Direct Finite Element Solver of Linear Complexity for Analyzing Electrically Large Problems

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***Abstract*—In this paper, we develop a fast direct finite element solver of linear (optimal) complexity for the electromagnetic analysis of electrically large problems. Both theoretical analysis and numerical experiments have demonstrated the solver's linear complexity in CPU time and memory consumption with prescribed accuracy satisfied.**

***Keywords*—Direct solver, electrically large analysis.**

I. Introduction

State-of-the-art fast finite-element methods (FEM) rely on iterative approaches to solve large-scale problems. The computational complexity of an iterative solver is $O(N_{it}N_{rhs}N)$ at best, where N_{it} is the number of iterations and N_{rhs} is the number of right hand sides. When N_{it} and N_{rhs} are large, state-of-the-art iterative solutions become inefficient since the entire iteration procedure has to be repeated for each right hand side. Among existing direct matrix solutions, the multifrontal method is a state-of-the-art direct sparse solver. Well-known sparse solver packages such as `UMFPACK`, `MUMPS`, and `Pardiso` in Intel MKL (Math Kernel Library) are all based on the multifrontal method. The complexity of the multifrontal method relies on the elimination ordering used to reduce fill-ins. Its best time complexity is $O(N^2)$ for solving the sparse matrix of a general 3-D problem. Recently, we have shown that the complexity of a direct FEM solution can be reduced to linear (optimal) complexity for general 3-D circuit extraction [1]. The contribution of this paper is a direct FEM solver of linear complexity for analyzing electrically large 3-D problems such as large-scale antennas.

II. Proposed Linear-Complexity Direct Solver

In the proposed method, we fully take advantage of the zeros in the original FEM matrix, and also maximize the zeros in the $\mathbf{L}$ and $\mathbf{U}$ factors. Instead of treating $\mathbf{L}/\mathbf{U}$ as a whole $\mathcal{H}$-matrix, we only store the nonzeros in $\mathbf{L}$ and $\mathbf{U}$ with a compact error-controlled $\mathcal{H}$-matrix representation, compute these nonzeros by developing fast $\mathcal{H}$-matrix based algorithms, while completely removing all the zeros in $\mathbf{L}$ and $\mathbf{U}$ from storage and computation. Since the geometry of the structure being simulated is known, we maximize the zeros in $\mathbf{L}$ and $\mathbf{U}$ by nested dissection ordering. Moreover, we organize the factorization of the original 3-D finite element matrix into a sequence of partial factorizations of 2-D dense matrices, and thereby control the rank to follow a 2-D based growth rate, which is much slower than a 3-D based growth rate [2] to facilitate electrically large analyses.

The overall algorithm of the proposed direct solver has six major steps as shown in Algorithm 1.

Algorithm 1: Proposed Direct Solver

1. Partition unknowns by nested dissection.
2. Build elimination tree $\mathcal{E}_{\mathcal{I}}$ from nested dissection ordering.
3. Do symbolic factorization by elimination tree $\mathcal{E}_{\mathcal{I}}$.
4. Generate a local $\mathcal{H}$-matrix representation for the frontal matrix associated with each node in the elimination tree.
5. Do numerical factorization across elimination tree $\mathcal{E}_{\mathcal{I}}$ by developing fast $\mathcal{H}$-matrix-based algorithms.
6. Compute the solution and do post-processing.

III. Numerical Results

We first validated the accuracy of the proposed solver on a cavity-backed patch antenna [3]. The antenna consists of a 5.0 cm $\times$ 3.4 cm rectangular metal patch residing on a dielectric substrate, which is housed in a 7.5 cm by 5.1 cm rectangular cavity recessed in a ground plane. The input impedance of the antenna from 1 to 4 GHz extracted from the proposed method is shown in Fig. 1, which agrees very well with the measured data as well as the FE-BI (finite-element boundary-integral) results in [3]. Notice that triangular prism elements are used here, whereas brick elements are used in [3].

We then simulated an array of such a patch antenna structure, and also increased the array element number from 2 by 2 to 34 by 34 (1,156 elements), resulting in 14,449 to 3.47 million unknowns. The frequency is 2.75 GHz. In Fig. 2, we plot the factorization time, memory, and solution error with respect to N. The solution error is measured by relative residual $||\mathbf{YX}-\mathbf{B}||/||\mathbf{B}||$, where $\mathbf{B}$ denotes the right hand side matrix whose column dimension ranges from 2 to 400. From Fig. 2, it can be seen clearly that the proposed method exhibits linear complexity in both CPU time and memory consumption with good accuracy achieved in the entire unknown range. The smaller error at the early stage is due to the fact that many blocks are full-matrix blocks when the unknown number is small, and the admissible block number has not saturated.

References

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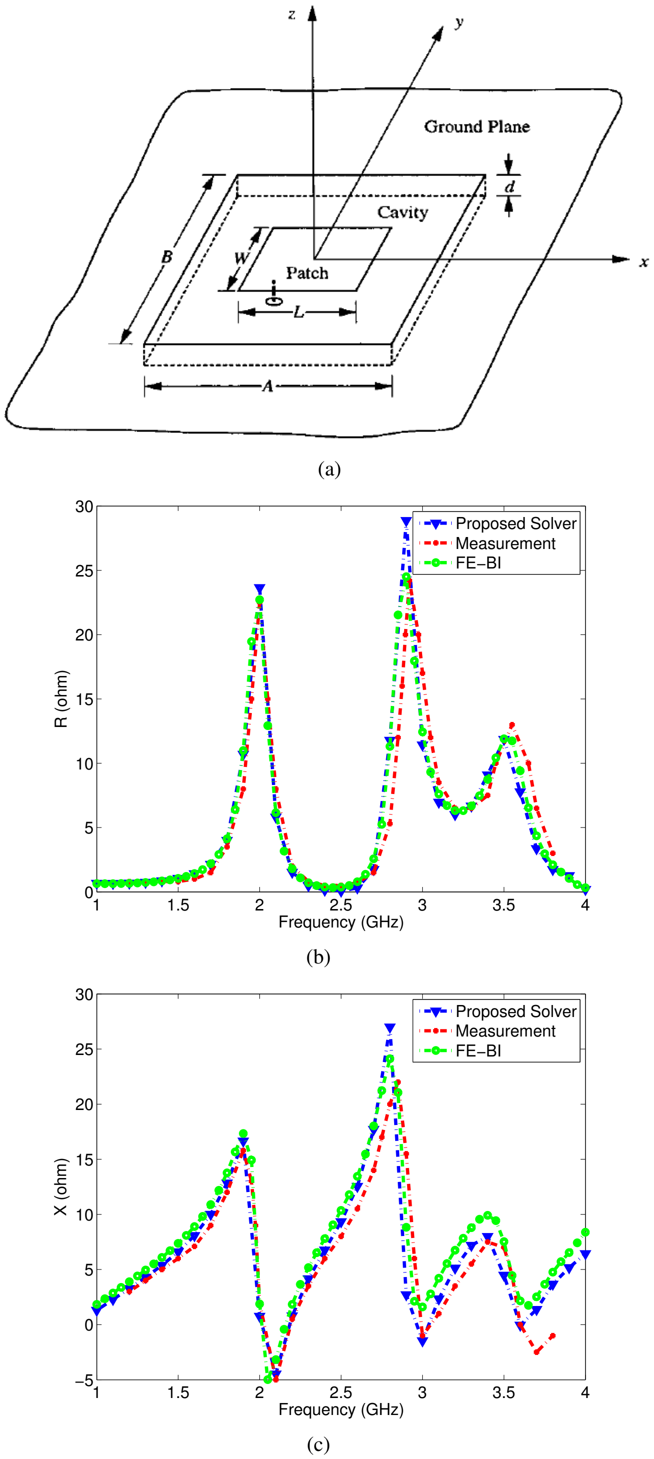

Fig. 1. Simulation of a cavity-backed microstrip patch antenna; (a) structure (after [3]), (b) input resistance R, and (c) input reactance X.

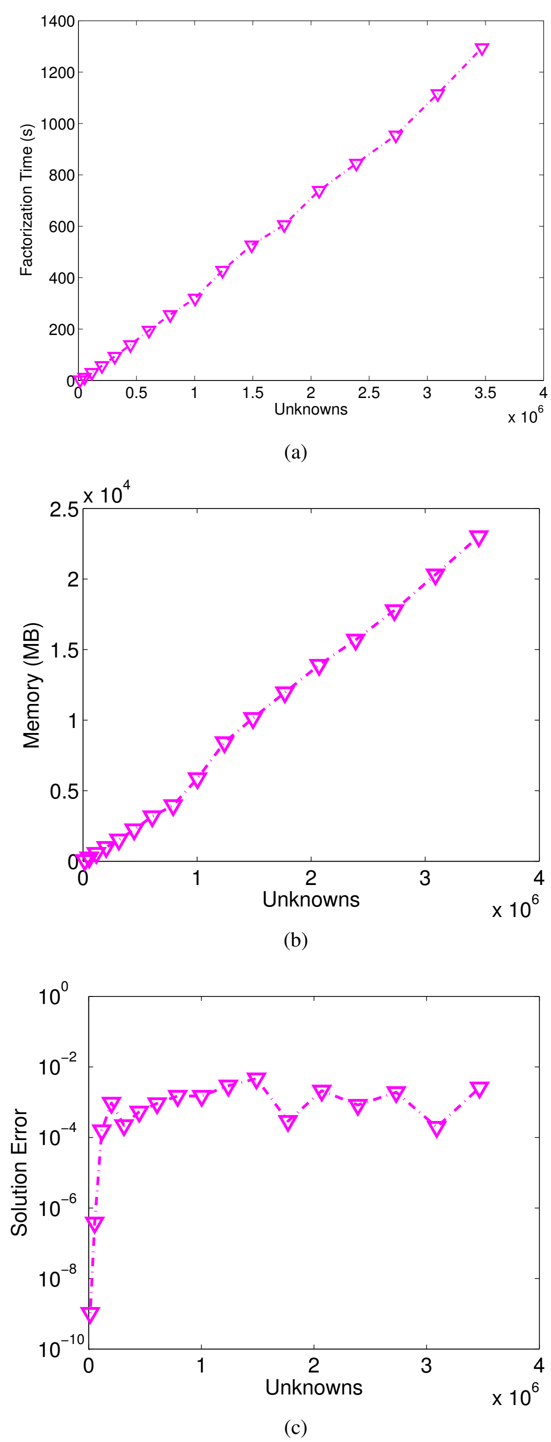

Fig. 2. Simulation of a suite of patch antenna arrays containing 4 to 1,156 elements with N ranging from 14,449 to 3.47 million at 2.75 GHz; (a) LU factorization time, (b) memory, and (c) solution error.