

# Time-Domain Finite-Element Modeling of Dispersive Media

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**Abstract**—A general formulation is described for time-domain finite-element modeling of electromagnetic fields in a general dispersive medium. The formulation is based on the second-order vector wave equation and incorporates the dispersion effect of a medium via a recursively evaluated convolution integral. This evaluation is kept to second order in accuracy using linear interpolation within each time step. Numerical examples are given to validate the proposed formulation.

**Index Terms**—Dispersive medium, finite-element method.

## I. INTRODUCTION

FOR any time-domain based numerical method to accurately perform wide-band electromagnetic simulations, one has to incorporate the effect of medium dispersion in its formulation. Over the past decade, several approaches have been proposed for the finite-difference time-domain (FDTD) method [1]–[6]. Little work has been reported on the dispersion modeling in the time-domain finite-element method (TDFEM) since TDFEM is not as well developed as FDTD. This situation, however, is changing rapidly; much interest has recently been attracted to TDFEM because of its modeling accuracy and flexibility [7]–[9]. In this work, a general formulation is developed to model the dispersion effect in TDFEM. This TDFEM is based on the second-order vector wave equation, in contrast to most FDTD schemes that solve the first-order Maxwell's equations. The required convolution integral is evaluated recursively without a need to store the fields of all past time steps. This evaluation is ensured of second order in accuracy by adopting a linear interpolation for the fields within each time step. The proposed formulation is shown to be valid for plasma, Debye, and Lorentz media with a single or multiple poles. Three-dimensional (3-D) numerical examples are given to demonstrate its efficacy.

## II. FORMULATION

The electric field in a general dispersive medium satisfies the second-order wave equation

$$\begin{aligned} \nabla \times \mu_r^{-1} \nabla \times \mathbf{E}(\mathbf{r}, t) + \mu_0 \epsilon \partial_t^2 \mathbf{E}(\mathbf{r}, t) + \mathcal{L}(\mathbf{E}(\mathbf{r}, t)) \\ = -\mu_0 \partial_t \mathbf{J}_s(\mathbf{r}, t) \end{aligned} \quad (1)$$

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where  $\mathbf{J}_s(\mathbf{r}, t)$  denotes the source current density and  $\mathcal{L}$  represents an operator on the field  $\mathbf{E}$ . For plasma

$$\begin{aligned} \mathcal{L}(\mathbf{E}(\mathbf{r}, t)) &= \mu_0 \epsilon_0 \omega_p^2 \{1 - \nu_c \varphi(t) * \} \mathbf{E}(\mathbf{r}, t) \\ \varphi(t) &= e^{-\nu_c t} \bar{u}(t) \end{aligned} \quad (2)$$

where

- $\omega_p$  plasma frequency;
- $\nu_c$  damping frequency;
- $\bar{u}(\cdot)$  unit step function;
- $*$  convolution.

For a Debye medium

$$\begin{aligned} \mathcal{L}(\mathbf{E}(\mathbf{r}, t)) &= \mu_0 \epsilon_0 (\epsilon_s - \epsilon_\infty) \tau^{-3} \{ \tau^2 \partial_t - \tau + \varphi(t) * \} \mathbf{E}(\mathbf{r}, t) \\ \varphi(t) &= e^{-t/\tau} \bar{u}(t) \end{aligned} \quad (3)$$

where  $\tau$  is the relaxation time,  $\epsilon_s$  and  $\epsilon_\infty$  denote the relative dielectric constants at zero (dc) and infinite frequencies, respectively. For a Lorentz medium

$$\begin{aligned} \mathcal{L}(\mathbf{E}(\mathbf{r}, t)) &= \mu_0 \epsilon_0 (\epsilon_s - \epsilon_\infty) \omega_0^2 \\ &\quad \times \{ G - \alpha^{-1} [2\delta \partial_t + \omega_0^2] \varphi(t) * \} \mathbf{E}(\mathbf{r}, t) \\ \varphi(t) &= e^{-\delta t} \sin(\alpha t) \bar{u}(t) \end{aligned} \quad (4)$$

where

- $\delta = \nu_c/2$  damping constant;
- $\omega_0$  resonant frequency;
- $\alpha = \sqrt{\omega_0^2 - \delta^2}$ , and  $G$  coefficient weighting the contribution from the induced polarization currents.

To illustrate the finite element solution of (1), we assume a mixed boundary condition on the surface of the volume of interest as

$$\mu_r^{-1} \hat{n} \times [\nabla \times \mathbf{E}(\mathbf{r}, t)] + c^{-1} \partial_t \hat{n} \times [\hat{n} \times \mathbf{E}(\mathbf{r}, t)] = \mathbf{U}(\mathbf{r}, t). \quad (5)$$

The corresponding weak-form solution is then given by

$$\begin{aligned} \iiint_V \{ \mu_r^{-1} [\nabla \times \mathbf{N}_i(\mathbf{r})] \cdot [\nabla \times \mathbf{E}(\mathbf{r}, t)] \\ + \mu_0 \epsilon \mathbf{N}_i(\mathbf{r}) \cdot \partial_t^2 \mathbf{E}(\mathbf{r}, t) + \mathbf{N}_i(\mathbf{r}) \cdot \mathcal{L}(\mathbf{E}(\mathbf{r}, t)) \\ + \mu_0 \mathbf{N}_i(\mathbf{r}) \cdot \partial_t \mathbf{J}_s(\mathbf{r}, t) \} dV \\ + \iint_S \{ c^{-1} [\hat{n} \times \mathbf{N}_i(\mathbf{r})] \cdot \partial_t [\hat{n} \times \mathbf{E}(\mathbf{r}, t)] \\ + \mathbf{N}_i(\mathbf{r}) \cdot \mathbf{U}(\mathbf{r}, t) \} dS = 0 \end{aligned} \quad (6)$$

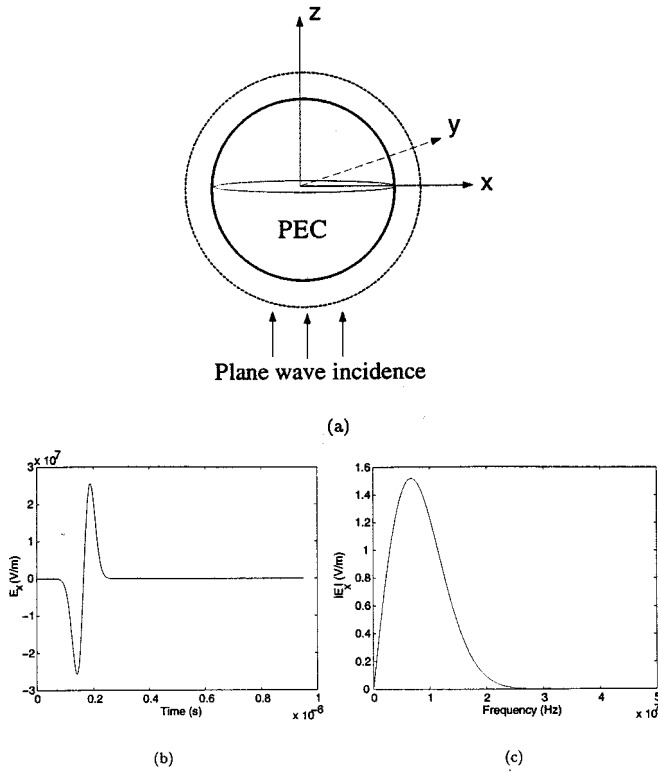


Fig. 1. The coated sphere and the incident electric field. (a) Geometry. (b)  $E_x$  versus time. (c)  $|E_x|$  versus frequency.

where  $\mathbf{N}_i(\mathbf{r})$  denotes the vector basis functions. Expanding the electric field as

$$\mathbf{E}(\mathbf{r}, t) = \sum_{j=1}^N u_j(t) \mathbf{N}_j(\mathbf{r}) \quad (7)$$

with  $N$  denoting the total number of unknowns, and substituting into (6), we obtain the ordinary differential equation

$$\mathbf{T} \frac{d^2 u}{dt^2} + \mathbf{R} \frac{du}{dt} + \mathbf{S} u + \mathbf{Y} \frac{d\psi}{dt} + \mathbf{Z} \psi + w = 0 \quad (8)$$

where  $\mathbf{T}$ ,  $\mathbf{R}$ ,  $\mathbf{S}$ ,  $\mathbf{Y}$ , and  $\mathbf{Z}$  denote matrices whose elements can be identified from (6). Also,  $u$  is a vector given by  $u = [u_1, u_2, \dots, u_N]^T$ ,  $\psi$  is a vector whose elements are given by

$$\psi_i(t) = \varphi(t) * u_i(t) \quad (9)$$

and finally,  $w$  is a vector contributed by  $\mathbf{J}_s(\mathbf{r}, t)$  and  $\mathbf{U}(\mathbf{r}, t)$ .

Since the susceptibility function of a general dispersive medium can be expressed as a rational function in frequency domain, its time-domain counterpart inherits the feature of exponential functions. Without loss of generality, we can write  $\varphi(t)$  as

$$\varphi(t) = \text{Re}[ae^{-bt}\bar{u}(t)] \quad (10)$$

where

$$\begin{aligned} a = 1 \text{ and } b = \nu_c & \quad \text{plasma;} \\ a = 1 \text{ and } b = \tau^{-1} & \quad \text{Debye medium;} \\ a = -j \text{ and } b = \delta - j\alpha & \quad \text{Lorentz medium.} \end{aligned}$$

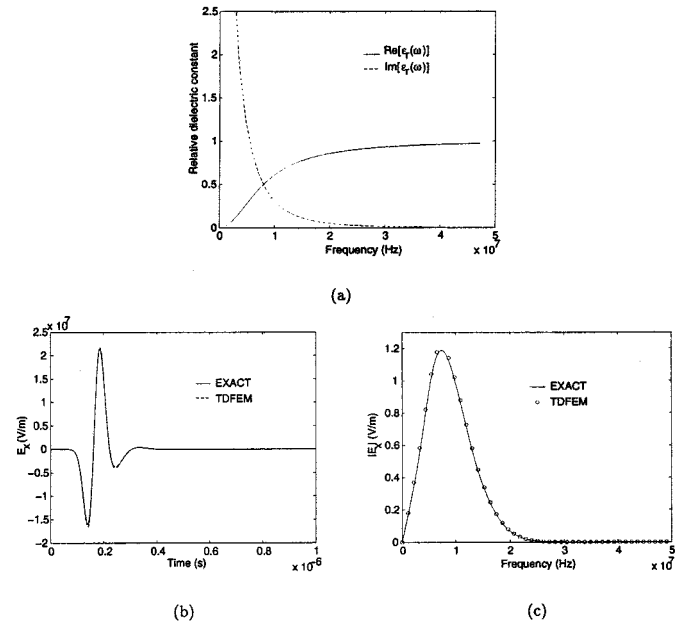


Fig. 2. Results for a metallic sphere coated with plasma. (a) Relative dielectric constant ( $\omega_p = \nu_c = 50$  Mrad/s). (b)  $E_x$  versus time. (c)  $|E_x|$  versus frequency.

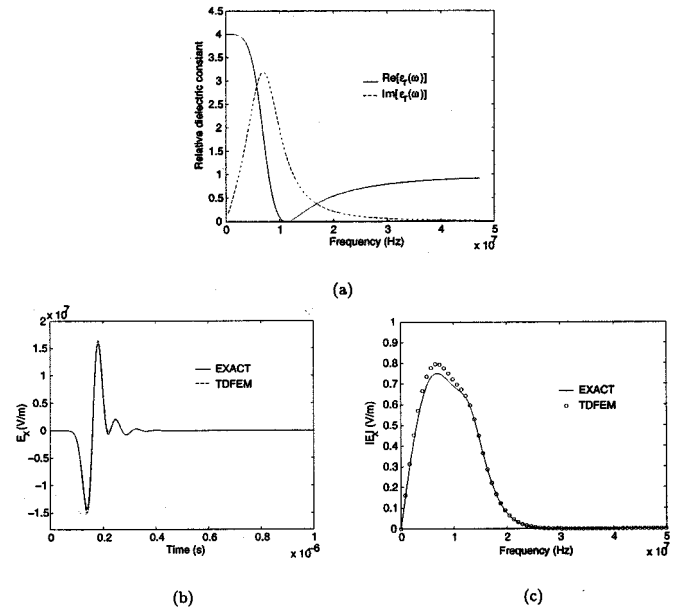


Fig. 3. Results for a metallic sphere coated with a Lorentz medium. (a) Relative dielectric constant ( $\omega_1 = 2\delta = 50$  Mrad/s,  $\epsilon_s = 4.0$ ,  $\epsilon_\infty = 1.0$ ,  $G = 1$ ). (b)  $E_x$  versus time. (c)  $|E_x|$  versus frequency.

As a result, the convolution in (9) can be evaluated recursively as

$$\begin{aligned} \hat{\psi}_i^{n+1} &= \text{Re}[\hat{\psi}_i^{n+1}] \\ \hat{\psi}_i^{n+1} &= e^{-b\Delta t} \hat{\psi}_i^n + ae^{-b(n+1)\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} e^{b\tau} u_i(\tau) d\tau. \end{aligned} \quad (11)$$

Instead of assuming  $u_i(t)$  to be constant within the time interval  $(n\Delta t, (n+1)\Delta t)$ , we employ linear interpolation to guarantee

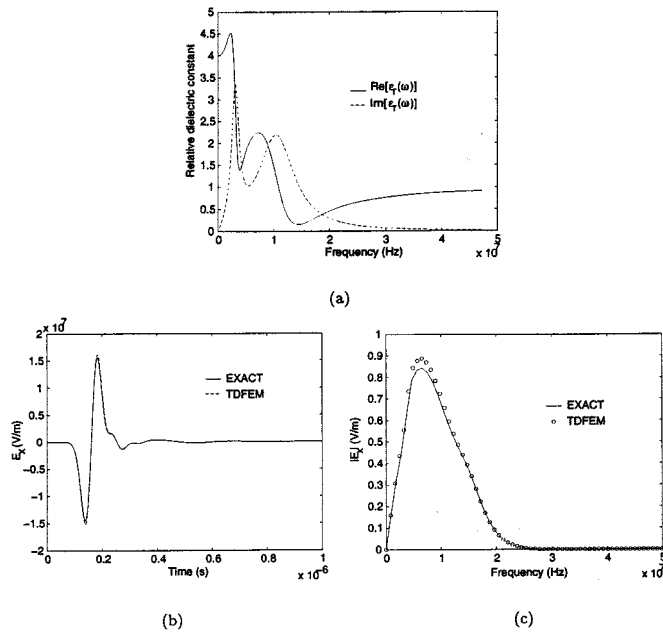


Fig. 4. Results for a metallic sphere coated with a second-order Lorentz medium. (a) Relative dielectric constant ( $\omega_1 = 70$  Mrad/s,  $2\delta_1 = 50$  Mrad/s,  $\omega_2 = 20$  Mrad/s,  $2\delta_2 = 10$  Mrad/s,  $G_1 = G_2 = 0.5$ ,  $\epsilon_s = 4.0$ ,  $\epsilon_\infty = 1.0$ ). (b)  $E_x$  versus time. (c)  $|E_x|$  versus frequency.

the second order in accuracy. As a consequence, we obtain the recursive relation

$$\hat{q}_i^{n+1} = e^{-b\Delta t} \hat{q}_i^n + 0.5a\Delta t (u_i^{n+1} + e^{-b\Delta t} u_i^n). \quad (12)$$

The above strategy can easily be extended to a general dispersive medium with an arbitrary order.

### III. NUMERICAL EXAMPLES

To validate the proposed formulation, we consider a metallic sphere coated with either plasma, a Lorentz, or a second-order Lorentz medium. The metallic sphere has a radius of 0.8 m and the coating has a thickness of 0.2 m. This coated sphere is illuminated by an  $x$ -polarized incident plane wave, whose electric field is plotted in Fig. 1, propagating along the  $z$ -direction. The

boundary integral equation is used to truncate the finite-element mesh accurately [9].

Figs. 2–4 display the calculated electric field  $E_x$  at the observation point  $\mathbf{r} = 0.1\hat{y} - 1.18\hat{z}$  m as a function of time and frequency. It is seen that the calculated results agree very well with the exact solution obtained from the Mie series.

### IV. CONCLUSION

A general approach was proposed to incorporate the dispersion effect in the TDFEM modeling of electromagnetic fields in a general dispersive medium. This approach employs a recursive evaluation of convolution integrals to avoid the storage of all past fields and adopts linear interpolation within each time step to achieve a second-order accuracy. Three-dimensional numerical examples were given to demonstrate its validity.

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