A New $\mathcal{H}^2$-Matrix-Based Representation of Electrodynamic Systems with Minimized Rank and Prescribed Accuracy

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Introduction

In recent years, fast solvers [1] such as fast multiple based methods, fast low-rank compression methods, and FFT-based methods have been developed, which dramatically reduce the memory complexity of the iterative integral equation (IE) solvers to $O(N)$, and the CPU time to $O(N\log N)$ for electrodynamic problems. The $\mathcal{H}^2$-matrix based mathematical framework has also been introduced and further developed to reduce the computational complexity of IE-based solutions of electrodynamic problems [2]. It is shown that given a wide range of electric sizes which lead to a wide range of N, the dense system of $O(N^2)$ parameters can be compactly stored in $O(N)$ units, and the dense matrix-vector multiplication can be performed in $O(N)$ operations. Moreover, the same order of accuracy can be kept across this range. The $\mathcal{H}^2$-matrix-based representation of the electrodynamic kernels in [2] is generated by an interpolation based scheme. The rank of each admissible block is determined by the number of interpolation points, which varies with the tree level based on a rank function. Such a rank function can be used to maintain the same order of accuracy in a wide range of electric sizes without compromising the linear computational cost. However, restricted by the interpolation based $\mathcal{H}^2$-representation, the resultant rank for each admissible block and for each electric size is often much larger than the minimal one that is required to satisfy a prescribed accuracy. This can greatly slow down the computation.

In this work, we generate a new $\mathcal{H}^2$-matrix-based representation of the dense system matrix arising from the IE-based analysis of electrodynamic problems. Such a representation features a minimized rank with prescribed accuracy. Instead of using an interpolation based scheme, we develop a new method to construct the nested cluster basis and the coupling matrices, the rank of which is minimized based on the accuracy requirement. Furthermore, the new method has a linear complexity, and hence the computational overhead is minimal for constructing an $\mathcal{H}^2$-based representation of the system matrix. The proposed method has been applied to solve large-scale electrodynamic problems. It is shown that a dense matrix-vector multiplication involving over 1 million unknowns can be completed in 20 seconds and 15 GB memory on a single 8222SE AMD Opteron processor.

Background

A Method of Moment based solution of electric field integral equation (EFIE) results in a dense system of linear equations $\mathbf{G}I=V$. When a block cluster (t, s) is admissible based on an admissibility condition [3, p.145], the matrix block $\mathbf{G}^{t,s}$ formed by t and s can be represented by a factorized form

$$\tilde{\mathbf{G}}^{t,s}_{m,n} := \mathbf{V}^{t}_{m,k}\mathbf{S}^{t,s}_{k,k}\mathbf{V}^{s^{\mathrm{T}}}_{n,k} \tag{1}$$

where $\mathbf{V}^t$ (or $\mathbf{V}^s$) is called cluster basis, which is stored in the cluster t of a cluster tree [2], and $\mathbf{S}^{t,s}$ is called coupling matrix stored in a block cluster (t, s) of a block cluster tree [2], $k \in \mathbb{N}$ is the rank of $\tilde{\mathbf{G}}^{t,s}$. If $\mathbf{V}^t$ is nested, $\mathbf{V}^t$ only needs to be stored in the leaf clusters. The low-rank approximation of $\mathbf{G}$ given in (1) with nested basis $\mathbf{V}$ forms an $\mathcal{H}^2$-matrix

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representation of $\mathbf{G}$. The rank k in $\mathbf{G}^{t,s}$ is p^d, where d is the dimension of the problem, and p is the number of interpolation points determined by a rank function [2] as follows

$$p(b)=\hat{a}+\hat{b}(L-l(b)) \quad \text{with } L=L_{\min}=\min\{level(\tau):\tau\in\mathcal{L}_{\mathcal{I}}\} \text{ and } p(b)=\hat{a} \quad \text{if } L\le l(b) \qquad (2)$$

It has been proved in [2] that the computational cost of matrix-vector products increases with the rank for the IE-based computation of electrodynamic problems. By reducing rank k for each admissible block, the computational cost can be significantly reduced. We found that the rank determined by the rank function (2) is, in general, larger than what is needed to satisfy a certain accuracy requirement. In the following section, we propose a new method to construct the nested cluster basis and coupling matrices for the $\mathcal{H}^2$-based representation of electrodynamic kernels. Such a method leads to a minimum rank that is determined by the accuracy requirement.

Proposed New $\mathcal{H}^2$-Representation with a Minimized Rank and Prescribed Accuracy

For a cluster t in a cluster tree, it forms admissible blocks (t, s_1), (t, s_2), …, (t, s_{q1}) based on the admissibility condition, as shown in Fig. 1. The cluster basis $\mathbf{V}^t$ in the cluster t is used for the representation of all the admissible blocks formed by t. Due to the nested property, $\mathbf{V}^t$ is also indirectly used for all the admissible blocks (t_P, s_q) formed by t's ancestors t_P as shown in Fig. 1. Conversely, given a cluster tree, for each cluster t in it, if we can construct a matrix $\mathbf{V}^t$ that can be used for representing all the admissible blocks horizontally formed by the cluster t in the same tree level as well as the admissible blocks vertically formed by its ancestors t_P across the tree level, then the cluster tree is nested and the matrix $\mathbf{V}^t$ can be viewed as a cluster basis in this cluster tree. This indicates that a nested cluster basis $\mathbf{V}^t$ in a cluster tree used for an $\mathcal{H}^2$ representation is not unique. The optimal $\mathbf{V}^t$ has the smallest rank for a given accuracy requirement.

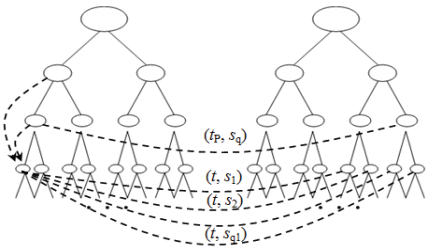

Fig. 1 A block cluster tree.

To develop such an optimal $\mathbf{V}^t$, for each cluster t, we collect all the t-related admissible matrix blocks, and combine them into a single matrix $\tilde{\mathbf{G}}^t=\left[\tilde{\mathbf{G}}^{t,s1},\tilde{\mathbf{G}}^{t,s2},\cdots,\tilde{\mathbf{G}}^{tp,sq}\right]$, where the matrix blocks with superscript t are the admissible blocks directly formed by t, and the matrix block with superscript t_p is the t-related admissible block contributed by t's ancestors t_p. By performing a schur decomposition of the Gram matrix $\tilde{\mathbf{G}}^t\tilde{\mathbf{G}}^{t\,\mathrm{H}}$, we obtain

$$\tilde{\mathbf{G}}_2^t=\tilde{\mathbf{G}}^t\tilde{\mathbf{G}}^{t\,\mathrm{H}}=\mathbf{P}_{nt,nt}^t\mathbf{D}_{nt,nt}\mathbf{P}_{nt,nt}^{t\,\mathrm{H}}, \qquad (3)$$

where nt is the cardinality of t, the columns of $\mathbf{P}$ are the eigenvectors of $\tilde{\mathbf{G}}_2^t$, $\mathbf{D}=diag(\sigma_1^2,\sigma_2^2,\cdots,\sigma_{nt}^2)$ corresponds to the squared singular values of $\tilde{\mathbf{G}}_2^t$ with $\sigma_1^2\ge\sigma_2^2\cdots\ge\sigma_{nt}^2\ge 0$. Based on an accuracy requirement ε, (3) can be truncated to rank $\tilde{k}$ as the following

$$\tilde{\mathbf{G}}_2^t=\mathbf{P}_{nt,nt}^t\mathbf{D}_{nt,nt}\mathbf{P}_{nt,nt}^{t\,\mathrm{H}} \xrightarrow{\text{rank } \tilde{k} \text{ trunctation}} \mathbf{P}_{nt,\tilde{k}}^t\mathbf{D}_{\tilde{k},\tilde{k}}\mathbf{P}_{nt,\tilde{k}}^{t\,\mathrm{H}}. \qquad (4)$$

Then the best rank-$\tilde{k}$ approximation of $\tilde{\mathbf{G}}^t$ in the column space of $\mathbf{P}^t$ satisfies

$$\begin{aligned}\left\|\tilde{\mathbf{G}}^t-\mathbf{P}^t\mathbf{P}^{t\,\mathrm{H}}\tilde{\mathbf{G}}^t\right\|_F^2&=\sum_{i=\tilde{k}+1}^{nt}\sigma_i^2=\left\|\left[\tilde{\mathbf{G}}^{t,s1},\tilde{\mathbf{G}}^{t,s2},\cdots,\tilde{\mathbf{G}}^{tp,sq}\right]-\mathbf{P}^t\mathbf{P}^{t\,\mathrm{H}}\left[\tilde{\mathbf{G}}^{t,s1},\tilde{\mathbf{G}}^{t,s2},\cdots,\tilde{\mathbf{G}}^{tp,sq}\right]\right\|_F^2\\&=\left\|\left[(\tilde{\mathbf{G}}^{t,s1}-\mathbf{P}^t\mathbf{P}^{t\,\mathrm{H}}\tilde{\mathbf{G}}^{t,s1}),\cdots,(\tilde{\mathbf{G}}^{tp,sq}-\mathbf{P}^t\mathbf{P}^{t\,\mathrm{H}}\tilde{\mathbf{G}}^{tp,sq})\right]\right\|_F^2=\underbrace{\left\|\tilde{\mathbf{G}}^{t,s1}-\mathbf{P}^t\mathbf{P}^{t\,\mathrm{H}}\tilde{\mathbf{G}}^{t,s1}\right\|_F^2+\cdots\cdots}_{\text{Direct error related to } t}+\underbrace{\left\|\tilde{\mathbf{G}}^{tp,sq}-\mathbf{P}^t\mathbf{P}^{t\,\mathrm{H}}\tilde{\mathbf{G}}^{tp,sq}\right\|_F^2}_{\text{Nested error from } t_p}.\end{aligned} \qquad (5)$$

The first equality is obtained based on the theorem given in [2, p.153] by utilizing the fact that the eigenvectors of $\tilde{\mathbf{G}}^t$ are the same as those of the Gram matrix $\tilde{\mathbf{G}}^t\tilde{\mathbf{G}}^{t\,\mathrm{H}}$. Similarly, we can compute a rank-$\tilde{k}$ approximation of $\tilde{\mathbf{G}}^s$ formed by a cluster *s*, which satisfies

$$\left\|\tilde{\mathbf{G}}^s-\mathbf{P}^s\mathbf{P}^{s\,\mathrm{H}}\tilde{\mathbf{G}}^s\right\|_F^2=\sum_{i=\tilde{k}+1}^{ns}\sigma_{si}^2=\left\|\tilde{\mathbf{G}}^{s,t1}-\mathbf{P}^s\mathbf{P}^{s\,\mathrm{H}}\tilde{\mathbf{G}}^{s,t1}\right\|_F^2+\cdots+\left\|\tilde{\mathbf{G}}^{sp,tq}-\mathbf{P}^s\mathbf{P}^{s\,\mathrm{H}}\tilde{\mathbf{G}}^{sp,tq}\right\|_F^2,\quad \tilde{\mathbf{G}}\text{ is symmetric }\rightarrow \tag{6}$$
$$=\left\|\tilde{\mathbf{G}}^{t1,s^{\mathrm{T}}}-\mathbf{P}^s\mathbf{P}^{s\,\mathrm{H}}\tilde{\mathbf{G}}^{t1,s^{\mathrm{T}}}\right\|_F^2+\cdots+\left\|\tilde{\mathbf{G}}^{tq,sp^{\mathrm{T}}}-\mathbf{P}^s\mathbf{P}^{s\,\mathrm{H}}\tilde{\mathbf{G}}^{tq,sp^{\mathrm{T}}}\right\|_F^2=\left\|\tilde{\mathbf{G}}^{t1,s}-\tilde{\mathbf{G}}^{t1,s}(\mathbf{P}^s\mathbf{P}^{s\,\mathrm{H}})^{\mathrm{T}}\right\|_F^2+\cdots+\left\|\tilde{\mathbf{G}}^{tq,sp}-\tilde{\mathbf{G}}^{tq,sp}(\mathbf{P}^s\mathbf{P}^{s\,\mathrm{H}})^{\mathrm{T}}\right\|_F^2.$$

Based on (5) and (6), the error of the rank-$\tilde{k}$ approximation of an admissible block $\tilde{\mathbf{G}}^{t,s}$ formed by a block cluster (*t*, *s*) is bounded by

$$\begin{aligned}&\|\tilde{\mathbf{G}}^{t,s}-\mathbf{P}^t\mathbf{P}^{t\,\mathrm{H}}\tilde{\mathbf{G}}^{t,s}(\mathbf{P}^s\mathbf{P}^{s\,\mathrm{H}})^{\mathrm{T}}\|_F^2=\|\tilde{\mathbf{G}}^{t,s}-\mathbf{P}^t\mathbf{P}^{t\,\mathrm{H}}\tilde{\mathbf{G}}^{t,s}+\mathbf{P}^t\mathbf{P}^{t\,\mathrm{H}}\tilde{\mathbf{G}}^{t,s}-\mathbf{P}^t\mathbf{P}^{t\,\mathrm{H}}\tilde{\mathbf{G}}^{t,s}(\mathbf{P}^s\mathbf{P}^{s\,\mathrm{H}})^{\mathrm{T}}\|_F^2\\&\le\left\|\tilde{\mathbf{G}}^{t,s}-\mathbf{P}^t\mathbf{P}^{t\,\mathrm{H}}\tilde{\mathbf{G}}^{t,s}\right\|_F^2+\left\|\mathbf{P}^t\mathbf{P}^{t\,\mathrm{H}}(\tilde{\mathbf{G}}^{t,s}-\tilde{\mathbf{G}}^{t,s}(\mathbf{P}^s\mathbf{P}^{s\,\mathrm{H}})^{\mathrm{T}})\right\|_F^2\le\sum_{i=\tilde{k}+1}^{nt}\sigma_i^2+\sum_{i=\tilde{k}+1}^{ns}\sigma_{si}^2\end{aligned}. \tag{7}$$

If the rank of matrix $\tilde{\mathbf{G}}^{t,s}$ is $\tilde{k}$, the error is 0. From (7), it can be seen that $\tilde{\mathbf{V}}^t=\mathbf{P}^t$ is an accurate new cluster basis for the cluster *t*. The new $\mathcal{H}^2$-based representation of $\tilde{\mathbf{G}}^{t,s}$ based on (7) is $\tilde{\mathbf{G}}_{new}^{t,s}=\tilde{\mathbf{V}}^t\tilde{\mathbf{V}}^{t\,\mathrm{H}}\tilde{\mathbf{G}}^{t,s}(\tilde{\mathbf{V}}^s\tilde{\mathbf{V}}^{s\,\mathrm{H}})^{\mathrm{T}}=\tilde{\mathbf{V}}^t(\tilde{\mathbf{V}}^{t\,\mathrm{H}}\tilde{\mathbf{G}}^{t,s}\overline{\tilde{\mathbf{V}}}^s)\tilde{\mathbf{V}}^{s\mathrm{T}}=\tilde{\mathbf{V}}^t\mathbf{S}_{new}^{t,s}\tilde{\mathbf{V}}^{s\mathrm{T}}$ with the new coupling matrix $\mathbf{S}_{new}^{t,s}=\tilde{\mathbf{V}}^{t\,\mathrm{H}}\tilde{\mathbf{G}}^{t,s}\overline{\tilde{\mathbf{V}}}^s$, where $\overline{\tilde{\mathbf{V}}}$ is the complex conjugate of $\tilde{\mathbf{V}}$. Such a choice of $\tilde{\mathbf{G}}_{new}^{t,s}$ has a controlled accuracy as shown by (7). The new cluster bases $\tilde{\mathbf{V}}^t$ constructed in the aforementioned way are not only orthogonal, but also have a rank minimized for a given accuracy requirement. In addition, it better represents the original electrodynamic problem since it contains frequency information carried by $\tilde{\mathbf{G}}^{t,s}$, and hence becoming frequency-dependent. Next we show how to efficiently compute the new cluster basis $\tilde{\mathbf{V}}^t$ and coupling matrices, and meanwhile keeping the nested property of $\tilde{\mathbf{V}}^t$.

Efficient Computation of New Cluster Basis $\tilde{\mathbf{V}}^t$. For horizontal admissible blocks formed by *t* in the same tree level, to compute (3), we evaluate

$$\sum_{s\in col(t)}\mathbf{V}^t\mathbf{S}^{t,s}\mathbf{V}^{s\,\mathrm{T}}\overline{\mathbf{V}}^s\mathbf{S}^{t,s\,\mathrm{H}}\mathbf{V}^{t\,\mathrm{H}}=\mathbf{V}^t(\sum_{s\in col(t)}\mathbf{S}^{t,s}\mathbf{B}^s\mathbf{S}^{t,s\,\mathrm{H}})\mathbf{V}^{t\,\mathrm{H}}=\mathbf{V}^t(\mathbf{S}_{sum}^t)\mathbf{V}^{t\,\mathrm{H}}, \tag{8}$$

where $\mathbf{B}^s=\mathbf{V}^{s\,\mathrm{T}}\overline{\mathbf{V}}^s$. To compute the vertical contributions to *t* from its parent t_p with two children (*t*, *t*'), we evaluate

$$\sum_{s\in col(tp)}\tilde{\mathbf{G}}^{(tp,s)}\tilde{\mathbf{G}}^{(tp,s)\,\mathrm{H}}=\sum_{s\in col(tp)}\mathbf{V}^t\mathbf{E}^t\mathbf{S}^{tp,s}\mathbf{V}^{s\,\mathrm{T}}\overline{\mathbf{V}}^s\mathbf{S}^{tp,s\,\mathrm{H}}(\mathbf{V}^t\mathbf{E}^t)^{\mathrm{H}}=\mathbf{V}^t\mathbf{E}^t(\sum_{s\in col(tp)}\mathbf{S}^{tp,s}\overline{\mathbf{B}}^s\mathbf{S}^{tp,s\,\mathrm{H}})\mathbf{E}^{t\,\mathrm{H}}\mathbf{V}^{t\,\mathrm{H}}=\mathbf{V}^t\mathbf{E}^t\mathbf{S}_{sum}^{tp}\mathbf{E}^{t\,\mathrm{H}}\mathbf{V}^{t\,\mathrm{H}} \tag{9}$$

Based on (8) and (9), to compute (3), we perform

$$\tilde{\mathbf{G}}_2^t=\mathbf{V}^t(\mathbf{S}_{sum}^t)\mathbf{V}^{t\,\mathrm{H}},\quad\text{with }\mathbf{S}_{sum}^t=\begin{cases}\sum_{s\in col(t)}\mathbf{S}^{t,s}\mathbf{B}^s\mathbf{S}^{t,s\,\mathrm{H}} & \text{if } t \text{ has no parent}\\ \sum_{s\in col(t)}\mathbf{S}^{t,s}\mathbf{B}^s\mathbf{S}^{t,s\,\mathrm{H}}+\mathbf{E}^t\mathbf{S}_{sum}^{t_P}\mathbf{E}^{t\,\mathrm{H}} & \text{if } t \text{ has a parent } t_p\end{cases}, \tag{10}$$

which can be computed from top to bottom for each cluster *t* in a cluster tree.

(a)Fast computation of $\mathbf{S}_{sum}^t$. In order to compute $\mathbf{S}_{sum}^t$ efficiently, we first compute $\mathbf{B}^s$ for all the clusters and store them. The computation of $\mathbf{B}^s$ is done from bottom to top: for a leaf cluster *s*, since $\mathbf{V}^s$ is stored, $\mathbf{B}^s$ is directly computed; for a non-leaf cluster *s* with two children s_1 and s_2, $\overline{\mathbf{B}}^s=\mathbf{V}^{s\,\mathrm{H}}\mathbf{V}^s=\mathbf{E}^{s1\,\mathrm{H}}\overline{\mathbf{B}}^{s1}\mathbf{E}^{s1}+\mathbf{E}^{s2\,\mathrm{H}}\overline{\mathbf{B}}^{s2}\mathbf{E}^{s2}$ where $\overline{\mathbf{B}}^s$ is the complex conjugate of $\mathbf{B}^s$. Since $\mathbf{B}^{s1}$ and $\mathbf{B}^{s2}$ have been computed in the children level, $\mathbf{B}^s$ can be computed directly by them. After $\mathbf{B}^s$ for all clusters are computed, we can compute $\mathbf{S}^{t,s}\mathbf{B}^s\mathbf{S}^{t,s\,\mathrm{H}}$. Since the computation of $\mathbf{S}_{sum}^t$ is done from top to bottom, $\mathbf{E}^t\mathbf{S}_{sum}^{t_P}\mathbf{E}^{t\,\mathrm{H}}$ can also be directly computed.

(b) Preserving nested property. After $\mathbf{S}_{sum}^t$ is obtained, $\tilde{\mathbf{G}}_2^t$ is known from (10). We can compute the new cluster basis $\tilde{\mathbf{V}}^t$ by (3). For a leaf cluster *t*, by performing a schur decomposition on $\tilde{\mathbf{G}}_2^t$, we get the new cluster basis $\tilde{\mathbf{V}}^t=\mathbf{P}^t$. In order to preserve the nested property, once we get the new cluster basis $\tilde{\mathbf{V}}^t$ for the cluster *t*, we use

$\tilde{\mathbf{V}}^t\tilde{\mathbf{V}}^{t\,\mathrm{H}}\tilde{\mathbf{G}}^{tp,s}$ to approximate $\tilde{\mathbf{G}}^{tp,s}$ for the computation of the new cluster basis $\tilde{\mathbf{V}}^{tp}$ in its parent cluster t_p, the error of which is bounded by the nested-error as shown in (5), and hence for a non-leaf cluster t_p with two children t and t', $\tilde{\mathbf{G}}_2^{tp}$ in (10) becomes

$$\tilde{\mathbf{G}}_2^{tp}=\begin{bmatrix}\tilde{\mathbf{V}}^t\tilde{\mathbf{V}}^{t\,\mathrm{H}} & \\ & \tilde{\mathbf{V}}^{t'}\tilde{\mathbf{V}}^{t'\,\mathrm{H}}\end{bmatrix}\cdot\mathbf{V}^{tp}(\mathbf{S}_{sum}^{tp})\mathbf{V}^{tp\,\mathrm{H}}\cdot\begin{bmatrix}\tilde{\mathbf{V}}^t\tilde{\mathbf{V}}^{t\,\mathrm{H}} & \\ & \tilde{\mathbf{V}}^{t'}\tilde{\mathbf{V}}^{t'\,\mathrm{H}}\end{bmatrix}^{\mathrm{H}}\overset{\text{Nested property}}{=}\begin{bmatrix}\tilde{\mathbf{V}}^t\tilde{\mathbf{V}}^{t\,\mathrm{H}} & \\ & \tilde{\mathbf{V}}^{t'}\tilde{\mathbf{V}}^{t'\,\mathrm{H}}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{V}^t\mathbf{E}^t\\ \mathbf{V}^{t'}\mathbf{E}^{t'}\end{bmatrix}(\mathbf{S}_{sum}^{tp})\begin{bmatrix}\mathbf{V}^t\mathbf{E}^t\\ \mathbf{V}^{t'}\mathbf{E}^{t'}\end{bmatrix}^{\mathrm{H}}\cdot\begin{bmatrix}\tilde{\mathbf{V}}^t\tilde{\mathbf{V}}^{t\,\mathrm{H}} & \\ & \tilde{\mathbf{V}}^{t'}\tilde{\mathbf{V}}^{t'\,\mathrm{H}}\end{bmatrix}^{\mathrm{H}}$$

$$=\begin{bmatrix}\tilde{\mathbf{V}}^t & \\ & \tilde{\mathbf{V}}^{t'}\end{bmatrix}\cdot\left(\begin{bmatrix}\tilde{\mathbf{B}}^t\mathbf{E}^t\\ \tilde{\mathbf{B}}^{t'}\mathbf{E}^{t'}\end{bmatrix}(\mathbf{S}_{sum}^{tp})\begin{bmatrix}\tilde{\mathbf{B}}^t\mathbf{E}^t\\ \tilde{\mathbf{B}}^{t'}\mathbf{E}^{t'}\end{bmatrix}^{\mathrm{H}}\right)\cdot\begin{bmatrix}\tilde{\mathbf{V}}^t & \\ & \tilde{\mathbf{V}}^{t'}\end{bmatrix}^{\mathrm{H}}=\begin{bmatrix}\tilde{\mathbf{V}}^t & \\ & \tilde{\mathbf{V}}^{t'}\end{bmatrix}\cdot(\hat{\mathbf{G}}_2^{tp})\cdot\begin{bmatrix}\tilde{\mathbf{V}}^t & \\ & \tilde{\mathbf{V}}^{t'}\end{bmatrix}^{\mathrm{H}}=\underbrace{\begin{bmatrix}\tilde{\mathbf{V}}^t & \\ & \tilde{\mathbf{V}}^{t'}\end{bmatrix}\cdot\hat{\mathbf{P}}^{tp}}_{with\ \mathbf{P}^{tp}=\tilde{\mathbf{V}}^{tp}}\mathbf{D}\hat{\mathbf{P}}^{tp\,\mathrm{H}}\cdot\begin{bmatrix}\tilde{\mathbf{V}}^t & \\ & \tilde{\mathbf{V}}^{t'}\end{bmatrix}^{\mathrm{H}}$$

where $\tilde{\mathbf{B}}^t=\tilde{\mathbf{V}}^{t\,\mathrm{H}}\mathbf{V}^t$ can be efficiently computed like $\mathbf{B}^t$ and $\hat{\mathbf{P}}^{tp}$ includes new transfer matrices.

Linear Cost . The computation of $\mathbf{B}$ and $\tilde{\mathbf{B}}$ for each cluster costs $O(k^3)$ where k is the original rank determined by (2). The computation of $\mathbf{S}_{sum}^t$ for each cluster costs $O(C_{sp}k^3)$. The computation of schur decomposition in each cluster is $O(\tilde{k}k^2)$ where $\tilde{k}$ is the new rank determined by the proposed scheme. Then the total cost for the proposed scheme is

$$\sum_{l=0}^{L}2^l C_{sp}O(k^3)\le C_{sp}O(\sum_{l=0}^{L}[\hat{a}+\hat{b}(L-l)]^{3d}\cdot 2^l)\le C_{sp}O((\hat{a}+\hat{b})^{3d})\cdot 2^L\cdot\sum_{l=0}^{L}(l+1)^{3d}\cdot 2^{-l}\le C_{sp}O((\frac{2d(\hat{a}+\hat{b})}{\ln 1.5})^{3d})\cdot N$$

Numerical Results

The parameters used in all simulations are *leafsize* = 20, η =1, ε = 10^{-4} for schur decomposition, and $\hat{a}=5$, $\hat{b}=2$ for rank function. The first example is a conducting sphere. The electric size is from 2λ to 19λ with unknowns from 3,688 to 340,640. Fig 2(a) shows the accuracy of the proposed rank-minimized $\mathcal{H}^2$-representation with respect to *N*. Fig. 2 (b) shows the matrix-vector multiplication time of the $\mathcal{H}^2$-based iterative solver. A clear linear scaling can be observed in the entire range of electric sizes. The second example is a square plate with electric size from 2λ to 60λ having unknowns from 1,160 to 1,078,800. In Fig. 3(a) and (b), we plot the memory complexity and the time complexity of the matrix-vector multiplication (MVM). Again, a linear trend can be observed in the entire range of electric sizes. The MVM involving over 1 million unknowns is done in 20 s with 15GB memory. In Fig. 4(a) and (b) we compare the performance of the proposed solver with the interpolation based $\mathcal{H}^2$ construction developed in [2]. Clearly, the proposed solver outperforms in CPU time without sacrificing accuracy.

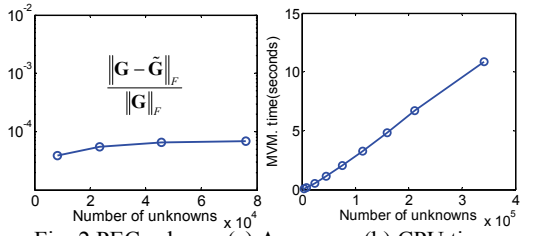

Fig. 2 PEC sphere. (a) Accuracy. (b) CPU time.

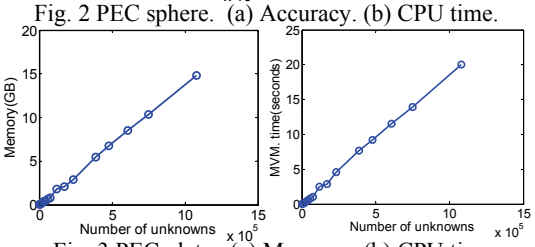

Fig. 3 PEC plate. (a) Memory. (b) CPU time.

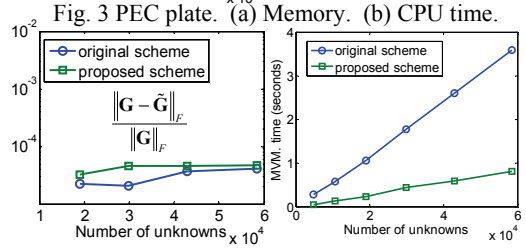

Fig. 4 Performance comparisons for the plate example. (a) Accuracy. (b) CPU time.

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