

# On the characterization of strict positive realness for general matrix transfer functions\*

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## Abstract

We present conditions which are necessary and sufficient for a transfer function (or transfer function matrix) to be strictly positive real. A counter example is given to illustrate that, in the rational matrix case, the conditions presented here differ from those previously presented in the literature, and that these same conditions represent an incomplete characterization of strict positive realness. The proof of our results differs from previous related proofs in that it only uses properties of analytic functions and matrices and does not require state-space realizations. Also the results are not restricted to rational transfer functions with real coefficients.

## 1 Introduction

The concept of Strict Positive Realness (SPR) of a transfer function matrix appears frequently in various aspects of engineering. Application oriented areas such as optimal control, adaptive control, VLSI design and in particular, stability theory, have all benefited greatly from the concept of SPR [7, 6, 8, 12, 2, 11, 6, 1]. It is therefore vitally important to characterize this property via conditions which can readily be computed or verified experimentally. While such conditions have been readily available in the literature [3, 13, 9, 10, 5] for some time, our main purpose here in this note is to demonstrate by means of an elementary counter-example that these conditions are in fact incomplete in the general matrix case. We then present an alternative characterization of a strictly positive real transfer function matrix that takes care of the problems highlighted by the counter-example.

**Definition 1 (SPR)** *A transfer function  $G : \mathbb{C} \rightarrow \mathbb{C}^{m \times m}$  is strictly positive real (SPR) if there exists a real scalar  $\epsilon > 0$  such that  $G$  is analytic in a region for which  $\Re(s) \geq -\epsilon$  and*

$$G(j\omega - \epsilon) + G(j\omega - \epsilon)^* \geq 0 \quad \text{for all } \omega \in \mathbb{R}. \quad (1)$$

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We say that  $G$  is **regular** if  $\det[G(j\omega) + G(j\omega)^*]$  is not identically zero for all  $\omega \in \mathbb{R}$ .

The first appearance of the above definition seems to be [11, 6] in the scalar case. The definition was motivated by a desire to obtain conditions on a rational transfer function which satisfied the requirements of the Kalman-Yacubovic-Popov Lemma. Reference [11] also provides an electrical network interpretation of SPR. Although the literature contains other definitions of SPR, the above definition is consistent with the requirements of the KYP lemma for rational transfer functions. Assuming  $G$  is stable, rational and proper, it was known that the dissipativity condition,  $G(j\omega) + G(j\omega)^* > 0$  for all finite  $\omega$ , was necessary but not sufficient for SPR. Requiring in addition that  $G(\infty) + G(\infty)^* > 0$  yields sufficiency but not necessity. Thus, starting with [11], a search was on for a side condition which in addition to the dissipativity condition yielded an equivalent characterization of SPR; some of this research is summarized in Section 3. Such a side condition eliminates the need for  $\epsilon$  in the characterization of SPR and is important for two main reasons: The dissipativity condition is something that can be verified experimentally by looking at the frequency response of a system and the  $\epsilon$ -free conditions are more computationally tractable. We present here a new side condition which along with stability and the dissipativity condition yields  $\epsilon$ -free conditions which are necessary and sufficient for a transfer function (or transfer function matrix) to be strictly positive real. This new side condition can be simply stated as

$$\boxed{\lim_{|\omega| \rightarrow \infty} \omega^{2\rho} \det[G(j\omega) + G(j\omega)^*] \neq 0} \quad (2)$$

where  $\rho$  is the nullity of  $G(\infty) + G(\infty)^*$ . As we shall illustrate by means of an elementary counter-example, when  $0 < \rho < m$ , the new side condition presented here is not equivalent to those previously presented in the literature. In those cases, the example presented highlights that the conditions given previously are only necessary but not sufficient to characterize an SPR transfer function matrix.

## 2 Main results

We assume throughout  $G$  is analytic at infinity, that is,  $G$  has a power series expansion in  $1/s$  (see (5)), and we let

$$D = G(\infty) := \lim_{|\omega| \rightarrow \infty} G(j\omega). \quad (3)$$

Examples of such functions include all proper rational functions; as another example consider  $G(s) = g_1(s) + g_2(s)e^{g_3(s)}$  where  $g_1, g_2$  and  $g_3$  are scalar, proper rational functions. Our main result (Lemma 1 below) contains a new side condition that involves  $\rho$ , the nullity of  $D + D^*$ , namely the dimension of the null space of  $D + D^*$ . This is the same as  $m - m_1$  where  $m_1$  is the rank of  $D + D^*$ . It is also the same as the number of zero eigenvalues of  $D + D^*$ .

**Lemma 1** *A transfer function  $G$  is SPR and regular if and only if the following conditions hold.*

- (a) [Stability] *There exists  $\beta > 0$  such that  $G$  is analytic in  $\{s \in \mathbb{C} : \Re(s) > -\beta\}$ .*

(b) [Dissipativity]

$$G(j\omega) + G(j\omega)^* > 0 \quad \text{for all } \omega \in \mathbb{R} \quad (4)$$

(c) [Asymptotic side condition]

$$\lim_{|\omega| \rightarrow \infty} \omega^{2\rho} \det[G(j\omega) + G(j\omega)^*] \neq 0$$

where  $\rho$  is the nullity of  $G(\infty) + G(\infty)^*$ .

In either case, the above limit is positive.

A proof of this lemma is given in Section 4.

## 2.1 Another characterization of the side condition

Here we provide another characterization of side condition (2). This is sometimes useful for computational purposes. Also, when  $G$  has a finite-dimensional state space realization, this condition can readily be checked using state space data. To this end, recall that  $G$  is analytic at infinity. Specifically we require that, for some  $\epsilon > 0$ , the function  $G$  has the following power series expansion

$$G(s) = D + \frac{1}{s}G_1 + \frac{1}{s^2}G_2 + \dots \quad (5)$$

for  $|s|$  large and  $\Re(s) \geq -\epsilon$ . Here  $D, G_1, G_2, \dots$  are constant  $m \times m$  matrices. As before, let  $\rho$  be the nullity of the matrix  $D + D^*$ , that is,  $\rho$  is the dimension of the null space of  $D + D^*$ . We distinguish between three cases:

$$\begin{aligned} \rho = 0 : & \quad \det(D + D^*) \neq 0 \\ \rho = m : & \quad D + D^* = 0 \\ 0 < \rho < m : & \quad \det(D + D^*) = 0 \text{ and } D + D^* \neq 0 \end{aligned}$$

When  $0 < \rho < m$ , we let  $U$  and  $V$  be any matrices of sizes  $m \times (m - \rho)$  and  $m \times \rho$ , respectively, where the columns of  $U$  form a basis for the range of  $D + D^*$  and the columns of  $V$  form a basis for the null space of  $D + D^*$ . Note that these matrices can be reliably and efficiently obtained from a singular value decomposition of  $D + D^*$ . Now let

$$E := \begin{cases} D & \text{if } \rho = 0 \\ -G_2 & \text{if } \rho = m \\ \begin{pmatrix} U^* D U & U^* G_1 V \\ -V^* G_1 U & -V^* G_2 V \end{pmatrix} & \text{if } 0 < \rho < m \end{cases} \quad (6)$$

It follows from Corollary 1 (stated later) that side condition (2) is equivalent to

$$\boxed{\det(E + E^*) \neq 0} \quad (7)$$

Also, under the strict dissipativity hypothesis (4), side condition (2) is also equivalent to  $E + E^* > 0$ .

Although the choice of  $U$  and  $V$  are not unique, the above results are independent of the choice of  $U$  and  $V$ .

**Remark 1** *It follows from the above discussion that Condition (7) provides another characterization of side condition (2). This is sometimes useful for computational purposes, especially when  $G$  is rational and one has a state space realization of  $G$ ; see next section.*

## 2.2 Proper rational transfer functions

Consider a proper rational transfer function with state-space realization  $(A, B, C, D)$ , that is,

$$G(s) = C(sI - A)^{-1}B + D \quad (8)$$

where  $A \in \mathbb{C}^{n \times n}$ ,  $B \in \mathbb{C}^{n \times m}$ ,  $C \in \mathbb{C}^{m \times n}$  and  $D \in \mathbb{C}^{m \times m}$ . If all the eigenvalues of  $A$  have negative real part then, there is a  $\beta > 0$  such that  $G$  is analytic in  $\{s \in \mathbb{C} : \Re(s) > -\beta\}$ . Also, for large  $s$ ,

$$G(s) = D + \frac{1}{s}CB + \frac{1}{s^2}CAB + \dots$$

Hence  $G_1 = CB$ ,  $G_2 = CAB$  and

$$E = \begin{cases} D & \text{if } \rho = 0 \\ -CAB & \text{if } \rho = m \\ \begin{pmatrix} U^*DU & U^*CBV \\ -V^*CBU & -V^*CABV \end{pmatrix} & \text{if } 0 < \rho < m \end{cases} \quad (9)$$

**Remark 2** *Recalling Remark 1, we see that satisfaction of the side condition (2) can readily be ascertained from any state space realization  $(A, B, C, D)$  of  $G$ .*

## 3 Previous conditions in the literature

For rational transfer functions corresponding to real  $A, B, C, D$  matrices, references [3, 13, 9, 10, 5] present conditions which they claim are necessary and sufficient for SPR. Reference [3] considers scalar transfer functions whereas [13] considers the special cases of  $D = 0$  or  $D > 0$  in the matrix case. For these special cases, their conditions are basically the same as those here. References [9, 10] consider the general rational matrix case. Their conditions are the same as here except in the case when  $0 < \rho < m$  where  $\rho$  is the nullity of  $D + D^*$ . In this case, instead of our side condition (2), they have the following condition:

$$\lim_{|\omega| \rightarrow \infty} \omega^2 [G(j\omega) + G(j\omega)^*] > 0 \quad \text{and} \quad D + D^* \geq 0$$

However, for the above limit to exist one must have  $D + D^* = 0$ , that is, when  $\rho = m$ . So, the limit does not exist when  $\rho < m$ . Hence the results in these references do not apply when  $0 < \rho < m$ . Reference [5] also considers the general rational matrix case. In Lemma 6.1 it is claimed that a regular  $G$  is SPR if and only if hypotheses (a) and (b) of Lemma 1 hold and the following side condition holds. Either  $G(\infty) + G(\infty)^T$  is positive definite or it is positive semi-definite and

$$\lim_{|\omega| \rightarrow \infty} \omega^2 V^T [G(j\omega) + G(j\omega)^*] V > 0$$

where  $V$  is any  $m \times \rho$  full rank matrix for which  $V^T [G(\infty) + G(\infty)^T] V = 0$  and  $\rho$  is the nullity of  $G(\infty) + G(\infty)^T$ . When  $\rho = 0$  or  $\rho = m$  or  $U^*(G_1 - G_1^*)V = 0$ , the above condition is the same as side condition (28) which in turn is equivalent to our main

side condition (2). When  $0 < \rho < m$  and  $U^*(G_1 - G_1^*)V \neq 0$ , the condition is less restrictive than (2). So, according to Lemma 1, this condition is necessary for SPR. However, in general it is not sufficient as the example below illustrates. For rational transfer functions with real coefficients where  $D + D^*$  has a specific structure, the early Russian literature (Yacubovich [4]) contains a condition similar to our side condition. Although [4] does not discuss SPR functions, the condition presented there is used in a KYP lemma presented in that paper. The conditions presented here can be viewed as a generalization of the conditions in [4] to general transfer function matrices.

**Example 1** We will show that the transfer function

$$G(s) = \begin{pmatrix} 1 & \frac{1}{s+1} \\ -\frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

satisfies the SPR requirements of Lemma 6.1 of [5] but is not SPR. The matrix

$$G(j\omega - \epsilon) + G(j\omega - \epsilon)^* = 2 \begin{pmatrix} 1 & \frac{-j\omega}{(1-\epsilon)^2 + \omega^2} \\ \frac{j\omega}{(1-\epsilon)^2 + \omega^2} & \frac{1-\epsilon}{(1-\epsilon)^2 + \omega^2} \end{pmatrix}$$

has determinant

$$\delta(\omega, \epsilon) = 4 \frac{(1-\epsilon)^3 - \epsilon\omega^2}{[(1-\epsilon)^2 + \omega^2]^2}. \quad (10)$$

Consider any  $\epsilon > 0$ ; it can be seen that  $\delta(\omega, \epsilon) < 0$  for large  $\omega$  and, hence the matrix  $G(j\omega - \epsilon) + G(j\omega - \epsilon)^*$  is not positive semi-definite. Since this is true for any  $\epsilon > 0$ , we conclude that  $G$  is not SPR.

Hypotheses (a) and (b) of Lemma 1 hold. Since

$$G(\infty) + G(\infty)^* = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad (11)$$

the nullity  $\rho$  of  $G(\infty) + G(\infty)^*$  is one and we can let

$$U = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and obtain that

$$\lim_{|\omega| \rightarrow \infty} \omega^2 V^* [G(j\omega) + G(j\omega)^*] V = \lim_{|\omega| \rightarrow \infty} \frac{2\omega^2}{\omega^2 + 1} = 2 > 0.$$

So, the requirements of [5] hold but  $G$  is not SPR.

Note that

$$G_1 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

which results in

$$U^*(G_1 - G_1^*)V = 2 \neq 0.$$

This is the situation in which the side condition of [5] is only necessary but not sufficient.

Since  $\rho = 1$  and

$$\det[G(j\omega) + G(j\omega)^*] = \frac{4}{(1 + \omega^2)^2}$$

we see that

$$\lim_{|\omega| \rightarrow \infty} \omega^{2\rho} \det[G(j\omega) + G(j\omega)^*] = \lim_{|\omega| \rightarrow \infty} \frac{4\omega^2}{(1 + \omega^2)^2} = 0.$$

Thus, hypothesis (c) of Lemma 1 is not satisfied and this lemma correctly predicts that  $G$  is not SPR.

**The example and the KYP Lemma.** It is of interest to see where the above example does not satisfy the requirements of the KYP Lemma in [5]. Suppose that a real quadruple  $(A, B, C, D)$  is a state space realization of a proper rational transfer function  $G$  with  $A$  Hurwitz,  $(A, B)$  controllable and  $(C, A)$  observable. The KYP Lemma (Lemma 6.3 in [5]) states that  $G$  is SPR if and only if there is a matrix  $P = P^T > 0$ , a scalar  $\epsilon > 0$  and matrices  $W$  and  $L$  such that

$$\begin{aligned} PA + A^T P &= -L^T L - \epsilon P \\ PB &= C^T - L^T W \\ W^T W &= D + D^T \end{aligned}$$

Rewriting these equations as

$$\begin{pmatrix} PA + A^T P + \epsilon P & PB - C^T \\ B^T P - C & -D - D^T \end{pmatrix} = - \begin{pmatrix} L & W \end{pmatrix}^T \begin{pmatrix} L & W \end{pmatrix},$$

one may readily see that the above KYP conditions are equivalent to the existence of a matrix  $P = P^T > 0$  and a scalar  $\epsilon > 0$  such that

$$\begin{pmatrix} PA + A^T P + \epsilon P & PB - C^T \\ B^T P - C & -D - D^T \end{pmatrix} \leq 0 \tag{12}$$

Consider now the following state space realization of the transfer function in the above example.

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

This realization is controllable and observable while  $A$  is Hurwitz. One can show that, for any  $\epsilon > 0$ , there is no matrix  $P$  satisfying inequality (12). However, for this example, one can obtain a unique  $P = P^T > 0$  satisfying (12) with  $\epsilon = 0$ .

**A note on the proof of Lemma 6.1 in [5].** In the proof of Lemma 6.1 in [5] it is claimed that when  $G(j\omega) + G(j\omega)^* > 0$  for all  $\omega$  and the side condition in [5] holds with  $D + D^*$  singular, one has

$$\lim_{|\omega| \rightarrow \infty} \sigma_1(\omega) > 0 \quad \text{where} \quad \sigma_1(\omega) := \sigma_{\max}[G(j\omega) + G(j\omega)^*].$$

and

$$\lim_{|\omega| \rightarrow \infty} \omega^2 \sigma_2(\omega) > 0 \quad \text{where} \quad \sigma_2(\omega) := \sigma_{\min}[G(j\omega) + G(j\omega)^*].$$

In the example above, the last inequality does not hold. In this example

$$\sigma_1(\omega)\sigma_2(\omega) = \delta(\omega, 0) = \frac{4}{(1 + \omega^2)^2}$$

It follows from (11) that

$$\lim_{|\omega| \rightarrow \infty} \sigma_1(\omega) = 2.$$

Hence

$$\lim_{|\omega| \rightarrow \infty} \omega^2 \sigma_2(\omega) = \lim_{|\omega| \rightarrow \infty} \frac{\omega^2 \delta(\omega, 0)}{\sigma_1(\omega)} = \lim_{|\omega| \rightarrow \infty} \frac{2\omega^2}{(1 + \omega^2)^2} = 0.$$

## 4 Proof of main result

Before proving the main result, we need some preliminary results.

### 4.1 A property of positive real transfer functions

Recall that a transfer function  $G$  is positive real (PR) if  $G$  is analytic in the set  $\{s \in \mathbb{C} : \Re(s) \geq 0\}$  and

$$G(j\omega) + G(j\omega)^* \geq 0 \quad \text{for all} \quad \omega \in \mathbb{R}. \quad (13)$$

**Lemma 2** *Suppose  $g$  is a positive real scalar transfer function with a power series expansion of the form*

$$g(s) = \frac{g_1}{s} + \frac{g_2}{s^2} + \dots \quad (14)$$

*Then  $g_1$  is real and non-negative; moreover,  $g_1 = 0$  implies  $g(s)$  is zero for all  $s$ .*

The Appendix contains a proof.

**Lemma 3** *Suppose  $G$  is a positive real transfer function with a power series expansion of the form (5). If  $V$  is any matrix for which  $(D + D^*)V = 0$  then,  $V^*G_1V$  is hermitian and positive semi-definite, that is*

$$V^*G_1^*V = V^*G_1V \geq 0. \quad (15)$$

*If in addition  $V$  is full column rank and  $G$  is regular then,  $V^*G_1V > 0$ .*

The Appendix contains a proof.

## 4.2 A preliminary lemma

Here we present a lemma which is fundamental to obtaining the main result of the paper. In this result,  $\epsilon$  is any real scalar and the function  $L_\epsilon$  of a complex variable  $s$  is defined by

$$L_\epsilon(s) := \begin{cases} G(s-\epsilon) & \text{if } \rho = 0 \\ -s^2 G(s-\epsilon) & \text{if } \rho = m \\ \begin{pmatrix} U^* G(s-\epsilon) U & s U^* G(s-\epsilon) V \\ -s V^* G(s-\epsilon) U & -s^2 V^* G(s-\epsilon) V \end{pmatrix} & \text{if } 0 < \rho < m \end{cases} \quad (16)$$

where  $\rho$  is the nullity of  $D + D^*$  and  $U, V$  are as previously defined. We also have the following definitions.

$$E_\epsilon := \begin{cases} D & \text{if } \rho = 0 \\ -G_2 - \epsilon G_1 & \text{if } \rho = m \\ \begin{pmatrix} U^* D U & U^* G_1 V \\ -V^* G_1 U & -V^* G_2 V - \epsilon V^* G_1 V \end{pmatrix} & \text{if } 0 < \rho < m \end{cases} \quad (17)$$

and

$$\tilde{G}_1 := \begin{cases} 0 & \text{if } \rho = 0 \\ G_1 & \text{if } \rho = m \\ V^* G_1 V & \text{if } 0 < \rho < m \end{cases} \quad (18)$$

**Lemma 4** (i) For all nonzero  $\omega \in \mathbb{R}$ ,

$$G(j\omega - \epsilon) + G(j\omega - \epsilon)^* \geq 0$$

if and only if

$$L_\epsilon(j\omega) + L_\epsilon(j\omega)^* \geq 0.$$

This result also holds for strict inequalities.

(ii) If  $\tilde{G}_1$  is hermitian and  $\{(\omega_k, \epsilon_k)\}_{k=1}^\infty$  is any sequence with  $\lim_{k \rightarrow \infty} \epsilon_k = \epsilon$  and  $\lim_{k \rightarrow \infty} |\omega_k| = \infty$  then,

$$\lim_{k \rightarrow \infty} [L_{\epsilon_k}(j\omega_k) + L_{\epsilon_k}(j\omega_k)^*] = E_\epsilon + E_\epsilon^*. \quad (19)$$

(iii)

$$\det[L(s) + L(-\bar{s})^*] = c(-s^2)^\rho \det[G(s) + G(-\bar{s})^*] \quad (20)$$

where  $c > 0$ .

The Appendix contains a proof.

Let  $L(s) = L_0(s)$  and note that  $E = E_0$ . It follows from Lemma 4 above that, for all nonzero  $\omega \in \mathbb{R}$ ,

$$G(j\omega) + G^*(j\omega) > 0 \iff L(j\omega) + L(j\omega)^* > 0 \quad (21)$$

$$\det[L(j\omega) + L(j\omega)^*] = c\omega^{2\rho} \det[G(j\omega) + G(j\omega)^*] \quad (22)$$

for some constant  $c > 0$ , and

$$\lim_{|\omega| \rightarrow \infty} [L(j\omega) + L(j\omega)^*] = E + E^*. \quad (23)$$

The above observations lead us to the following result.

**Corollary 1** *The following conditions are equivalent.*

$$\lim_{|\omega| \rightarrow \infty} \omega^{2\rho} \det[G(j\omega) + G(j\omega)^*] \neq 0 \quad (24)$$

$$\lim_{|\omega| \rightarrow \infty} \det[L(j\omega) + L(j\omega)^*] \neq 0 \quad (25)$$

$$\det[E + E^*] \neq 0 \quad (26)$$

If  $G(j\omega) + G(j\omega)^* > 0$  for all  $\omega \in \mathbb{R}$  then the above conditions are equivalent to the following conditions.

$$\lim_{|\omega| \rightarrow \infty} \omega^{2\rho} \det[G(j\omega) + G(j\omega)^*] > 0 \quad (27)$$

$$\lim_{|\omega| \rightarrow \infty} [L(j\omega) + L(j\omega)^*] > 0 \quad (28)$$

$$E + E^* > 0 \quad (29)$$

### 4.3 Proof of Lemma 1

*Necessity.* Suppose  $G$  is SPR and regular. Then there exists  $\epsilon > 0$  such that  $G$  is analytic in a region for which  $\Re(s) \geq -\epsilon$  and

$$G(j\omega - \epsilon) + G(j\omega - \epsilon)^* \geq 0 \quad \text{for all } \omega \in \mathbb{R}. \quad (30)$$

Using properties of analytic functions and the fact that  $G$  is regular, one can show that  $G$  is analytic in the region  $\{s \in \mathbb{C} : \Re(s) > -\epsilon\}$  and  $G(s) + G(s)^* > 0$  for all  $s$  in this region; in particular, we must have

$$G(j\omega) + G(j\omega)^* > 0 \quad (31)$$

for all  $\omega \in \mathbb{R}$ .

To complete the proof of necessity, we will show that

$$E + E^* > 0 \quad (32)$$

which, using Corollary 1, implies the desired result (2). It follows from the SPR condition (30) and part (i) of Lemma 4 that

$$L_\epsilon(j\omega) + L_\epsilon(j\omega)^* \geq 0 \quad (33)$$

for all nonzero  $\omega$ . Since  $G$  is PR (recall (31)) and regular, Lemma 3 tells us that  $\tilde{G}_1$  is hermitian; also  $\tilde{G}_1 > 0$  for  $\rho > 0$ . Considering limits as  $|\omega| \rightarrow \infty$  in (33) and recalling part (ii) of Lemma 4 we obtain that

$$E_\epsilon + E_\epsilon^* \geq 0. \quad (34)$$

If  $\rho = 0$  then  $D + D^* > 0$  and  $E = D$ ; hence (32) holds. Consider now  $\rho = m$ . In this case,  $E_\epsilon = -G_2 - \epsilon G_1 = E - \epsilon \tilde{G}_1$  and (34) reduces to

$$E + E^* \geq 2\epsilon \tilde{G}_1.$$

Since  $\tilde{G}_1 > 0$ , we have the desired result (32).

Now consider the remaining case in which  $0 < \rho < m$ . Partition  $M := E + E^*$  in accordance with the partition of  $E$  as

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$

Then inequality (34) can be written as

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} - 2\epsilon\tilde{G}_1 \end{pmatrix} \geq 0 \quad (35)$$

Since  $\epsilon > 0$  and  $\tilde{G}_1 > 0$ , there exists  $\epsilon > 0$  such that  $2\epsilon\tilde{G}_1 \geq \epsilon I$ ; thus inequality (35) yields that

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} - \epsilon I \end{pmatrix} \geq 0.$$

The above inequality implies that  $U^*(D + D^*)U = M_{11} \geq 0$ . Since the columns of  $U$  span the range of  $D + D^*$ , we obtain that  $D + D^* \geq 0$ . Since the intersection of the range of  $U$  and the null space of  $D + D^*$  is the zero vector we must have  $M_{11} = U^*(D + D^*)U > 0$ . It now follows from Lemma 5 in the Appendix that  $M > 0$ , that is,  $E + E^* > 0$ , which is the desired result.

*Sufficiency.* Clearly hypothesis (b) implies that  $G$  is regular.

To demonstrate the existence of  $\epsilon > 0$  such that (1) holds, we first show that there exists  $\epsilon_1$  and  $\omega_1 > 0$  with  $0 < \epsilon_1 < \beta$  such that

$$G(j\omega - \epsilon) + G(j\omega - \epsilon)^* > 0 \quad \text{for all} \quad 0 \leq \epsilon \leq \epsilon_1 \quad \text{and} \quad |\omega| \geq \omega_1. \quad (36)$$

To see this consider the function

$$F(\epsilon, \eta) = \begin{cases} L_\epsilon(\frac{\omega}{\eta}) + L_\epsilon(\frac{\omega}{\eta})^* & \text{for} \quad \eta \neq 0 \\ E_\epsilon + E_\epsilon^* & \text{for} \quad \eta = 0 \end{cases}$$

Since  $G$  is PR, the matrix  $\tilde{G}_1$  is hermitian and it follows from part (ii) of Lemma 4 that  $F$  is continuous. Recall from Corollary 1 that, under hypothesis (b), hypothesis (c) is equivalent to  $E + E^* > 0$ , that is  $F(0, 0) > 0$ . Since  $F$  is continuous, there exists  $\epsilon_1$  and  $\eta_1 > 0$  such that  $0 < \epsilon_1 < \beta$  and  $F(\epsilon, \eta) > 0$  for  $|\epsilon| \leq \epsilon_1$  and  $|\eta| \leq \eta_1$ . Letting  $\omega_1 = 1/\eta_1$  yields the desired result (36).

We now show that there exists  $\epsilon_2$  with  $0 < \epsilon_2 < \beta$  such that

$$\det[G(j\omega - \epsilon) + G(j\omega - \epsilon)^*] > 0 \quad \text{for all} \quad |\omega| \leq \omega_1 \quad \text{and} \quad 0 \leq \epsilon \leq \epsilon_2 \quad (37)$$

To achieve this introduce the continuous function

$$f(\epsilon) = \min\{\det[G(j\omega - \epsilon) + G(j\omega - \epsilon)^*] : |\omega| \leq \omega_1\} \quad |\epsilon| < \beta.$$

It follows from hypothesis (b) that  $f(0) > 0$ . Hence there exists  $\epsilon_2$  such that  $0 \leq \epsilon_2 < \beta$  and  $f(\epsilon) > 0$  for  $0 \leq \epsilon \leq \epsilon_2$ . This yields (37).

If we let  $\epsilon_3 = \min\{\epsilon_1, \epsilon_2\}$  we obtain the desired result that (1) holds with  $\epsilon = \epsilon_3$  and  $G$  is analytic in the region for which  $\Re(s) \geq -\epsilon_3$ . ■

## 5 Conclusions

This paper presents conditions which are necessary and sufficient for a transfer function (or transfer function matrix) to be strictly positive real. A counter-example is given to illustrate that, in the general rational matrix case, the conditions presented here differ from those previously presented in the literature, and that these same conditions represent an incomplete characterization of strict positive realness. The proof of the results presented here differs from previous proofs of related results; it only uses properties of analytic functions and matrices and does not rely on state space realizations. Also the results are not restricted to rational transfer functions with real coefficients.

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## Appendix

### 5.1 Proof of Lemma 2

PROOF. Since  $g$  is positive real, we have  $g(j\omega) + g(j\omega)^* \geq 0$  for all  $\omega \in \mathbb{R}$ . Since  $g$  is analytic for  $\Re(s) \geq 0$ , we must also have

$$g(s) + g(s)^* \geq 0 \quad \text{for} \quad \Re(s) \geq 0 \quad (38)$$

Now consider any function  $g$  which satisfies (38) and which has a power series of the form

$$g(s) = \frac{g_n}{s^n} + \frac{g_{n+1}}{s^{n+1}} + \dots \quad (39)$$

with  $g_n \neq 0$ . We will show that  $n = 1$  and  $g_1$  is real with  $g_1 > 0$ .

To this end, consider any real  $r$  and  $\theta$  with  $r \geq 0$  and  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ . Then  $\Re(s) \geq 0$  for  $s = re^{-j\theta}$  and it follows from inequality (38) that

$$r^n [g(re^{-j\theta}) + g(re^{-j\theta})^*] \geq 0.$$

Considering the power series expansion (39) we see that

$$\lim_{r \rightarrow \infty} r^n g(re^{-j\theta}) = e^{jn\theta} g_n.$$

So, we must have

$$e^{jn\theta} g_n + e^{-jn\theta} g_n^* \geq 0. \quad (40)$$

Considering  $\theta = \pi/2n$  and  $\theta = -\pi/2n$  yields  $j(g_n - g_n^*) \geq 0$  and  $-j(g_n - g_n^*) \geq 0$ , respectively. Hence  $j(g_n - g_n^*) = 0$  and we must have  $g_n^* = g_n$ , that is,  $g_n$  is real. We now show that  $g_n \geq 0$ . Considering  $\theta = 0$  in (40) yields  $2g_n \geq 0$ ; hence  $g_n \geq 0$ . Suppose  $n > 1$ ; then, considering  $\theta = \pi/n$  in (40) results in  $-2g_n \geq 0$ . So, we obtain the contradiction that  $g_n = 0$  when  $n > 1$ . Hence  $n = 1$  and  $g_1 > 0$ .

It now follows that if  $g$  satisfies (38) and has a power series of the form (14) with  $g_1 = 0$  then,  $g_n = 0$  for all  $n$ ; hence  $g$  is identically zero. ■

### 5.2 Proof of Lemma 3

Suppose  $V$  is  $m \times m_2$  and consider any nonzero  $w \in \mathbb{C}^{m_2}$ . Let  $g$  be the scalar-valued transfer function given by

$$g(s) = u^* \hat{G}(s) u \quad \text{where} \quad \hat{G}(s) = G(s) + \frac{1}{2}(D^* - D) \quad \text{and} \quad u = Vw. \quad (41)$$

Then,

$$g(s) + g(s)^* = u^*[G(s) + G(s)^*]u \quad (42)$$

and, since  $G$  is positive real, we must have  $g(j\omega) + g(j\omega)^* \geq 0$  for all  $\omega \in \mathbb{R}$ ; hence  $g$  is positive real.

Recalling power series expansion (5) for  $G$  yields

$$\hat{G}(s) = \frac{1}{2}(D^* + D) + \frac{1}{s}G_1 + \frac{1}{s^2}G_2 + \dots$$

Since  $(D^* + D)V = 0$ , we have  $u^*(D^* + D)u = 0$ ; hence we obtain the following power series expansion for  $g$ :

$$g(s) = \frac{g_1}{s} + \frac{g_2}{s^2} + \dots \quad (43)$$

where  $g_n = u^*G_n u$  for  $n = 1, 2, \dots$ . Since  $g$  is PR, it follows from Lemma 2 that  $g_1 = w^*V^*G_1Vw$  is real and non-negative; moreover,  $g_1 = 0$  implies  $g(s)$  is zero for all  $s$ . Since  $w^*V^*G_1Vw$  is real and non-negative for any  $w \in \mathbb{C}^{m_2}$ , we conclude that  $V^*G_1V$  is hermitian and  $V^*G_1V \geq 0$ .

Suppose  $G$  is regular and PR. We first show that for any nonzero  $w \in \mathbb{C}^{m_2}$ , the scalar transfer function defined in (41) is not identically zero. Since  $G$  is regular, there exists  $\omega \in \mathbb{R}$  such that  $\det[G(j\omega) + G(j\omega)^*] \neq 0$ , that is,  $G(j\omega) + G(j\omega)^*$  is nonsingular. Since  $G$  is PR, we also have  $G(j\omega) + G(j\omega)^* \geq 0$ ; hence  $G(j\omega) + G(j\omega)^* > 0$ . The assumption that  $V$  is full column rank and  $w \neq 0$  implies that  $u = Vw \neq 0$ . Recalling (42) we now obtain that  $g(j\omega) + g(j\omega)^* > 0$ ; hence  $g$  is not identically zero.

Since  $g$  is not identically zero we must have  $w^*V^*G_1Vw = g_1 > 0$ . Since  $w^*V^*G_1Vw > 0$  for any nonzero  $w \in \mathbb{C}^{m_2}$ , we conclude that  $V^*G_1V > 0$ .  $\blacksquare$

### 5.3 Proof of Lemma 4

(i) Consider any  $\omega \in \mathbb{R}$ . For  $\rho = 0$ , we have  $L_\epsilon(j\omega) = G(j\omega - \epsilon)$ . Thus,

$$L_\epsilon(j\omega) + L_\epsilon(j\omega)^* = G(j\omega - \epsilon) + G(j\omega - \epsilon)^*$$

and clearly, the result holds.

For  $\rho = m$ , we have  $L_\epsilon(j\omega) = \omega^2 G(j\omega - \epsilon)$ . Thus,

$$L_\epsilon(j\omega) + L_\epsilon(j\omega)^* = \omega^2 [G(j\omega - \epsilon) + G(j\omega - \epsilon)^*]$$

and clearly, the result holds.

When  $0 < \rho < m$ , we have  $L_\epsilon(j\omega) = S^*G(j\omega - \epsilon)S$  where  $S = (U \ j\omega V)$ . Hence

$$L_\epsilon(j\omega) + L_\epsilon(j\omega)^* = S^* [G(j\omega - \epsilon) + G(j\omega - \epsilon)^*] S.$$

When  $\omega$  is nonzero,  $S$  is invertible and

$$G(j\omega - \epsilon) + G(j\omega - \epsilon)^* = S^{-*} [L_\epsilon(j\omega) + L_\epsilon(j\omega)^*] S^{-1}$$

These last two identities yield the desired result.

(ii) Suppose  $\tilde{G}_1$  is hermitian and  $\{(\omega_k, \epsilon_k)\}_{k=1}^\infty$  is any sequence with  $\lim_{k \rightarrow \infty} \epsilon_k = \epsilon$  and  $\lim_{k \rightarrow \infty} |\omega_k| = \infty$ . Recall that, for  $|s|$  sufficiently large,

$$G(s) = D + \frac{1}{s}G_1 + \frac{1}{s^2}G_2 + \frac{1}{s^3}G_3 \dots \quad (44)$$

When  $\rho = 0$ , we have  $L_\epsilon(j\omega) = G(j\omega - \epsilon)$  and  $E_\epsilon = D$ . It follows from the above power series expansion that

$$\lim_{k \rightarrow \infty} [G(j\omega_k - \epsilon_k) + G(j\omega_k - \epsilon_k)^*] = D + D^*, \quad (45)$$

that is, the desired result (19) holds.

When  $\rho = m$ , we have  $L_\epsilon(j\omega) = \omega^2 G(j\omega - \epsilon)$ ,  $E_\epsilon = -\epsilon G_1 - G_2$  and  $\tilde{G}_1 = G_1$ . Thus  $G_1$  is hermitian and we need to show that

$$\lim_{k \rightarrow \infty} \omega_k^2 [G(j\omega_k - \epsilon_k) + G(j\omega_k - \epsilon_k)^*] = -2\epsilon G_1 - G_2 - G_2^*. \quad (46)$$

Since  $D + D^* = 0$  and  $G_1^* = G_1$ ,

$$G(s) + G(s)^* = \left( \frac{1}{s} + \frac{1}{s^*} \right) G_1 + \frac{1}{s^2} G_2 + \frac{1}{(s^*)^2} G_2^* + \frac{1}{s^3} G_3 + \frac{1}{(s^*)^3} G_3^* + \dots$$

When  $s = s_k := j\omega_k - \epsilon_k$ ,

$$\frac{1}{s_k} = \frac{1}{j\omega_k - \epsilon_k} = \frac{-j\omega_k - \epsilon_k}{\omega_k^2 + \epsilon_k^2} \quad \text{and} \quad \left( \frac{1}{s_k} + \frac{1}{s_k^*} \right) G_1 = \frac{-2\epsilon_k}{\omega_k^2 + \epsilon_k^2} G_1$$

which results in

$$\lim_{k \rightarrow \infty} \omega_k^2 \left( \frac{1}{s_k} + \frac{1}{s_k^*} \right) G_1 = -2\epsilon G_1.$$

Noting that

$$\lim_{k \rightarrow \infty} \frac{\omega_k^2}{(j\omega_k - \epsilon_k)^2} = \lim_{k \rightarrow \infty} \frac{\omega_k^2}{(-j\omega_k - \epsilon_k)^2} = -1$$

we obtain the desired result (46).

Now consider the remaining case in which  $0 < \rho < m$ . Recalling that the columns of  $V$  are in the null space of  $D + D^*$ , we have  $(D + D^*)V = 0$ ; hence the power series expansion (44) yields that

$$\begin{aligned} j\omega U^* G(j\omega - \epsilon)V + j\omega U^* G(j\omega - \epsilon)^*V &= \left( \frac{j\omega}{j\omega - \epsilon} \right) U^* G_1 V + \left( \frac{j\omega}{-j\omega - \epsilon} \right) U^* G_1^* V \\ &+ \frac{j\omega}{(j\omega - \epsilon)^2} U^* G_2 V + \frac{j\omega}{(-j\omega - \epsilon)^2} U^* G_2^* V + \dots \end{aligned}$$

Hence

$$\lim_{k \rightarrow \infty} [j\omega_k U^* G(j\omega_k - \epsilon_k)V + j\omega_k U^* G(j\omega_k - \epsilon_k)^*V] = U^* G_1 V - U^* G_1^* V. \quad (47)$$

Since  $V^* G_1 V$  is hermitian, we can use the same arguments as used in the case when  $\rho = m$  to obtain that

$$\lim_{k \rightarrow \infty} \omega_k^2 [V^* G(j\omega_k - \epsilon_k)V + V^* G(j\omega_k - \epsilon_k)^*V] = -2\epsilon V^* G_1 V - V^* G_2 V - V^* G_2^* V \quad (48)$$

The desired result (19) now follows from (45) and (47)-(48).

(iii) For  $\rho = 0$  and  $\rho = m$ , we have  $L(s) = G(s)$  and  $L(s) = -s^2 G(s)$ , respectively. So clearly the result holds in these cases. When  $0 < \rho < m$ , we can express  $L(s)$  as

$$L(s) = \begin{pmatrix} I_{m_1} & 0 \\ 0 & -s^* I_\rho \end{pmatrix}^* T^* G(s) T \begin{pmatrix} I_{m_1} & 0 \\ 0 & s I_\rho \end{pmatrix} \quad (49)$$

where  $m_1$  is the rank of  $D + D^*$  and  $T = [U \ V]$ . Hence

$$L(s) + L(-s^*)^* = \begin{pmatrix} I_{m_1} & 0 \\ 0 & -s^* I_\rho \end{pmatrix}^* T^* [G(s) + G(-s^*)^*] T \begin{pmatrix} I_{m_1} & 0 \\ 0 & s I_\rho \end{pmatrix}$$

and using properties of determinants, we see that

$$\det[L(s) + L(-s^*)^*] = \det(T^* T) (-s^2)^\rho \det [G(s) + G(-s^*)^*] .$$

Since  $\det(T^* T) > 0$  we are done. ■

## 5.4 A final lemma

**Lemma 5** *Suppose*

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

*is hermitian with  $M_{11}$  square and positive definite and there exists  $\epsilon > 0$  such that*

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} - \epsilon I \end{pmatrix} \geq 0. \tag{50}$$

*Then,  $M$  is positive definite.*

PROOF. Since  $M_{11} > 0$ , inequality (50) implies that

$$M_{22} - \epsilon I - M_{21} M_{11}^{-1} M_{12} \geq 0 .$$

This inequality and  $\epsilon > 0$  result in

$$M_{22} - M_{21} M_{11}^{-1} M_{12} > 0 .$$

Since  $M_{11} > 0$ , the last inequality implies that  $M > 0$ . ■