On The Capacity Of 2-User 1-Hop Relay Erasure Networks — The Union of Feedback, Scheduling, Opportunistic Routing, and Network Coding

Wei-Cheng Kuo, Chih-Chun Wang; {wkuo, chihw}@purdue.edu
Center of Wireless Systems and Applications (CWSA)
School of Electrical and Computer Engineering, Purdue University, USA

***Abstract*—This work studies the capacity of 2-user 1-hop relay networks, for which the sources, destinations, and the common relay are interconnected by broadcast packet erasure channels. In contrast with the existing results, this paper allows (i) transmission from a source being heard directly by its 2-hop-away destination, the so-called *opportunistic routing* scenario, (ii) *instant channel status feedback among all network nodes*, and (iii) per-slot scheduling decisions that are functions of the traffic loads and the past channel status. A new pair of inner and outer bounds is provided, and a condition is identified for the scenario in which the bounds coincide. Numerical experiments show that for commonly encountered scenarios, the gap between the inner and the outer bounds is less than 0.2%, which demonstrates the effectiveness of the proposed bounding techniques.**

I. Problem Formulation

For any positive integer M, define $[M] \triangleq \{1, \cdots, M\}$. We consider a 1-hop relay network with 2 source-destination pairs (s_1, d_1) and (s_2, d_2) and a common relay r interconnected by packet erasure channels (PECs). See Fig. 1(a) for illustration. Assume slotted transmission. Within an overall time budget of n time slots, s_i would like to convey nR_i packets $\mathbf{W}_i \triangleq (W_{i,1}, \cdots, W_{i,nR_i})$, $j \in [nR_i]$, to d_i for all $i \in \{1, 2\}$. For each $i \in \{1, 2\}$, $j \in [nR_i]$, the information packet $W_{i,j}$ is chosen independently and uniformly randomly from $\mathsf{GF}(q)$.

For any time t, consider a random 8-dimensional *channel status vector* for the entire network:

$$\mathbf{Z}(t) = (Z_{s_1\to d_1}(t), Z_{s_1\to d_2}(t), Z_{s_1\to r}(t), Z_{s_2\to d_1}(t), Z_{s_2\to d_2}(t), Z_{s_2\to r}(t), Z_{r\to d_1}(t), Z_{r\to d_2}(t)) \in \{*, 1\}^8,$$

where "$*$" and "1" represent erasure and successful reception, respectively. That is, when s_1 transmits a packet $X_{s_1}(t) \in \mathsf{GF}(q)$ in time t, relay r receives $Y_{s_1\to r}(t) = X_{s_1}(t)$ if $Z_{s_1\to r}(t) = 1$ and receives $Y_{s_1\to r}(t) = *$ if $Z_{s_1\to r}(t) = *$. For simplicity, we use $Y_{s_1\to r}(t) = X_{s_1}(t) \circ Z_{s_1\to r}(t)$ as shorthand. To model interference, we further assume that only one node can be scheduled in each time slot. We use $\sigma(t) \in \{s_1, s_2, r\}$ to denote the *scheduling decision* at time t. For convenience, when s_1 is not scheduled at time t, we simply set $Y_{s_1\to r}(t) = *$. As a result, the scheduling decision can be incorporated into the following expression of $Y_{s_1\to r}(t)$:

$$Y_{s_1\to r}(t) = X_{s_1}(t) \circ Z_{s_1\to r}(t) \circ 1_{\{\sigma(t)=s_1\}}.$$

Similar notation is used for all other received signals. For example, $Y_{r\to d_2}(t) = X_r(t) \circ Z_{r\to d_2}(t) \circ 1_{\{\sigma(t)=r\}}$ is what d_2 receives from r in time t, where $X_r(t)$ is the packet sent by r in time t.

We assume memoryless, stationary, erasure networks. Namely, we allow arbitrary joint distribution for the coordinates of $\mathbf{Z}(t)$ but assume that $\mathbf{Z}(t)$ is independently and identically distributed over the time axis t. We also assume $\mathbf{Z}(t)$ is independent of the information messages $\mathbf{W}_1$ and $\mathbf{W}_2$.

The following notation turns out to be useful. We use brackets $[\cdot]_1^t$ to denote the collection from time 1 to t. For example, $[\sigma, \mathbf{Z}, Y_{s_1\to d_1}]_1^t \triangleq \{\sigma(\tau), \mathbf{Z}(\tau), Y_{s_1\to d_1}(\tau) : \forall \tau \in [1, t]\}$. For any $S \subseteq \{s_1, s_2, r\}$, $T \subseteq \{r, d_1, d_2\}$, we define

$$\mathbf{Y}_{S\to T}(t) \triangleq \{Y_{s\to d}(t) : \forall s \in S, \forall d \in T\}.$$

For example, $\mathbf{Y}_{\{s_1,r\}\to\{d_1,d_2\}}(t)$ is the collection of $Y_{s_1\to d_1}(t)$, $Y_{s_1\to d_2}(t)$, $Y_{r\to d_1}(t)$, and $Y_{r\to d_2}(t)$.

Given the traffic load (R_1, R_2), a joint scheduling and network coding (NC) scheme is defined by n scheduling decision functions

$$\forall t \in [n],\ \sigma(t) = f_{\sigma,t}([\mathbf{Z}]_1^{t-1}), \tag{1}$$

and $3n$ encoding functions at s_1, s_2, and r, respectively: For all $t \in [n]$

$$X_{s_i}(t) = f_{s_i,t}(\sigma(t), \mathbf{W}_i, [\mathbf{Z}]_1^{t-1}), \quad \forall i \in \{1, 2\}, \tag{2}$$

$$X_r(t) = f_{r,t}(\sigma(t), [\mathbf{Y}_{\{s_1,s_2\}\to r}, \mathbf{Z}]_1^{t-1}), \tag{3}$$

and 2 decoding functions at d_1 and d_2, respectively:

$$\hat{\mathbf{W}}_i = f_{d_i}([\sigma, \mathbf{Y}_{\{s_1,s_2,r\}\to d_i}, \mathbf{Z}]_1^n), \quad \forall i \in \{1, 2\}. \tag{4}$$

Namely, the scheduling decision at time t is based on the channel status of the entire network in time 1 to $(t-1)$. Encoding at s_i depends on the scheduling decision, the information messages, and past channel status. Encoding at r depends on the scheduling decision, what r received in the past, and past channel status. In the end, d_i decodes $\mathbf{W}_i$ based on the past scheduling decisions, what d_i has received, and the past channel status of the entire network.

This setting models the scenario in which there is a dedicated low-rate control channel that can broadcast the scheduling decision $\sigma(t)$ and the previous network status $\mathbf{Z}(t-1)$ causally to all network nodes. The total amount of control information is less than $(8 + \log_2(3))$ bits per time slot, which is much smaller than the actual content of each packet $\approx 10^4$

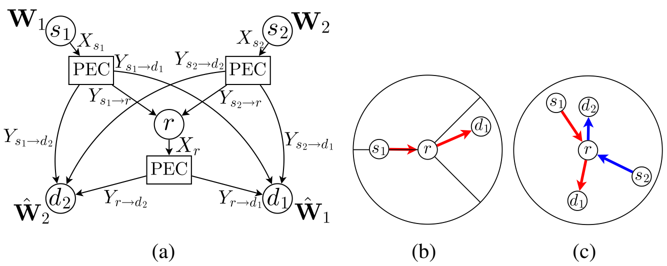

Fig. 1. (a) The 2-user 1-hop relay network. (b) The relative location of (s_i, d_i). (c) Topology of two (s_i, d_i) pairs.

TABLE I
COMPARISON WITH PREVIOUS WORKS

[10]:	This work:
S1: Deterministic sequential scheduling. **S2:** No channel status feedback for s_1 and s_2. Feedback is sent to r for only one time instant (via the use of reception reports). **S3:** Not modeling the feedback mechanism of COPE [4].	**S1:** Dynamic scheduling. **S2:** Continuous channel status feedback among the entire network. **S3:** Strictly include COPE as a special example.
R1: Provide inner and outer bounds for general M users. **R2:** For $M \leq 3$, inner and outer bounds coincide when disallowing 2-hop overhearing.	**R1:** Provide inner and outer bounds for 2 users. **R2:** For M=2, inner and outer bounds coincide when disallowing 2-hop overhearing.

bits and thus can be easily implemented by buffering and piggybacking on the data packets.

Definition 1: Fix the distribution of $\mathbf{Z}(t)$. A rate vector (R_1, R_2) is achievable if for any $\epsilon > 0$, there exists a joint scheduling and NC scheme with sufficiently large n and $\mathsf{GF}(q)$ such that

$$\max_{\forall i\in\{1,2\}} \mathsf{Prob}(\mathbf{W}_i \neq \hat{\mathbf{W}}_i) < \epsilon.$$

The capacity region is defined as the closure of all achievable rate vectors (R_1, R_2).

Remark 1: In (1), the scheduling decision $\sigma(t)$ does not depend on the information messages $\mathbf{W}_i$, which means that we prohibit the use of timing channels [1], [7]. Even when we allow the usage of timing channels, we conjecture that the overall capacity improvement is negligible. A heuristic argument is that each successful packet transmission gives $\log_2(q)$ bits of information while the timing information (to transmit or not) gives roughly 1 bit of information. When focusing on sufficiently large $\mathsf{GF}(q)$, additional gain of timing information is thus likely to be absorbed in our timing-information-free capacity characterization.

Remark 2: In our setting, r is the only node that can mix packets from two different data sessions. Further relaxation such that s_1 and s_2 can hear each other and perform coding accordingly is beyond the scope of this work.

II. Related Works

The capacity for the setting of multiple unicast sessions, termed the intersession network coding problem (INC), remains largely unsolved. Recently, [10] models the practical INC protocol [4] as a 1-hop relay network interconnected by PECs and studies the corresponding capacity region.

There are two major differences between the setting of this work and in [10]. First, a deterministic sequential scheduling policy is used in [10], which schedules nodes s_1, s_2, and r in strict order. Namely, once we finish transmission of s_2, the subsequent time slots can only be used for transmitting packets from r. For comparison, our setting allows dynamically choosing the schedule $\sigma(t)$ for each time slot t. Second, in [10] no feedback is allowed when s_1 and s_2 transmit. More specifically, suppose jointly s_1 and s_2 take $t_{s_1} + t_{s_2}$ time slots to finish transmission. Then only in the beginning of time $(t_{s_1} + t_{s_2} + 1)$ are we allowed to send the channel status $[\mathbf{Z}]_1^{t_{s_1}+t_{s_2}}$ to r. No further feedback is allowed until time n, the end of overall transmission. For comparison, our setting allows constantly broadcasting network-wide channel status $[\mathbf{Z}]_1^{t-1}$ to s_1, s_2, and r, as discussed in Section I. This setting thus includes the Automatic Repeat reQuest (ARQ) mechanism as a special case [3]. Broadcasting $[\mathbf{Z}]_1^{t-1}$ also eliminates the need of estimating/learning the reception status of the neighbors. Our capacity region thus contains all achievable rates of the COPE protocol [4]. In Table I we use S1 to S3 to highlight the differences of the settings, and use R1 and R2 to highlight the main results.

Since we allow the schedule $\sigma(t)$ to depend on the past reception status $[\mathbf{Z}]_1^{t-1}$, we also include any store-&-forward-based scheduling policies as special cases, such as the back-pressure and the maximal weighted matching schemes (see [6] for references). Our results thus quantify the best achievable rates with jointly designed scheduling and coding policies.

Remark: From the relay's perspective, the past channel output $[\mathbf{Y}_{r\to\{d_1,d_2\}}]_1^{t-1}$ is a function of $[X_r, \sigma, \mathbf{Z}]_1^{t-1}$ that can be computed from $[\mathbf{Y}_{\{s_1,s_2\}\to r}]_1^{t-2}$ and $[\mathbf{Z}]_1^{t-1}$. As a result, our setting, particularly (3), also includes the setting of channel output feedback [3], [9] as a special example.

In Section III, we provide the outer/inner bounds for a 2-user 1-hop relay erasure network, and identify the condition for which the bounds coincide. Section IV provides the results of numerical experiments. Our results show that for scenarios commonly encountered in practice, the gap between the outer and inner bounds is less than .2%. The bounding techniques in this work thus effectively determine the joint scheduling-coding capacity. Section V concludes this paper.

III. Main Results

All our results assume a fixed distribution of the channel status vector $\mathbf{Z}(t)$, which can be described by specifying the joint probability mass function for the 2^8 possible outcomes.[1] In this work, we use $p_{s\to T_1\overline{T_2}}$ to denote the probability that a packet sent from s is received by all nodes in T_1 but not by any node in T_2 (may or may not by any node outside $T_1 \cup T_2$). For example, $p_{s_1\to d_1 r\overline{d_2}}$ denotes the probability that a packet sent from s_1 is received by d_1 and r but not by d_2. Additionally, we use $p_{s;T}$ to denote the probability that a packet sent by s is received by *at least one node in* T. For example, $p_{r;\{d_1,d_2\}}$ is the probability that a packet sent by r is received by at least one node in $\{d_1, d_2\}$. By definition, $p_{s_2;r} = p_{s_2\to r}$.

[1] By allowing the coordinates of $\mathbf{Z}(t)$ to be correlated, our setting also models the scenario in which destinations d_1 and d_2 are situated in the same physical node and thus have perfectly correlated channel success events.

A. Capacity Outer Bound

Proposition 1: For any achievable (R_1, R_2), there exist three non-negative scalars t_{s_1}, t_{s_2}, and t_r satisfying

$$t_{s_1} + t_{s_2} + t_r \leq 1 \tag{5}$$

$$\forall i \in \{1,2\}, \quad R_i \leq t_{s_i} p_{s_i;\{d_i,r\}} \tag{6}$$

$$\frac{(R_1 - t_{s_1} p_{s_1;d_1})^+}{p_{r;d_1}} + \frac{\left(R_2 - t_{s_2} p_{s_2;\{d_1,d_2\}}\right)^+}{p_{r;\{d_1,d_2\}}} \leq t_r \tag{7}$$

$$\frac{\left(R_1 - t_{s_1} p_{s_1;\{d_1,d_2\}}\right)^+}{p_{r;\{d_1,d_2\}}} + \frac{(R_2 - t_{s_2} p_{s_2;d_2})^+}{p_{r;d_2}} \leq t_r. \tag{8}$$

where $(\cdot)^+ \triangleq \max(0,\cdot)$ is the projection to non-negative reals.

Proof: For any joint scheduling and NC scheme, we choose t_{s_i} (resp. t_r) as the normalized *expected number of time slots for which s_i (resp. r) is scheduled*. As a result, (5) must hold. Without loss of generality, in the subsequent discussion we also assume that both the n and $\mathsf{GF}(q)$ used by the joint scheduling and NC scheme are sufficiently large.

The intuition behind (6) is the following observation. Only for the time instants with $\sigma(t) = s_i$ can s_i convey information directly to d_i or indirectly through r. Therefore, the achievable rate R_i is upper bounded by $t_{s_i} p_{s_i;\{d_i,r\}}$, the expected number of distinct packets received by either d_i or r.

We can also prove (6) formally.

$$\begin{aligned}
&I(\mathbf{W}_1; \hat{\mathbf{W}}_1) \leq I(\mathbf{W}_1; [Y_{s_1\to d_1}, Y_{s_2\to d_1}, Y_{r\to d_1}, \mathbf{Z}, \sigma]_1^n) \\
&\leq I(\mathbf{W}_1; [Y_{s_1\to d_1}, Y_{s_2\to d_1}, Y_{s_1\to r}, Y_{s_2\to r}, \mathbf{Z}, \sigma]_1^n) \\
&= \sum_{t=1}^{n} I(\mathbf{W}_1; \mathbf{Y}_{\{s_1,s_2\}\to\{d_1,r\}}(t), \mathbf{Z}(t-1), \sigma(t) | B_t) \qquad (9) \\
&= \sum_{t=1}^{n} (I(\mathbf{W}_1; \mathbf{Y}_{s_1\to\{d_1,r\}}(t), \mathbf{Z}(t-1), \sigma(t) | B_t) \\
&\quad + I(\mathbf{W}_1; \mathbf{Y}_{s_2\to\{d_1,r\}}(t) | \mathbf{Y}_{s_1\to\{d_1,r\}}(t), B_t) \qquad (10)
\end{aligned}$$

where $B_t \triangleq ([\mathbf{Y}_{\{s_1,s_2\}\to\{d_1,r\}}, \sigma]_1^{t-1}, [\mathbf{Z}]_1^{t-2})$; consider the summation in (10). Since the first term is non-zero only when $\sigma(t) = s_1$, it is upper bounded by

$$\begin{aligned}
&I(\mathbf{W}_1; \mathbf{Y}_{s_1\to\{d_1,r\}}(t), \mathbf{Z}(t-1), \sigma(t) | B_t) \\
&\leq \sum_{\text{all } b_t} \mathsf{Prob}(B_t = b_t) \cdot \mathsf{Prob}(\sigma(t) = s_1 | B_t = b_t) \cdot p_{s_1;\{d_1,r\}} \\
&= \mathsf{Prob}(\sigma(t) = s_1) \cdot p_{s_1;\{d_1,r\}} \qquad (11)
\end{aligned}$$

Hence the summation of the first term in (10) is upper bounded by

$$\sum_{t=1}^{n} \mathsf{Prob}(\sigma(t) = s_1) \cdot p_{s_1;\{d_1,r\}} = n t_{s_1} p_{s_1;\{d_1,r\}} \tag{12}$$

By the independence between $\mathbf{W}_1$ and $\mathbf{Y}_{s_2\to\{d_1,d_2\}}(t)$ conditioning on $\mathbf{Y}_{s_1\to\{d_1,d_2\}}(t)$ and B_t, the second term of the summation in (10) is zero. On the other hand, Fano's inequality gives us

$$I(\mathbf{W}_1; \hat{\mathbf{W}}_1) \geq n R_1 (1-\epsilon) - H(\epsilon) \tag{13}$$

Combining (12) and (13), we have

$$R_1(1-\epsilon) - \frac{H(\epsilon)}{n} \leq t_{s_1} p_{s_1;\{d_1,r\}} \tag{14}$$

Let $\epsilon \to 0$ and therefore (14) implies (6) for $i = 1$. With symmetric derivation, we can derive (6) for $i = 2$.

We prove (7) by similar techniques as used in [2], [8]. That is, we first add an auxiliary pipe that sends all information available at d_2 directly to d_1. Then the decoding function (see (4)) at d_1 becomes

$$\hat{\mathbf{W}}_1 = f_{d_1}([\sigma, \mathbf{Y}_{\{s_1,s_2,r\}\to\{d_1,d_2\}}, \mathbf{Z}]_1^n). \tag{15}$$

For any $t \in [n]$, define

$$U(t) \triangleq (\mathbf{W}_2, [\mathbf{Y}_{\{s_1,s_2\}\to\{d_1,d_2\}}]_1^t, [\mathbf{Y}_{r\to\{d_1,d_2\}}, \sigma]_1^{t-1}, [\mathbf{Z}]_1^{t-2}).$$

By taking logarithm with base q and by Fano's inequality, for some $\epsilon_1 > 0$ that goes to 0 as $\epsilon \to 0$, we must have

$$\begin{aligned}
&n R_1 = H(\mathbf{W}_1 | \mathbf{W}_2) \leq I(\mathbf{W}_1; \hat{\mathbf{W}}_1 | \mathbf{W}_2) + n\epsilon_1 \\
&\leq I(\mathbf{W}_1; [\sigma, \mathbf{Y}_{\{s_1,s_2,r\}\to\{d_1,d_2\}}, \mathbf{Z}]_1^n | \mathbf{W}_2) + n\epsilon_1 \qquad (16) \\
&= n\epsilon_1 + \sum_{t=1}^{n} I(\mathbf{W}_1; \mathbf{Y}_{\{s_1,s_2,r\}\to\{d_1,d_2\}}(t), \sigma(t), \mathbf{Z}(t-1) | A_t) \\
&\leq n\epsilon_1 + \sum_{t=1}^{n} (I(\mathbf{W}_1; \mathbf{Y}_{s_1\to\{d_1,d_2\}}(t) | A_t) \\
&\qquad + I(\mathbf{W}_1; \mathbf{Y}_{s_2\to\{d_1,d_2\}}(t) | \mathbf{Y}_{s_1\to\{d_1,d_2\}}(t), A_t) \\
&\qquad + I(\mathbf{W}_1; \mathbf{Y}_{r\to\{d_1,d_2\}}(t), \sigma(t), \mathbf{Z}(t-1) | \\
&\qquad\qquad \mathbf{Y}_{\{s_1,s_2\}\to\{d_1,d_2\}}(t), A_t)) \qquad (17)
\end{aligned}$$

where $A_t \triangleq (\mathbf{W}_2, [\mathbf{Y}_{\{s_1,s_2,r\}\to\{d_1,d_2\}}, \sigma]_1^{t-1}, [\mathbf{Z}]_1^{t-2})$; (16) follows from (15); and (17) follows from the chain rule.[2]

Consider the summation in (17). Since the first term is non-zero only when $\sigma(t) = s_1$, it is upper bounded by

$$\sum_{t=1}^{n} \mathsf{Prob}(\sigma(t) = s_1) \cdot p_{s_1;\{d_1,d_2\}} = n t_{s_1} p_{s_1;\{d_1,d_2\}}.$$

By the independence between $\mathbf{W}_1$ and $\mathbf{Y}_{s_2\to\{d_1,d_2\}}(t)$ conditioning on $\mathbf{Y}_{s_1\to\{d_1,d_2\}}(t)$ and A_t, the second term of the summation is zero. Since mutual information is non-negative, we now have

$$\begin{aligned}
&\left(n R_1 - n t_{s_1} p_{s_1;\{d_1,d_2\}} - n\epsilon_1\right)^+ \\
&\leq \sum_{t=1}^{n} I(\mathbf{W}_1; \mathbf{Y}_{r\to\{d_1,d_2\}}(t), \sigma(t), \mathbf{Z}(t-1) | U(t)) \\
&\leq \sum_{t=1}^{n} I(X_r(t); \mathbf{Y}_{r\to\{d_1,d_2\}}(t), \sigma(t), \mathbf{Z}(t-1) | U(t)) \qquad (18) \\
&\leq \sum_{t=1}^{n} \left(\log_q(3) + \log_q(2^8)\right. \\
&\qquad \left. + I(X_r(t); \mathbf{Y}_{r\to\{d_1,d_2\}}(t) | U(t), \sigma(t), \mathbf{Z}(t-1))\right), \qquad (19)
\end{aligned}$$

where (18) follows from the fact that conditioning on $U(t)$, $\mathbf{W}_1 \mapsto X_r(t) \mapsto (\mathbf{Y}_{r\to\{d_1,d_2\}}(t), \sigma(t), \mathbf{Z}(t-1))$ form a

[2] For simplicity, in (9) and (17), we ignore the last term involving $\mathbf{Z}(n)$ since it is a negligible term when a sufficiently large n is used.

Markov chain; and (19) follows from the fact that the cardinality of $\sigma(t)$ and $\mathbf{Z}(t-1)$ are 3 and 2^8, respectively.

By Fano's inequality, for some $\epsilon_2 > 0$ that goes to 0 as $\epsilon \to 0$, with similar steps as in (17), we can also prove that

$$\begin{aligned} nR_2 &= H(\mathbf{W}_2) \\ &\le I(\mathbf{W}_2; [\mathbf{Y}_{\{s_1,s_2,r\}\to d_2}, \mathbf{Z}, \sigma]_1^n) + n\epsilon_2 \\ &= n\epsilon_2 + \sum_{t=1}^n I(\mathbf{W}_2; \mathbf{Y}_{\{s_1,s_2,r\}\to d_2}(t), \mathbf{Z}(t-1), \sigma(t)|D_t) \\ &= n\epsilon_2 + \sum_{t=1}^n (I(\mathbf{W}_2; \mathbf{Y}_{\{s_1,s_2\}\to d_2}(t)|D_t) \\ &\quad + I(\mathbf{W}_2; Y_{r\to d_2}(t), \mathbf{Z}(t-1), \sigma(t)|\mathbf{Y}_{\{s_1,s_2\}\to d_2}(t), D_t)) \\ &= n\epsilon_2 + \sum_{t=1}^n (I(\mathbf{W}_2; Y_{s_2\to d_2}(t)|D_t) \\ &\quad + I(\mathbf{W}_2; Y_{r\to d_2}(t), \mathbf{Z}(t-1), \sigma(t)|\mathbf{Y}_{\{s_1,s_2\}\to d_2}(t), D_t)) \end{aligned} \tag{20}$$

where $D_t \triangleq ([\mathbf{Y}_{\{s_1,s_2,r\}\to d_2}, \sigma]_1^{t-1}, [\mathbf{Z}]_1^{t-2})$; and (20) follows from the independence between $\mathbf{W}_2$ and $Y_{s_1\to d_2}$ conditioning on D_t. Notice that the summation of the first term in (20) can be upper bounded by

$$\sum_{t=1}^n \mathsf{Prob}(\sigma(t) = s_2) \cdot p_{s_1;d_2} = nt_{s_2} p_{s_1;d_2} \tag{21}$$

Hence

$$\begin{aligned} (20) \le & n\epsilon_2 + nt_{s_2} p_{s_2\to d_2} \\ & + \sum_{t=1}^n (H(Y_{r\to d_2}(t), \mathbf{Z}(t-1), \sigma(t)|\mathbf{Y}_{\{s_1,s_2\}\to d_2}(t), D_t) \\ & - H(Y_{r\to d_2}(t), \mathbf{Z}(t-1), \sigma(t)|\mathbf{W}_2, \mathbf{Y}_{\{s_1,s_2\}\to d_2}(t), D_t)) \end{aligned} \tag{22}$$

$$\begin{aligned} \le\ & n\epsilon_2 + nt_{s_2} p_{s_2\to d_2} \\ & + \sum_{t=1}^n (H(Y_{r\to d_2}(t), \mathbf{Z}(t-1), \sigma(t)) \\ & \quad - H(Y_{r\to d_2}(t), \mathbf{Z}(t-1), \sigma(t)|U(t))) \end{aligned} \tag{23}$$

$$= n\epsilon_2 + nt_{s_2} p_{s_2\to d_2} + \sum_{t=1}^n I(U(t); Y_{r\to d_2}(t), \mathbf{Z}(t-1), \sigma(t)) \tag{24}$$

where (23) follows from the property of entropy; since mutual information is non-negative, we now have

$$\begin{aligned} & (nR_2 - nt_{s_2} p_{s_2;d_2})^+ \\ & \le n\epsilon_2 + \sum_{t=1}^n I(U(t); Y_{r\to d_2}(t), \mathbf{Z}(t-1), \sigma(t)) \end{aligned} \tag{25}$$

$$\begin{aligned} & \le n\epsilon_2 + \sum_{t=1}^n (\log_q(3) + \log_q(2^8) \\ & \qquad + I(U(t); Y_{r\to d_2}(t)|\sigma(t), \mathbf{Z}(t-1))) \end{aligned} \tag{26}$$

where (26) follows from the fact that the cardinality of $\sigma(t)$ and $\mathbf{Z}(t-1)$ are 3 and 2^8, respectively.

Now we can define the time sharing random variable Q_1, where $Q_1 = t \in [n]$ with probability $\frac{1}{n}$, and define $\mathbf{Q} = (Q_1, \mathbf{Z}(Q_1 - 1))$. By using $q \to \infty$ to absorb the $\log_q(3) + \log_q(2^8)$ term, we can rewrite (19) and (26) as

$$\begin{aligned} & (R_1 - t_{s_1} p_{s_1;\{d_1,d_2\}})^+ \\ & \le \epsilon_1 + \sum_{q_1=1}^n \frac{1}{n} I(X_r(q_1); \mathbf{Y}_{r\to\{d_1,d_2\}}(q_1)| \\ & \quad U(q_1), \mathbf{Z}(q_1-1), Q_1 = q_1, \sigma(q_1)) \\ & = I(X_r(Q_1); \mathbf{Y}_{r\to\{d_1,d_2\}}(Q_1)|U(Q_1), \mathbf{Q}, \sigma(Q_1)) + \epsilon_1 \end{aligned} \tag{27}$$

$$\begin{aligned} & (R_2 - t_{s_2} p_{s_2;d_2})^+ \\ & \le \epsilon_2 + \sum_{q_1=1}^n \frac{1}{n} I(U(q_1); Y_{r\to d_2}(q_1)| \\ & \quad \mathbf{Z}(q_1-1), Q_1 = q_1, \sigma(q_1)) \\ & = I(U(Q_1); Y_{r\to d_2}(Q_1)|\mathbf{Q}, \sigma(Q_1)) + \epsilon_2 \end{aligned} \tag{28}$$

Notice that only if $\sigma(Q_1) = r$, the mutual information in (27) and (28) not equals to zero (i.e. $Y_{r\to d_1}(Q_1)$ and $Y_{r\to d_2}(Q_1)$ are non-empty). And since Q_1 is uniformly distributed over $[n]$, $\mathsf{Prob}(\sigma(Q_1) = r) = t_r$. Define $X_r \triangleq X_r(Q_1)$, $Y_{r\to d_1} \triangleq Y_{r\to d_1}(Q_1)$, $Y_{r\to d_2} \triangleq Y_{r\to d_2}(Q_1)$, $U \triangleq U(Q_1)$, then we have

$$(R_1 - t_{s_1} p_{s_1;\{d_1;d_2\}})^+ \le t_r I(X_r; \mathbf{Y}_{r\to\{d_1,d_2\}}|U, \mathbf{Q}) + \epsilon_1 \tag{29}$$

$$(R_2 - t_{s_2} p_{s_2;d_2})^+ \le t_r I(U; Y_{r\to d_2}|\mathbf{Q}) + \epsilon_2 \tag{30}$$

(29) and (30) with $\epsilon \to 0$ imply that for any achievable (R_1, R_2), its derived vector $\left((R_1 - t_{s_1} p_{s_1;\{d_1,d_2\}})^+, (R_2 - t_{s_2} p_{s_2;d_2})^+\right)$ must lie in the convex hull of

$$\begin{aligned} \mathcal{C} \triangleq & \{(r_1, r_2) : r_1 \le t_r I(X_r; \mathbf{Y}_{r\to\{d_1,d_2\}}|U), \\ & \quad r_2 \le t_r I(U; Y_{r\to d_2}), \text{for all auxiliary RVs } U\}, \end{aligned} \tag{31}$$

where $U \mapsto X_r \mapsto \mathbf{Y}_{r\to\{d_1,d_2\}}$ form a Markov chain and the conditional distribution $P_{\mathbf{Y}_{r\to\{d_1,d_2\}}|X_r}$ follows the same erasure distribution as the PEC faced by r (conditioning on node r is scheduled). Equivalently, we can define the capacity function

$$\begin{aligned} & \forall \lambda > 0, \\ & \mathcal{C}(\lambda) = \max_{p(u)p(x_r|u)} \{t_r I(U; Y_{r\to d_2}) + \lambda t_r I(X_r; \mathbf{Y}_{r\to\{d_1,d_2\}}|U)\} \end{aligned} \tag{32}$$

We notice that (31) has the same form as the feedback-free erasure channel capacity with success probabilities $p_{r;\{d_1,d_2\}}$ and $p_{r;d_2}$ and overall time budget nt_r slots. We can thus have following lemmas

Lemma 1: If $\left((R_1 - t_{s_1} p_{s_1;\{d_1,d_2\}})^+, (R_2 - t_{s_2} p_{s_2;d_2})^+\right)$ is in $\mathcal{C}$, then

$$\frac{(R_1 - t_{s_1} p_{s_1;\{d_1,d_2\}})^+}{p_{r;\{d_1,d_2\}}} + \frac{(R_2 - t_{s_2} p_{s_2;d_2})^+}{p_{r;d_2}} \le t_r$$

Proof: Notice that $U \mapsto X_r \mapsto \mathbf{Y}_{r\to\{d_1,d_2\}} \mapsto Y_{r\to d_2}$ forms a Markov chain. Then follows from [11], for the distribution $p(u)p(x|u)$ which maximizes the capacity function $\mathcal{C}(\lambda)$ for some $\lambda > 0$, we can assume $\|\mathcal{U}\| = \min\{\|\mathcal{X}_r\|, \|\mathcal{Y}_{r\to d_1 1} \cup \mathcal{Y}_{r\to d_2 2}\|, \|\mathcal{Y}_{r\to d_2}\|\} = \|\mathcal{X}_r\|$. Let

$$E = \begin{cases} 0 & \text{if } u = x_r; \\ 1 & \text{if } u \neq x_r. \end{cases}$$

with $\mathsf{Prob}(E=1) = \beta$ and $\mathsf{Prob}(E=0) = 1-\beta$, then we have

$$\begin{aligned}(R_1 - t_{s_1} p_{s_1;\{d_1,d_2\}})^+ &\leq t_r p_{r;\{d_1,d_2\}} H(\beta) \\ (R_2 - t_{s_2} p_{s_2;d_2})^+ &\leq t_r p_{r;d_2}(1 - H(\beta))\end{aligned}$$

Hence

$$\frac{(R_1 - t_{s_1} p_{s_1;\{d_1,d_2\}})^+}{p_{r\ \{d_1,d_2\}}} + \frac{(R_2 - t_{s_2} p_{s_2;d_2})^+}{p_{r;d_2}} \leq t_r$$

■

Lemma 2: If $((R_1 - t_{s_1} p_{s_1;\{d_1,d_2\}})^+, (R_2 - t_{s_2} p_{s_2;d_2})^+)$ is in the convex hull of $\mathcal{C}$, then

$$\frac{(R_1 - t_{s_1} p_{s_1;\{d_1,d_2\}})^+}{p_{r;\{d_1,d_2\}}} + \frac{(R_2 - t_{s_2} p_{s_2;d_2})^+}{p_{r;d_2}} \leq t_r$$

Proof: Since $\left((R_1 - t_{s_1} p_{s_1;\{d_1,d_2\}})^+, (R_2 - t_{s_2} p_{s_2;d_2})^+\right)$ is in the convex hull of $\mathcal{C}$, there exists finite number of pairs $(r_{1,1}, r_{1,2})$, $(r_{2,1}, r_{2,2})$,..., $(r_{N,1}, r_{N,2})$ in $\mathcal{C}$. And N nonnegative real numbers c_1, c_2,..., c_N satisfying $\sum_{i=1}^N c_i = 1$ such that $\left((R_1 - t_{s_1} p_{s_1;\{d_1,d_2\}})^+, (R_2 - t_{s_2} p_{s_2;d_2})^+\right) = c_1(r_{1,1}, r_{1,2}) + c_2(r_{2,1}, r_{2,2}) + ...c_N(r_{N,1}, r_{N,2})$.

Follows from *Lemma 1* and the fact that $\sum_{i=1}^N c_i = 1$, we get

$$\frac{(R_1 - t_{s_1} p_{s_1;\{d_1,d_2\}})^+}{p_{r;\{d_1,d_2\}}} + \frac{(R_2 - t_{s_2} p_{s_2;d_2})^+}{p_{r;d_2}} \leq t_r$$

■

Finally, the following lemma gives us the capacity outer bound

Lemma 3: If $((R_1 - t_{s_1} p_{s_1;\{d_1,d_2\}})^+, (R_2 - t_{s_2} p_{s_2;d_2})^+)$ is in the closure of the convex hull of $\mathcal{C}$, then

$$\frac{(R_1 - t_{s_1} p_{s_1;\{d_1,d_2\}})^+}{p_{r;\{d_1,d_2\}}} + \frac{(R_2 - t_{s_2} p_{s_2;d_2})^+}{p_{r;d_2}} \leq t_r$$

Proof: Since $\left((R_1 - t_{s_1} p_{s_1;\{d_1,d_2\}})^+, (R_2 - t_{s_2} p_{s_2;d_2})^+\right)$ is in the closure of the convex hull of $\mathcal{C}$, there exists a sequence of pairs $(r_{i,1}, r_{i,2})$ in the convex hull of $\mathcal{C}$, $i = 1, 2, 3, ...$, such that $\left((R_1 - t_{s_1} p_{s_1;\{d_1,d_2\}})^+, (R_2 - t_{s_2} p_{s_2;d_2})^+\right) = \lim_{i\to\infty}(r_{i,1}, r_{i,2})$. From *Lemma 2*, for any $i = 1, 2, 3, ...$,

$$\frac{r_{i,1}}{p_{r;\{d_1,d_2\}}} + \frac{r_{i,2}}{p_{r;d_2}} \leq t_r$$

Taking the limit as $i \to \infty$ on both sides, we get

$$\frac{(R_1 - t_{s_1} p_{s_1;\{d_1,d_2\}})^+}{p_{r;\{d_1,d_2\}}} + \frac{(R_2 - t_{s_2} p_{s_2;d_2})^+}{p_{r;d_2}} \leq t_r$$

■

Lemma 3 gives us Eq. (8). Eq. (7) can then be obtained by symmetry. The proof is complete. ■

B. Capacity Inner Bound

Proposition 2: A rate vector (R_1, R_2) is achievable if there exist three non-negative scalars t_{s_1}, t_{s_2}, and t_r satisfying

$$t_{s_1} + t_{s_2} + t_r \leq 1 \tag{33}$$

$$\forall i \in \{1,2\}, \quad R_i < t_{s_i} p_{s_i;\{r,d_i\}} \tag{34}$$

$$\frac{(R_1 - t_{s_1} p_{s_1;d_1})^+}{p_{r;d_1}} + \frac{\left(\tilde{R}_2 - t_{s_2} p_{s_2;\{d_1,d_2\}}\right)^+}{p_{r;\{d_1,d_2\}}} < t_r \tag{35}$$

$$\frac{\left(\tilde{R}_1 - t_{s_1} p_{s_1;\{d_1,d_2\}}\right)^+}{p_{r;\{d_1,d_2\}}} + \frac{(R_2 - t_{s_2} p_{s_2;d_2})^+}{p_{r;d_2}} < t_r \tag{36}$$

where for any distinct $i \neq j$ in $\{1,2\}$

$$\tilde{R}_i \triangleq R_i \left(1 + \frac{p_{s_i\to d_j\overline{d_i r}}}{p_{s_i;\{d_1,d_2,r\}}} \cdot \frac{p_{s_i\to d_i\overline{d_j r}}}{p_{s_i;\{d_i,r\}}}\right). \tag{37}$$

By comparing Propositions 1 and 2, one can verify that the outer bound and the closure of the inner bound coincide when $R_i = \tilde{R}_i$ for $i = 1, 2$. For example, if we do not allow d_1 to directly hear s_1, then $p_{s_1\to d_1\overline{d_2 r}} = 0$ and thus $R_1 = \tilde{R}_1$. In practice, the relay r and the overhearing node d_2 are generally closer to s_1 than d_1. The probability $p_{s_1\to d_1\overline{d_2 r}}$ that a packet sent by s_1 is not heard by any of $\{d_2, r\}$ but by d_1 is thus small. Intuitively, the gap between the two bounds would be small as well, which is later verified in Section IV.

Sketch of the proof: An inner-bound-achieving scheme is described as follows. For any (R_1, R_2), let t_{s_1}, t_{s_2}, and t_r be the scalars that satisfy Proposition 2. For any two distinct $i \neq j$ in $\{1,2\}$, consider 5 different queues $Q_\emptyset^{[i]}$, $Q_{d_i}^{[i]}$, $Q_{\overline{r d_1 d_2}}^{[i]}$, $Q_{d_j\overline{d_i r}}^{[i]}$, and $Q_{d_j r\overline{d_i}}^{[i]}$. Each of these five queues is used to store non-zero nR_i-dimensional row vectors, and the corresponding operations are described as follows. Removing a vector $\mathbf{v}$ from a queue Q means $Q \leftarrow Q\backslash\mathbf{v}$. Adding a non-zero vector $\mathbf{v}$ to Q means $Q \leftarrow Q \cup \{\mathbf{v}\}$. Adding a zero vector $\mathbf{v} = \mathbf{0}$ to Q means Q remains unchanged. If Q is empty, then choosing $\mathbf{v} \in Q$ means setting $\mathbf{v} = \mathbf{0}$. By reversing the roles of i and j, we totally have 10 queues to consider.

Initialization: For any $i \neq j$, the queues $Q_{d_i}^{[i]}$, $Q_{\overline{r d_1 d_2}}^{[i]}$, $Q_{d_j\overline{d_i r}}^{[i]}$, and $Q_{d_j r\overline{d_i}}^{[i]}$ are initialized as empty sets. We let $Q_\emptyset^{[i]}$ store the nR_i elementary basis vectors for the (s_i, d_i) session. Namely, each of the nR_i vectors in $Q_\emptyset^{[i]}$ has exactly one coordinate being one and all other coordinates being zero. Our scheme consists of three major phases that are executed in sequence.

§ **Phase 1.1:** (s_1 sends uncoded packets.)

1: **while** $\left|Q_\emptyset^{[1]}\right| > 0$ **do**

2: In the beginning of time t, arbitrarily choose one $\mathbf{v} \in Q^{[1]}_{\emptyset}$. s_1 then sends a coded symbol $\mathbf{v}\mathbf{W}_1^{\mathrm{T}}$, where $\mathbf{W}_1^{\mathrm{T}}$ is the column vector of all s_1 information packets.
3: In the end of time t, let $\mathcal{R}_t$ denote the set of destinations successfully receiving $\mathbf{v}\mathbf{W}_1^{\mathrm{T}}$. We have two cases depending on $\mathcal{R}_t$: **Case 1:** $d_1 \in \mathcal{R}_t$: remove $\mathbf{v}$ from $Q^{[1]}_{\emptyset}$ and add it to $Q^{[1]}_{d_1}$. We use $Q^{[1]}_{\emptyset} \overset{\mathbf{v}}{\rightarrowtail} Q^{[1]}_{d_1}$ as shorthand; **Case 2:** $d_1 \notin \mathcal{R}_t$ and $\mathcal{R}_t \neq \emptyset$: do $Q^{[1]}_{\emptyset} \overset{\mathbf{v}}{\rightarrowtail} Q^{[1]}_{\mathcal{R}_t\overline{\{d_1,d_2,r\}\backslash\mathcal{R}_t}}$;
4: **end while**

§ **Phase 1.2:** (s_1 sends packets that benefit $\{r, d_1\}$ while promoting *overhearing at* d_2.)

1: **while** $\left|Q^{[1]}_{d_2\overline{d_1 r}}\right| > 0$ **do**
2: In the beginning of time t, arbitrarily choose one $\mathbf{v}_2 \in Q^{[1]}_{d_2\overline{d_1 r}}$ and one $\mathbf{v}_r \in Q^{[1]}_{r\overline{d_1 d_2}}$. Set $\mathbf{v} = c_2\mathbf{v}_2 + c_r\mathbf{v}_r$ where c_2 and c_r are chosen independently and uniformly randomly from $\mathsf{GF}(q)$. s_1 then sends $\mathbf{v}\mathbf{W}_1^{\mathrm{T}}$.
3: In the end of time t, there are five possible actions: **Action 1:** $Q^{[1]}_{r\overline{d_1 d_2}} \overset{\mathbf{v}_r}{\rightarrowtail} Q^{[1]}_{d_2 r\overline{d_1}}$; **Action 2:** $Q^{[1]}_{d_2\overline{d_1 r}} \overset{\mathbf{v}_2}{\rightarrowtail} Q^{[1]}_{d_2 r\overline{d_1}}$; **Action 3:** $Q^{[1]}_{d_2\overline{d_1 r}} \overset{\mathbf{v}_2}{\rightarrowtail} Q^{[1]}_{d_1}$; **Action 4:** Remove $\mathbf{v}_2$ from $Q^{[1]}_{d_2\overline{d_1 r}}$ and add $\mathbf{v}$ to $Q^{[1]}_{d_1}$. We use $Q^{[1]}_{d_2\overline{d_1 r}} \overset{\mathbf{v}_2}{\rightarrowtail}\overset{\mathbf{v}}{\hookrightarrow} Q^{[1]}_{d_1}$ as shorthand; **Action 5:** $Q^{[1]}_{r\overline{d_1 d_2}} \overset{\mathbf{v}_r}{\rightarrowtail}\overset{\mathbf{v}}{\hookrightarrow} Q^{[1]}_{d_1}$.
4: Let $\mathcal{R}_t$ denote the receiving set. We then have six cases depending on $\mathcal{R}_t$: **Case 1:** $\{d_1, d_2\} \subseteq \mathcal{R}_t$: do Actions 1 and 4; **Case 2:** $\mathcal{R}_t = \{d_1, r\}$: if $\mathbf{v}_r \neq \mathbf{0}$, do Actions 2 and 5, else do Action 3; **Case 3:** $\mathcal{R}_t = \{d_1\}$: do Action 4; **Case 4:** $\mathcal{R}_t = \{d_2, r\}$: do Actions 1 and 2; **Case 5:** $\mathcal{R}_t = \{d_2\}$: do Action 1; **Case 6:** $\mathcal{R}_t = \{r\}$, do Action 2.
5: **end while**

§ **Phase 1.3:** (s_1 maximizes overhearing by retransmitting packets heard by r but not by d_1.)

1: **while** s_1 has not used up its allocated time budget nt_{s_1} time slots **do**
2: In the beginning of time t, arbitrarily choose one $\mathbf{v}_r \in Q^{[1]}_{r\overline{d_1 d_2}}$ and one $\mathbf{v}_{d_2 r} \in Q^{[1]}_{d_2 r\overline{d_1}}$. If $\mathbf{v}_r \neq \mathbf{0}$, s_1 sends $\mathbf{v}_r\mathbf{W}_1^{\mathrm{T}}$; else s_1 sends $\mathbf{v}_{d_2 r}\mathbf{W}_1^{\mathrm{T}}$.
3: In the end of time t, there are two cases depending on $\mathcal{R}_t$: **Case 1:** $d_1 \in \mathcal{R}_t$: if $\mathbf{v}_r \neq \mathbf{0}$, do $Q^{[1]}_{r\overline{d_1 d_2}} \overset{\mathbf{v}_r}{\rightarrowtail} Q^{[1]}_{d_1}$, else do $Q^{[1]}_{d_2 r\overline{d_1}} \overset{\mathbf{v}_{d_2 r}}{\rightarrowtail} Q^{[1]}_{d_1}$; **Case 2:** $d_1 \notin \mathcal{R}_t$ but $d_2 \in \mathcal{R}_t$: do $Q^{[1]}_{r\overline{d_1 d_2}} \overset{\mathbf{v}_r}{\rightarrowtail} Q^{[1]}_{d_2 r\overline{d_1}}$.
4: **end while**

§ **Phases 2.1 to 2.3** are symmetric versions of Phases 1.1 to 1.3 with reversed roles of (s_1, d_1) and (s_2, d_2).

§ **Phase 3.1:** (r sends s_1-packets that are not heard by d_2.)

1: **while** $\left|Q^{[1]}_{r\overline{d_1 d_2}}\right| > 0$ **do**
2: In the beginning of time t, arbitrarily choose one $\mathbf{v} \in Q^{[1]}_{r\overline{d_1 d_2}}$ and r sends $\mathbf{v}\mathbf{W}_1^{\mathrm{T}}$.
3: In the end of time t, there are two cases depending on $\mathcal{R}_t$: **Case 1:** $d_1 \in \mathcal{R}_t$: do $Q^{[1]}_{r\overline{d_1 d_2}} \overset{\mathbf{v}}{\rightarrowtail} Q^{[1]}_{d_1}$; **Case 2:** $\mathcal{R}_t = \{d_2\}$: do $Q^{[1]}_{r\overline{d_1 d_2}} \overset{\mathbf{v}}{\rightarrowtail} Q^{[1]}_{d_2 r\overline{d_1}}$.
4: **end while**

§ **Phase 3.2** is symmetric to Phase 3.1 with reversed roles of d_1 and d_2.

§ **Phase 3.3:** (r mixes overheard s_1- and s_2-packets.)

1: **while** $\max\{\left|Q^{[1]}_{d_2 r\overline{d_1}}\right|, \left|Q^{[2]}_{d_1 r\overline{d_2}}\right|\} > 0$ **do**
2: In the beginning of time t, arbitrarily choose one $\mathbf{v}_1 \in Q^{[1]}_{d_2 r\overline{d_1}}$ and one $\mathbf{v}_2 \in Q^{[2]}_{d_1 r\overline{d_2}}$. r sends $\mathbf{v}_1\mathbf{W}_1^{\mathrm{T}} + \mathbf{v}_2\mathbf{W}_2^{\mathrm{T}}$.
3: In the end of time t, for $i = 1, 2$ and $j \neq i$, if $d_i \in \mathcal{R}_t$, then do $Q^{[i]}_{d_j r\overline{d_i}} \overset{\mathbf{v}_i}{\rightarrowtail} Q^{[i]}_{d_i}$.
4: **end while**

The intuition of the proposed scheme is as follows. Claim(i) For any $i \neq j$, queues $Q^{[i]}_{d_i}$, $Q^{[i]}_{r\overline{d_1 d_2}}$, $Q^{[i]}_{d_j\overline{d_i r}}$, $Q^{[i]}_{d_j r\overline{d_i}}$, and $Q^{[i]}_{\emptyset}$ store the s_i-packets (vectors) that are known to d_i, known to r but not to $\{d_1, d_2\}$, known to d_j but not to $\{d_i, r\}$, known to $\{d_j, r\}$ but not known to d_i, and known to no nodes, respectively. Since each vector removal from a queue is company by a vector addition to another queue, we have claim(ii) for any time t, the above 5 queues contain totally nR_i vectors. By the random linear network coding (RLNC) arguments as used in [9], we can prove that claim(iii) with sufficiently large $\mathsf{GF}(q)$, these nR_i vectors remain linearly independent and form the basis of the *information space* of s_i.

The main coding parts are in Phases 1.2, 2.2, and 3.3. For example, the goal of Phase 1.2 is two-fold. Firstly, s_1 would like to send packets that are beneficial to both $\{d_1, r\}$ so that if d_1 hears it, d_1 can obtain new information; or if r hears it, r can relay it to d_1 in the subsequent phase. One such candidate is those packets in $Q^{[1]}_{d_2\overline{d_1 r}}$, packets that have not been heard by any of $\{d_1, r\}$. Secondly, s_1 would also like to maximize $\left|Q^{[1]}_{d_2 r\overline{d_1}}\right|$ since this queue contains those s_1-packets at r that are overheard by d_2 and thus can later be mixed at r with those s_2-packets overheard by d_1 (as described in Line 2 of Phase 3.3). To promote overhearing, a good candidate is to send a packet in $Q^{[1]}_{r\overline{d_1 d_2}}$ so that if d_2 receives it, that packet is now heard by both $\{d_2, r\}$. We then use RLNC to achieve these two goals simultaneously and send a linear sum of the packets from $Q^{[1]}_{d_2\overline{d_1 r}}$ and $Q^{[1]}_{r\overline{d_1 d_2}}$.

In addition of (i) to (iii), proving the correctness of our construction also consists of proving the following statements for sufficiently large n: claim(iv) We can finish Phases i.1 and i.2 of s_i within the allocated nt_{s_i} time slots for $i = 1, 2$; claim(v) We can finish Phases 3.1, 3.2 and 3.3 of r within the allocated nt_r time slots; claim(vi) After Phase 3.3, we have $\left|Q^{[i]}_{d_i}\right| = nR_i$ for $i = 1, 2$. Then by (i) and (iii), all basis vectors in $Q^{[i]}_{d_i}$ are known to d_i. The information at s_i can thus be successfully decoded at d_i.

Proof of claim(i): For $i \in \{1, 2\}$, $j = 3 - i$, we can see that every vector in $Q^{[i]}_{d_i}$ is indeed received by d_i. In Phase i.1, we also notice that if the vector is not yet received by any of $\{d_1, d_2, r\}$, then it is in $Q^{[i]}_{\emptyset}$. For $Q^{[i]}_{r\overline{d_1,d_2}}$, in Phase i.1, if the vector is in $Q^{[i]}_{r\overline{d_1,d_2}}$, then it is indeed received by r only. Notice that in Phase i.2, every time we move out the vector, $\mathbf{v}_r$ from $Q^{[i]}_{r\overline{d_1,d_2}}$, it is indeed received by d_j or $\mathbf{v}$ is received by d_i and

r receives $\mathbf{v}_2$ (hence we do not need $\mathbf{v}_r$ anymore). Similar argument can be applied on Phases $i.3$ and $3.i$. Hence $Q^{[i]}_{r\overline{d_1,d_2}}$ contains the vectors which are received by r only. Similar argument can be applied on $Q^{[i]}_{d_j\overline{d_i r}}$. For $Q^{[i]}_{rd_j\overline{d_i}}$, follows from the actions in the scheme, we notice that every time we put one vector into $Q^{[i]}_{d_j r\overline{d_i}}$, both r and d_j knows the vector. ∎

Proof of claim(iv): Since we assume n is large, in the following discussion, we always implicitly use Large Number Law. For $i \in \{1,2\}$, $j = 3-i$, let $N_{i.1}$ and $N_{i.2}$ denotes the number of required time slots to finish Phase $i.1$ and Phase $i.2$, respectively. In Phase $i.1$, notice that if $\mathcal{R}_t \neq \emptyset$, then we move out one vector from $Q^{[i]}_{\emptyset}$. Since initially, there are nR_i vectors in $Q^{[i]}_{\emptyset}$, hence

$$N_{i.1} = \frac{nR_i}{p_{s_i;\{d_1,d_2,r\}}}$$

In Phase $i.2$, notice that in Case 1, 2, 3, 4, and 6, we move out one vector from $Q^{[i]}_{d_j\overline{d_i r}}$ and the probability that any of Case 1, 2, 3, 4, and 6 happens is $p_{s_i;\{d_i,r\}}$. Since after Phases $i.1$, there are $\frac{nR_i p_{s_i\to d_j\overline{d_i r}}}{p_{s_i;\{d_1,d_2,r\}}}$ vectors in $Q^{[i]}_{d_j\overline{d_i r}}$. Hence

$$N_{i.2} = \frac{nR_i}{p_{s_i;\{d_1,d_2,r\}}}\frac{p_{s_i\to d_j\overline{d_i r}}}{p_{s_i;\{d_i,r\}}}$$

It remains to show that $N_{i.1}+N_{i.2} \leq nt_{s_i}$

$$\begin{aligned} N_{i.1}+N_{i.2} =& \frac{nR_i}{p_{s_i;\{d_1,d_2,r\}}} + \frac{nR_i}{p_{s_i;\{d_1,d_2,r\}}}\frac{p_{s_i\to d_j\overline{d_i r}}}{p_{s_i;\{d_i,r\}}} && (38)\\ =& \frac{nR_i}{p_{s_i;\{d_i,r\}}} < nt_{s_i} && (39)\end{aligned}$$

where (39) follows from the fact that $p_{s_i;\{d_i,r\}} + p_{s_i\to d_j\overline{d_i r}} = p_{s_i;\{d_1,d_2,r\}}$ and (34). ∎

Proof of claim(v): Let $N_{3.1}$, $N_{3.2}$, and $N_{3.3}$ be the number of required time slots to finish Phase 3.1, 3.2, and 3.3, respectively. To show $N_{3.1}+N_{3.2}+N_{3.3} \leq nt_r$, we first need to quantify (i) $\left|Q^{[1]}_{r\overline{d_1 d_2}}\right|$ at the beginning of Phase 3.1, (ii) $\left|Q^{[2]}_{r\overline{d_1 d_2}}\right|$ at the beginning of Phase 3.2, and (iii) $\left|Q^{[2]}_{d_1 r\overline{d_2}}\right|$ at the beginning of Phase 3.3 and (iv) $\left|Q^{[2]}_{d_1 r\overline{d_2}}\right|$ at the beginning of Phase 3.3.

We first discuss the first quantity. Notice that after Phase 1.1,

$$\left|Q^{[1]}_{r\overline{d_1 d_2}}\right| = N_{1.1}p_{s_i\to r\overline{d_1 d_2}} = \frac{nR_1}{p_{s_1;\{d_1,d_2,r\}}}p_{s_i\to r\overline{d_1 d_2}} \quad (40)$$

Also notice that in Phase 1.2, if any of Case 1, 2, 4, and 5 happens, then we move out one vector from $Q^{[1]}_{r\overline{d_1 d_2}}$. The probability that any of Case 1, 2, 4, and 5 happens is $(p_{s_1;\{d_1,d_2\}} - p_{s_1\to d_1\overline{d_2 r}})$. Hence after Phase 1.2, we move out

$$\begin{aligned} &N_{1.2}(p_{s_1;\{d_1,d_2\}} - p_{s_1\to d_1\overline{d_2 r}})\\ &= \frac{nR_1}{p_{s_1;\{d_1,d_2,r\}}}\frac{p_{s_1\to d_2\overline{d_1 r}}}{p_{s_1;\{d_1,r\}}}(p_{s_1;\{d_1,d_2\}} - p_{s_1\to d_1\overline{d_2 r}}) && (41)\end{aligned}$$

The remaining time slot for Phase 1.3 is

$$nt_{s_1} - N_{1.1} - N_{1.2} = nt_{s_1} - \frac{nR_1}{p_{s_1;\{d_1,r\}}} \quad (42)$$

In Phase 1.3, if any of d_1 and d_2 in $\mathcal{R}_t$, then we move out one vector from $Q^{[1]}_{r\overline{d_1 d_2}}$. Hence after Phase 1.3, we move out

$$(nt_{s_1} - \frac{nR_1}{p_{s_1;\{d_1,r\}}})p_{s_1;\{d_1,d_2\}} \quad (43)$$

With (40), (41), and (43), the first quantity is

$$\begin{aligned} &(\frac{nR_1}{p_{s_1;\{d_1,d_2,r\}}}p_{s_i\to r\overline{d_1 d_2}}\\ &\quad - \frac{nR_1}{p_{s_1;\{d_1,d_2,r\}}}\frac{p_{s_1\to d_2\overline{d_1 r}}}{p_{s_1;\{d_1,r\}}}(p_{s_1;\{d_1,d_2\}} - p_{s_1\to d_1\overline{d_2 r}})\\ &\quad - (nt_{s_1} - \frac{nR_1}{p_{s_1;\{d_1,r\}}})p_{s_1;\{d_1,d_2\}})^+ && (44)\\ =&(nR_1 - nt_{s_1}p_{s_1;\{d_1,d_2\}}\\ &\quad + \frac{nR_1}{p_{s_1;\{d_1,d_2,r\}}}\cdot\frac{p_{s_1\to d_2\overline{d_1 r}}}{p_{s_1;\{d_1,r\}}}p_{s_1\to d_1\overline{d_2 r}})^+ && (45)\\ =&\left(\tilde{R}_1 - t_{s_1}p_{s_1;\{d_1,d_2\}}\right)^+ && (46)\end{aligned}$$

where (45) follows from the fact that $\frac{nR_1}{p_{s_1;\{d_1,r\}}} = \frac{nR_1}{p_{s_1;\{d_1,d_2,r\}}} + \frac{nR_1}{p_{s_1;\{d_1,d_2,r\}}}\cdot\frac{p_{s_1\to d_2\overline{d_1 r}}}{p_{s_1;\{d_1,r\}}}$. Since in Phase 3.1, if any of d_1 and d_2 in $\mathcal{R}_t$, then we move out one vector from $Q^{[1]}_{r\overline{d_1 d_2}}$. Hence

$$N_{3.1} = \frac{\left(\tilde{R}_1 - t_{s_1}p_{s_1;\{d_1,d_2\}}\right)^+}{p_{r;\{d_1,d_2\}}}$$

Similarly, we have the second quantity, $\left|Q^{[2]}_{r\overline{d_1 d_2}}\right|$ at the beginning of Phase 3.2, is

$$\begin{aligned} &(nR_2 - nt_{s_2}p_{s_2;\{d_1,d_2\}} + \frac{nR_2}{p_{s_2;\{d_1,d_2,r\}}}\frac{p_{s_2\to d_1\overline{d_2 r}}}{p_{s_2;\{d_2,r\}}}p_{s_2\to d_2\overline{d_1 r}})^+\\ &= \left(\tilde{R}_2 - t_{s_2}p_{s_2;\{d_1,d_2\}}\right)^+ && (47)\end{aligned}$$

Hence

$$N_{3.2} = \frac{\left(\tilde{R}_2 - t_{s_2}p_{s_2;\{d_1,d_2\}}\right)^+}{p_{r;\{d_1,d_2\}}} \quad (48)$$

We then discuss the third and forth quantity, $\left|Q^{[1]}_{d_2 r\overline{d_1}}\right|$ and $\left|Q^{[2]}_{d_1 r\overline{d_2}}\right|$ at the beginning of Phase 3.3. Notice that after Phase 3.1 and 3.2, $Q^{[1]}_{r\overline{d_1 d_2}}$ and $Q^{[2]}_{r\overline{d_1 d_2}}$ are empty sets. Hence by claim(ii), $\left|Q^{[1]}_{d_2 r\overline{d_1}}\right| = nR_1 - \left|Q^{[1]}_{d_1}\right|$ and $\left|Q^{[2]}_{d_1 r\overline{d_2}}\right| = nR_2 - \left|Q^{[2]}_{d_2}\right|$. Thus it is enough to quantify $Q^{[1]}_{d_1}$ and $Q^{[2]}_{d_2}$ at the beginning of Phases 3.3.

For $i \in \{1,2\}$ and $j = 3-i$, since during Phase i, we do not repeat sending the vectors which are already received by d_i, hence after Phase i, there are $nt_{s_i}p_{s_i;d_i}$ vectors in $Q^{[i]}_{s_i}$. In Phase $3.i$, d_i can further receive $N_{3.i}p_{r;d_i}$ vectors. Hence at the beginning of Phase 3.3,

$$\begin{aligned} \left|Q^{[i]}_{d_i}\right| &= \min\{nt_{s_i}p_{s_i;d_i} + N_{3.i}p_{r;d_i}, nR_i\}\\ \left|Q^{[i]}_{d_j r\overline{d_i}}\right| &= (nR_i - nt_{s_i}p_{s_i;d_i} - N_{3.i}p_{r;d_i})^+ && (49)\end{aligned}$$

TABLE II
SUM-RATE CAPACITY OF DIFFERENT NETWORK CODING SCHEMES.

OpR/Sch	OpR	Sch	Only NC
0.516/0.523	0.475/0.483	0.425	0.364
OpR/Sch/CSF	OpR/CSF	Sch/CSF	CSF
0.530/0.530	0.490/0.490	0.433	0.373

Notice that in Phase 3.3, if $d_i \in \mathcal{R}_t$, then do $Q^{[i]}_{d_j r \overline{d_i}} \overset{\mathbf{v}_i}{\rightarrowtail} Q^{[i]}_{d_i}$. Hence we have

$$N_{3.3} = \max\{\frac{\left|Q^{[1]}_{d_2 r \overline{d_1}}\right|}{p_{r;d_1}}, \frac{\left|Q^{[2]}_{d_1 r \overline{d_2}}\right|}{p_{r;d_2}}\} \tag{50}$$

Now we have the required 4 quantities and $N_{3.1}$, $N_{3.2}$, and $N_{3.3}$. It remains to show that $N_{3.1} + N_{3.2} + N_{3.3} \leq nt_r$. Suppose $\frac{\left|Q^{[1]}_{d_2 r \overline{d_1}}\right|}{p_{r;d_1}} \geq \frac{\left|Q^{[2]}_{d_1 r \overline{d_2}}\right|}{p_{r;d_2}}$, we have

$$\begin{aligned} & N_{3.1} + N_{3.2} + N_{3.3} \\ =& N_{3.1} + N_{3.2} + \frac{(nR_1 - nt_{s_1} p_{s_1;d_1} - N_{3.1} p_{r;d_1})^+}{p_{r;d_1}} \\ =& \frac{(nR_1 - nt_{s_1} p_{s_1;d_1})^+}{p_{r;d_1}} + \frac{\left(\tilde{R}_2 - t_{s_2} p_{s_2;\{d_1,d_2\}}\right)^+}{p_{r;\{d_1,d_2\}}} \\ <& nt_r \end{aligned} \tag{51}$$

Similarly, suppose $\frac{\left|Q^{[1]}_{d_2 r \overline{d_1}}\right|}{p_{r;d_1}} < \frac{\left|Q^{[2]}_{d_1 r \overline{d_2}}\right|}{p_{r;d_2}}$, we have

$$\begin{aligned} & N_{3.1} + N_{3.2} + N_{3.3} \\ =& N_{3.1} + N_{3.2} + \frac{(nR_2 - nt_{s_2} p_{s_2;d_2} - N_{3.2} p_{r;d_2})^+}{p_{r;d_2}} \\ =& \frac{(nR_2 - nt_{s_2} p_{s_2;d_2})^+}{p_{r;d_2}} + \frac{\left(\tilde{R}_1 - t_{s_1} p_{s_1;\{d_1,d_2\}}\right)^+}{p_{r;\{d_1,d_2\}}} \\ <& nt_r \end{aligned} \tag{52}$$

■

Proof of claim(vi): Notice that after Phase 3.3, all the queues except $Q^{[1]}_{d_1}$ and $Q^{[2]}_{d_2}$ are empty sets. Follows from claim(ii), there are nR_1 vectors in $Q^{[1]}_{d_1}$ and nR_2 vectors in $Q^{[2]}_{d_2}$. ■

IV. NUMERICAL RESULTS

In the numerical experiments, we first place the relay r in the center of a unit disk. Then we place the source s_i and destination d_i uniformly randomly in the disk. To model a practical scenario, we impose a condition that d_i must be in the 90-degree pie area opposite of s_i. See Fig. 1(b) for illustration. Repeat the node placement for $i = 1, 2$. See Fig. 1(c) for one realization of our random node placement.

After node placement, we use the distance D between each pair of nodes and the Rayleigh model to decide the packet overhearing probability:

$$\mathsf{Prob(success)} = \int_{T^*}^{\infty} \frac{2x}{\gamma} e^{-\frac{x^2}{\gamma}} dx \quad \text{where } \gamma \triangleq \frac{1}{(4\pi)^2 D^\alpha},$$

α is the path loss factor, and T^* is the decodable SNR threshold. To fit the experience in practice, we choose $\alpha = 2.5$ and $T^* = 0.006$ such that the probability that a 1-hop neighbor (resp. a 2-hop neighbor) hears a transmission is around 0.7–0.8 (resp. 0.2–0.3). We assume the success events between different node pairs are independent.

Once the success probabilities are determined, we can maximize $(R_1 + R_2)$ subject to the outer and inner bounds in Propositions 1 and 2. To enforce fairness, we also impose an additional constraint $R_i = \beta \min\left(p_{s_i;\{d_i,r\}}, p_{s_i;d_i} + p_{r;d_i}\right)$ for $i = 1, 2$ with a common β, which requires R_i being proportional to the min-cut value from s_i to d_i assuming no other sessions are transmitting and s_i and r are scheduled with the same frequency. We repeat the above process for 1000 times and then take the average of the maximized sum-rate.

Our capacity characterization can be easily modified for schemes with different capabilities. More explicitly, we use "OpR" to stand for opportunistic routing. "CSF" stands for instant channel status feedback. "Sch" allows dynamically scheduling different nodes, in contrast with a fixed schedule for which each node is scheduled with frequency 1/3. Table II summarizes the sum-rate capacity for different NC schemes. For schemes equipped with OpR, we list both the inner and outer bounds. For schemes without OpR, our results describe the full capacity. As can be seen, the largest throughput gain follows from allowing OpR. Note that CSF strictly increases the capacity for all cases. Moreover, with CSF the gap between the inner and outer bounds is much smaller (less than 10^{-3}) when OpR is used. When compared to the "NC only" scheme, the joint use of OpR, CSF, and Sch enhances capacity by 45%. This thus illustrates the importance of jointly designing OpR, CSF, and Sch together with the NC solution.

V. CONCLUSION AND FUTURE WORK

This paper has provided a pair of inner and outer bounds for the capacity of 2-user 1-hop relay erasure network that allows opportunistic routing, instant channel status feedback, and per-slot scheduling. The conditions for which the outer and inner bounds coincide have been identified. In our numerical experiments, the gap between the outer and inner bounds is within 0.2%. In the future, we will generalize the results for K-user 1-hop relay erasure networks with arbitrary K values.

This work was supported in parts by NSF grants CCF-0845968 and CNS-0905331.

REFERENCES

[1] N. Anantharam and S. Verdu, "Bits through queues," *IEEE Trans. Inf. Theory*, vol. 42, no. 1, pp. 4–18, January 1996.

[2] A. El Gamal, "The feedback capacity of degraded broadcast channels," *IEEE Trans. Inf. Theory*, vol. 25, no. 2, pp. 379–381, March 1978.

[3] L. Georgiadis and L. Tassiulas, "Broadcast erasure channel with feedback — capacity and algorithms," in *Proc. 5th Workshop on Network Coding, Theory, & Applications (NetCod)*. Lausanne, Switzerland, June 2009, pp. 54–61.

[4] S. Katti, H. Rahul, W. Hu, D. Katabi, M. Médard, and J. Crowcroft, "XORs in the air: Practical wireless network," in *Proc. ACM Special Interest Group on Data Commun. (SIGCOMM)*, 2006.

[5] W.-C. Kuo and C.-C. Wang, "On the capacity of 2-user 1-hop relay erasure networks — the union of feedback, scheduling, opportunistic routing, and network coding," *Technical Report*, http://cobweb.ecn.purdue.edu/∼chihw/publications.htm.

[6] X. Lin, N. Shroff, and R. Srikant, "A tutorial on cross-layer optimization in wireless networks," *IEEE J. Sel. Areas Commun.*, vol. 24, no. 8, pp. 1452–1463, August 2006.

[7] T. Lutz, G. Kramer, and C. Hausl, "Capacity for half-duplex line networks with two sources," in *Proc. IEEE Int'l Symp. Inform. Theory*. Austin, Texas, USA, June 2010.

[8] L. Ozarow and S. Leung-Yan-Cheong, "An achievable region and outer bound for the Gaussian broadcast channel with feedback," *IEEE Trans. Inf. Theory*, vol. 30, no. 4, July 1984.

[9] C.-C. Wang, "Capacity of 1-to-K broadcast packet erasure channels with channel output feedback," in *Proc. 48th Annual Allerton Conf. on Comm., Contr., and Computing*. Monticello, Illinois, USA, September 2010.

[10] ——, "On the capacity of wireless 1-hop intersession network coding — a broadcast packet erasure channel approach," in *Proc. IEEE Int'l Symp. Inform. Theory*. Austin, TX, USA, June 2010.

[11] R. G. Gallager, "Capacity and coding for degraded broadcast channels," *Problemy Peredaci Informatsil*, vol. 10, no. 3, pp. 3-14, 1974.