Image Restoration Using Recursive Markov Random Field Models Driven by Cauchy Distributed Noise

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***Abstract*—A recursive restoration algorithm based on a Markov random field model driven by Cauchy noise is described and demonstrated to provide better edge preservation than similar algorithms using Gaussian or Laplacian noises without increased computational cost.**

***Index Terms*—Cauchy noise, image restoration, Markov random fields, recursive filters.**

I. Introduction

KADABA *et al.* [1], [2] have used Markov random field models driven by heavy-tailed noise distributions and recursive processing algorithms in both image segmentation [2] and image restoration [1] problems. The heavy-tailed noise distribution in the restoration work was empirically modeled as either a Laplacian or a Bernoulli–Laplacian mixture, in which case certain conditional variance calculations could be done symbolically. In this letter, we extend this work to Cauchy distributions where only limited symbolic calculations are possible but numerical quadrature calculation of the variances leads to algorithms that have essentially the same low computational burden as the original algorithms along with striking preservation of edge detail in the images.

The recursive algorithms described here are based on related one-dimensional (1-D) algorithms used in equalization of digital communication channels [3] where the impulse response of the channel lasts many symbol times and the symbols are non-Gaussian. These algorithms are recursive Bayesian filters, optimal over intervals of one sample and which include an approach to complexity reduction based on modeling the sufficiently distant past as conditionally Gaussian given data up to the present. The motivation for extending this approach to nonsymmetric half-plane (NSHP) causal Markov random fields [1] was based on an observation by Jain [4] and others that image residuals are typically not Gaussian.

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II. Method

Let d_k be the residual at step k. In order to use the recursive algorithm of [1], it is necessary to compute the conditional mean and conditional variance of d_k given the data $Y_k = \{y_j : j \le k\}$. Using Bayes rule and the independence of the current residual d_k and the past measurements Y_{k-1}, we have $f(d_k|Y_k) = f(y_k|d_k, Y_{k-1})f(d_k)/\int f(y_k|d_k, Y_{k-1})f(d_k)d\,d_k$. Let

$$I_1 = \int f(y_k|d_k, Y_{k-1})f(d_k)d\,d_k \tag{1}$$

$$I_2 = \int d_k f(y_k|d_k, Y_{k-1})f(d_k)d\,d_k \tag{2}$$

$$I_3 = \int d_k^2 f(y_k|d_k, Y_{k-1})f(d_k)d\,d_k. \tag{3}$$

Then the mean and variance can be computed by $E(d_k|Y_k) = I_2/I_1$ and $\mathrm{Var}(d_k|Y_k) = I_3/I_1 - (I_2/I_1)^2$ (see [1]).

In the algorithm of [1], the probability density function (pdf) $f(y_k|d_k, Y_{k-1})$ is Gaussian with a mean and variance that are recursively propagated in the algorithm. In the results described in [1], the heavy-tailed pdf $f(d_k)$ is Laplacian, $f(d_k) = (\beta/2)\exp(-\beta|d_k|)$. In this letter, we consider the case where $f(d_k)$ is Cauchy $f(d_k) = (\alpha/\pi)/(\alpha^2 + d_k^2)$, which has much heavier tails. More generally, it is possible to consider pdfs proportional to $1/(1+|d_k/\alpha|^\beta)$ or $1/(1+d_k^2/\alpha)^\beta$, where the second choice, specialized to $\alpha = n$ and $\beta = (n+1)/2$ (n an integer), is the student pdf with n degrees of freedom [5(7.8.4)]. For none of these choices can the integrals in (1)–(3) be evaluated by standard tables or by Mathematica. Therefore, we have considered both numerical evaluation and series evaluation.

Series evaluation of (1)–(3) is based on the identity [6, [8.957 (1)]]$\exp(-t^2 + 2tx) = \sum_{k=0}^{\infty} (t^k/k!)H_k(x)$, where $H_k(\cdot)$ is the kth Hermite polynomial [6, Section 8.95]. Using the notation of [1], let $\tilde{y}_k$ be an innovation in y_k, which can be written $\tilde{y}_k = y_k - C^- \hat{S}^-_{k|k-1}$. Let w_k^- be the variance of $\tilde{y}_k$ and $\alpha' = \alpha/\sqrt{2w_k^-}$. Then

$$I_n = (2w_k^-)^{(n-2)/2} \sum_{m=0}^{\infty} \frac{t^m}{m!} \int x^{n-1} \cdot \exp(-x^2) H_m(x) \frac{\alpha'/\pi}{\alpha'^2 + x^2}\, dx. \tag{4}$$

Since $H_{2m}(x)$ is even and $H_{2m+1}(x)$ is odd, either the m even terms are zero (if $n-$ 1 is odd) or the m odd terms are zero (if $n-$ 1 is even). Expanding $H_m(x)$ in monomials in x and using

Mathematica to symbolically compute the integrals provides a symbolic solution for any truncation of the m summation. We truncate the infinite series at $m = 10$, which was determined by experiment and practical limits on computation. When the truncated series for the variance is less than 120 (it can even be negative), we replace the variance by 120 which was determined by experiment.

We base our quadrature approximation of (1)–(3) on truncation of the region of integration based on the Gaussian factor $f(y_k|d_k, Y_{k-1})$ in the integrand of I_n. In particular, we replace the region $R_\infty = (-\infty, +\infty)$ by the region $R_{\text{finite}} = (\tilde{y}_k - 4\sqrt{w_k^-}, \tilde{y}_k + 4\sqrt{w_k^-})$, where w_k^- is the variance of $\tilde{y}_k$. Then, using the region R_{finite}, we apply a 400 point Simpson rule.

We identify the autoregression coefficients in the Markov random field model by the same method used in [1]. After identifying the autoregression coefficients, we compute the residuals and the sample variance of the residuals, which is denoted by $\widehat{s^2}$.

We desire to use simple moment-based estimators of the scale parameter α in the Cauchy pdf. Since the Cauchy pdf does not have moments, we use the finite range of image data (e.g., $\{0, 1, \cdots, 255\}$ levels) to motivate truncating the Cauchy pdf to the region $[-D, +D]$ with the result that $f_D(\cdot)$, the truncated pdf, is $f_D(d) = \gamma f(d)$ for $|d| \le D$ and $= 0$ for $|d| > D$, where $\gamma = \pi/[2\arctan(D/\alpha)]$. The first moment of f_D is zero by symmetry and the second moment is $s^2(D, \alpha) = [D - \alpha\arctan(D/\alpha)]\alpha/\arctan(D/\alpha)$. We pick $D = 6[\widehat{s^2}]^{1/2}$ and α as the smallest positive root of $\widehat{s^2} = s^2(6[\widehat{s^2}]^{1/2}, \alpha)$.

III. Computational Results and Discussion

All of our results concern the autoregressive model $x_{(m,n)} = \sum_{(i,j)\in\mathcal{N}_1} \theta_{(i,j)} x_{(m,n)+(i,j)} + d_{(m,n)}$ with a nonsymmetric half plane neighborhood $\mathcal{N}_1 = \{(0,-1), (-1,-1), (-1,0), (-1,1)\}$. We consider four restoration methods that differ in the statistical model for the residual and method of computation: Gaussian with analytical expectation formulas [1], Laplacian with analytical expectation formulas [1], Cauchy with expectations computed by numerical quadratures (this letter, abbreviated by Cauchy/quadrature), and Cauchy with expectations computed by series solution [(4), abbreviated by Cauchy/series].

The Cauchy/quadrature method has several attractive features relative to these competing methods. First, as shown in the histogram of the residuals [Fig. 1(a)] for the Macaw image [Fig. 1(b)], the Cauchy pdf better represents the tails of the histogram than the other pdfs shown. Second, the computational cost of the Cauchy/quadrature method is acceptable. The Gaussian approach requires the least computation. If its computational time per pixel is normalized to 1, then the other three algorithms have times of 2.04 (Laplacian), 2.29 (Cauchy/quadrature), and 52.7 (Cauchy/series), all measured in clock time using Matlab programs. Because of its excessive cost, we do not consider the Cauchy/series approach further. Note that the cost of the Cauchy/quadrature method could be reduced by use of more sophisticated quadrature methods such as Gaussian or adaptive quadratures. Third, the visual impression of the Cauchy/quadrature method is superior, as can be seen in Fig. 1, especially in regions with detailed edge structure such as the regions surrounding the eyes.

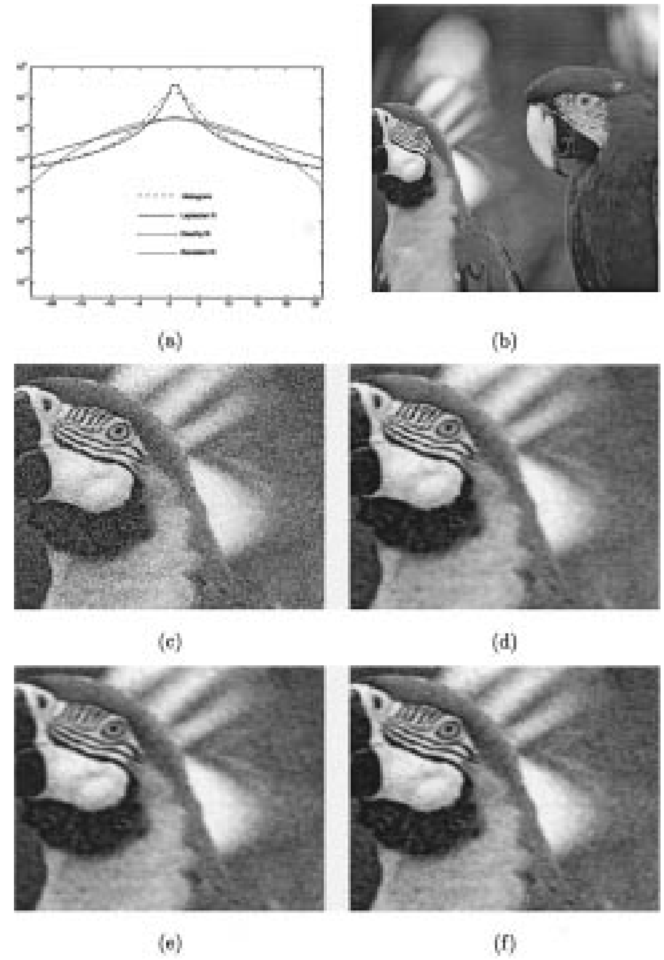

Fig. 1. Macaws image. (a) Histogram of the residuals, (b) true image, (c) degraded image (SNR = 5 dB), and (d)–(f) restorations using different statistical models for the residuals: (d) Cauchy ($\text{SNR}_{\text{imp}} = 9.20$ dB), (e) Gaussian ($\text{SNR}_{\text{imp}} = 8.78$ dB), and (f) Laplacian ($\text{SNR}_{\text{imp}} = 8.73$ dB). Since we have limited space and the most impressive performance gain due to the Cauchy approach is in regions of the image with detailed edge structure, we only show subimages for the noisy input and three restorations. The SNR numbers, however, are computed for the entire image. The SNR is the square of the l_2 norm of the true image divided by the variance of the Gaussian measurement noise. The SNR_{imp} is the square of the l_2 norm of the difference between the true and degraded images divided by the square of the l_2 norm of the difference between the true and reconstructed images.

References

[1] S. R. Kadaba, S. B. Gelfand, and R. L. Kashyap, "Recursive estimation of images using non-Gaussian autoregressive models," *IEEE Trans. Image Processing*, vol. 7, pp. 1439–1452, Oct. 1998.
[2] ——, "Bayesian decision feedback for segmentation of binary images," *IEEE Trans. Image Processing*, vol. 5, pp. 1163–1178, July 1996.
[3] G. K. Lee, S. B. Gelfand, and M. P. Fitz, "Bayesian techniques for known channel deconvolution," *IEEE Trans. Commun.*, vol. 44, pp. 448–455, Apr. 1996.
[4] A. K. Jain, *Fundamentals of Digital Image Processing*. Englewood Cliffs, NJ: Prentice-Hall, 1989.
[5] S. S. Wilks, *Mathematical Statistics*. London, U.K.: Wiley, 1962.
[6] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 4 ed. New York: Academic, 1980.