$, define the following event:

$$\mathcal{A}_n = \left\{\max\{I_d(\tau) : \tau \in (t_{i_0}, t_{i_0+1}]\} < n\right\}. \tag{15}$$

That is, $\mathcal{A}_n$ is the event that the entire trajectory of $I_d(t)$ in the interval $(t_{i_0}, t_{i_0+1}]$ is strictly below a ceiling value n. Define $\mathbb{1}_{\mathcal{A}_n}$ as the indicator function of $\mathcal{A}_n$.

In our proof we first show that

$$\mathbb{E}\left\{\mathbb{1}_{\mathcal{A}_n} \cdot (t_{i_0+1} - t_{i_0} - (\Delta+1))^+\right\} = \frac{1}{n}\sum_{\ell=1}^{n-1} f\left(\frac{\ell}{n}\right) \tag{16}$$

for some function $f(\cdot)$ that does not depend on n. The reason that (16) is easier to derive is that *multiplying $\mathbb{1}_{\mathcal{A}_n}$ restricts the state space of interest from all non-negative integers to a finite range* $[0, n]$. The (restricted) $I_d(t)$ thus becomes a finite-state Markov chain, which is much more tractable than the original unrestricted infinite-state-space problem. For example, the finite transition matrix admits an *almost tri-diagonal* structure because the random movement $(M_t - Y_t)$ in (7) only takes values in $\{-1, 0, 1\}$. We can then use the eigenspace decomposition results of tri-diagonal matrices [23] to complete the derivation of (16).

By the monotone convergence theorem, we have

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{E}\left\{\mathbb{1}_{\mathcal{A}_n} \cdot (t_{i_0+1} - t_{i_0} - (\Delta+1))^+\right\}\\ &= \mathbb{E}\left\{(t_{i_0+1} - t_{i_0} - (\Delta+1))^+\right\}. \end{aligned} \tag{17}$$

Since letting $n \to \infty$ turns the summation in (16) to an integral $\int_{x=0}^1 f(x)dx$, we have Lemma 4. In fact, the expression of $f(x)$, which we did not specify earlier, is exactly the expression of the integrand in (14). □

Since (9), (13), and (14) jointly governs the exact relationship among $(R, \epsilon, \Delta, p_e)$, we can conduct the asymptotic

	$\Delta = 100$		$\Delta = 200$	
	RLSCs	BMDS	RLSCs	BMDS
$R_{\text{exact}}(\Delta)$: Exact Finite-Length	0.3181	0.2286	0.3680	0.2982
$R_{\text{approx}}(\Delta)$: Ch. Disper. Approx.	0.3053	0.1910	0.3623	0.2815
$\frac{C-R_{\text{approx}}(\Delta)}{C-R_{\text{exact}}(\Delta)}$	1.0704	1.1385	1.0432	1.0828

TABLE I: Tightness comparison with $\epsilon = 0.5$ and $p_e = 10^{-3}$.

analysis that characterizes the tradeoff between R and Δ by fixing ϵ and p_e in (9), (13), and (14). The final result is then the channel dispersion equation in (12).

IV. Numerical Evaluation

In Fig. 3, we compare BMDS and RLSCs by plotting their exact finite length results and the corresponding approximations using (4) plus (2) and using (12), respectively. For any fixed (R, ϵ, Δ) tuple, the exact $p_{e,\text{actual}}$ of BMDS is computed by numerically evaluating (2) and (3). The exact $p_{e,\text{actual}}$ of RLSCs is evaluated either by the previous results [20, Proposition 2] using matrix operations or by combining (9), (13), (14) and numerically computing the integral value in (14). Either method gives identical $p_{e,\text{actual}}$. We then fix the channel $\epsilon = 0.5$ and the target error probability $p_e = 10^{-3}$, choose a deadline $\Delta \in [100, 1000]$ (and $n_B = \lfloor 0.5\Delta \rfloor$), and use the bisection method to find the largest R that still satisfies $p_{e,\text{actual}} < p_e$. Varying Δ, we plot the corresponding $R(\Delta)$ as the "Exact Finite-Length" curve(s) in Fig. 3.

The validity of BMDS and RLSC channel dispersion equations is clearly visible in Fig. 3. For RLSCs, the approximation (12) is quite tight even for moderate delay, say $\Delta = 100$ to 400. The complexity savings of the new approximation are very significant when compared to the previous finite length result [20, Proposition 2], which involves inverting a matrix of size growing roughly-linearly with respect to Δ, a numerically challenging task for large Δ. Even when compared to the new, much simplified integral-based finite-length analysis (9), (13), and (14), the complexity savings are still substantial since it circumvents the need of finding $R(\Delta)$ by bisection.

The RLSC channel dispersion approximation (12) is also much tighter than (4) of BMDS. TABLE I compares the exact $R_{\text{exact}}(\Delta)$ and their channel dispersion approximations $R_{\text{approx}}(\Delta)$. It also lists the ratios of the approximate versus exact gap-to-optimality. As can be seen, RLSC approximation is tighter than its BMDS counterpart, e.g., 107% versus 114% when $\Delta = 100$. One possible explanation is that the n_B used in (4) is roughly half of the Δ used in (12). As a result, the "approximation power" of (4) at the same Δ is only half of that of (12). However, if we compare the ratio of $\frac{C-R_{\text{approx}}(\Delta)}{C-R_{\text{exact}}(\Delta)}$ of RLSCs at $\Delta = 100$ with the ratio for BMDS but at $\Delta = 200$, one can see that the RLSC approximation is still tighter. Additional investigation is needed to further compare the approximation power of (4) and (12).

If we focus on the horizontal line of $R = 0.4$ in Fig. 3, the delay needed by RLSCs is 358 while the delay needed for BMDS is 902. That is, under the same $p_e = 10^{-3}$ and $R = 0.4$, RLSCs reduce the end-to-end delay by 60.3%, which implies that RLSCs not only eliminate the queueing delay completely (thus 50%) but also offer better error protection. This observation is consistent with our analytical discussion of $S^{-1}(p_e)$ versus $Q^{-1}(p_e)$ in Sec. III-C. All the aforementioned observations of Fig. 3 hold consistently when we repeat the evaluation with different parameter values, including $(\epsilon, p_e) = (0.5, 10^{-6})$, $(0.8, 10^{-3})$, $(0.25, 10^{-2})$, etc.

For pedagogical purposes, Fig. 4 repeats the experiment of Fig. 3 but with $p_e = 0.25$, an impractically large value. Focusing on the exact $R(\Delta)$, at $R = 0.48$, the delays required for RLSCs and BMDS are 689 and 926, respectively. This implies that even though RLSCs eliminate queueing delay (50% savings), they offer weaker error protection. Additional delay is needed to maintain the same $p_e = 0.25$ and the combined delay savings are only 25.6%.

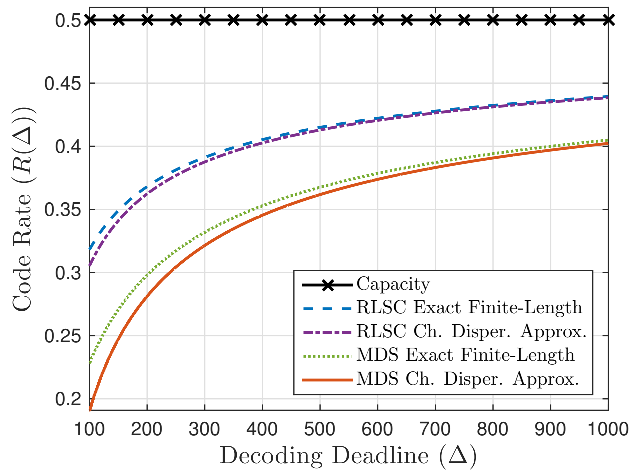

Fig. 3: Rate-delay tradeoff with $\epsilon = 0.5$ and $p_e = 10^{-3}$.

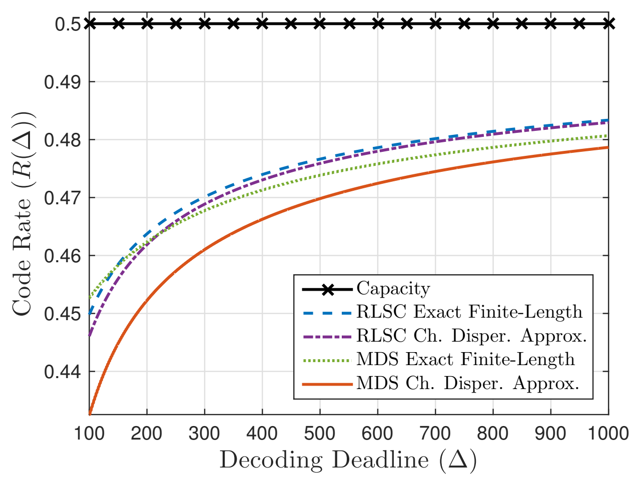

Fig. 4: Rate-delay tradeoff with $\epsilon = 0.5$ and $p_e = 0.25$.

V. Conclusion

This work has studied the PECs with Bernoulli packet arrivals. New channel dispersion characterizations have been developed for RLSCs, the first of such kind for any sequential coding schemes.

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