ECE 580 Spring 2009

Funwork #2

Solutions

1. Consider the problem of solving a jigsaw puzzle that consists of N pieces. Is this problem P, non-P, or NP? Justify your answer.

   Answer: This problem comes from the article "Million-Dollar Minesweeper" by Ian Stewart. The article is posted on the web site of the Clay Mathematics Institute. The answer to the question appears on page 3 of the article: "Solving the puzzle can be very hard, but if someone claims they've solved it, it usually takes no more than a quick glance to check whether they're right. To get a quantitative estimate of the running time, just look at each piece in turn and make sure that it fits the limited number of neighbours that adjoin it. The number of calculations required to do this is roughly proportional to the number of pieces, so the check runs in polynomial time. But you can't solve the puzzle that way. Neither can you try every potential solution in turn and check each as you go along, because the number of potential solutions grows much faster than any fixed power of the number of pieces." So a jigsaw puzzle is, in general, a type NP problem.

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2. Exercise **5.9** from TEXT on page 75. Verify your calculations using MATLAB's Symbolic Toolbox.

   Answer:

   (a) We have

   $$f(\boldsymbol{x}) = f(\boldsymbol{x}_0) + Df(\boldsymbol{x}_0)(\boldsymbol{x}-\boldsymbol{x}_0) + \frac{1}{2}(\boldsymbol{x}-\boldsymbol{x}_0)^T D^2 f(\boldsymbol{x}_0)(\boldsymbol{x}-\boldsymbol{x}_0) + \cdots$$

   We compute

   $$Df(\boldsymbol{x}) = \begin{bmatrix} e^{-x_2} & -x_1 e^{-x_2} + 1 \end{bmatrix},$$

   and

   $$D^2 f(\boldsymbol{x}) = \begin{bmatrix} 0 & -e^{-x_2} \\ -e^{-x_2} & x_1 e^{-x_2} \end{bmatrix}.$$

Hence,

$$\begin{aligned} f(\boldsymbol{x}) &= 2 + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 \end{bmatrix} \\ &\quad + \frac{1}{2} \begin{bmatrix} x_1 - 1 & x_2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 \end{bmatrix} + \cdots \\ &= 1 + x_1 + x_2 - x_1 x_2 + \frac{1}{2} x_2^2 + \cdots \end{aligned}$$

(b) We compute

$$Df(\boldsymbol{x}) = \begin{bmatrix} 4x_1^3 + 4x_1 x_2^2 & 4x_1^2 x_2 + 4x_2^3 \end{bmatrix},$$

and

$$D^2 f(\boldsymbol{x}) = \begin{bmatrix} 12x_1^2 + 4x_2^2 & 8x_1 x_2 \\ 8x_1 x_2 & 4x_1^2 + 12x_2^2 \end{bmatrix}.$$

Hence,

$$\begin{aligned} f(\boldsymbol{x}) &= 4 + \begin{bmatrix} 8 & 8 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \\ &\quad + \frac{1}{2} \begin{bmatrix} x_1 - 1 & x_2 - 1 \end{bmatrix} \begin{bmatrix} 16 & 8 \\ 8 & 16 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} + \cdots \\ &= 12 + 8x_1^2 + 8x_2^2 - 16x_1 - 16x_2 + 8x_1 x_2 + \cdots \end{aligned}$$

(c) We compute

$$Df(\boldsymbol{x}) = \begin{bmatrix} e^{x_1 - x_2} + e^{x_1 + x_2} + 1 & -e^{x_1 - x_2} + e^{x_1 + x_2} + 1 \end{bmatrix},$$

and

$$D^2 f(\boldsymbol{x}) = \begin{bmatrix} e^{x_1 - x_2} + e^{x_1 + x_2} & -e^{x_1 - x_2} + e^{x_1 + x_2} \\ -e^{x_1 - x_2} + e^{x_1 + x_2} & e^{x_1 - x_2} + e^{x_1 + x_2} \end{bmatrix}.$$

Hence,

$$\begin{aligned} f(\boldsymbol{x}) &= 2 + 2e + \begin{bmatrix} 2e + 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 \end{bmatrix} \\ &\quad + \frac{1}{2} \begin{bmatrix} x_1 - 1 & x_2 \end{bmatrix} \begin{bmatrix} 2e & 0 \\ 0 & 2e \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 \end{bmatrix} + \cdots \\ &= 1 + x_1 + x_2 + e\left(1 + x_1^2 + x_2^2\right) + \cdots \end{aligned}$$

An example of the MATLAB code

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%ECE 580 Funwork 2 Problem 2(a)
clear
syms x1 x2
f=x1*exp(-x2)+x2+1;
D1=jacobian(f);
D2=jacobian(D1);

% Second-order Taylor series expansion
syms x_1 x_2
f_T=f+D1*[x_1-x1;x_2-x2]+0.5*[x_1-x1,x_2-x2]*D2*[x_1-x1;x_2-x2];
f_1=subs(f_T,x1,1);
f_2=subs(f_1,x2,0);
f_T=expand(f_2)
pretty(f_T)
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3. Compute the linear, $l(x_1, x_2)$, and quadratic, $q(x_1, x_2)$, approximations of the function

$$f = f(x_1, x_2) = x_1^3 + x_1 x_2 - x_1^2 x_2^2,$$

at the point $\boldsymbol{x}^{(0)} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

Answer:

(i) We use the first-order Taylor series expansion to obtain a linear approximation of f,

$$l(\boldsymbol{x}) = f\left(\boldsymbol{x}^{(0)}\right) + \nabla f\left(\boldsymbol{x}^{(0)}\right)^T \left(\boldsymbol{x} - \boldsymbol{x}^{(0)}\right),$$

where

$$\nabla f\left(\boldsymbol{x}^{(0)}\right) = \left.\begin{bmatrix} 3x_1^2 + x_2 - 2x_1 x_2^2 \\ x_1 - 2x_1^2 x_2 \end{bmatrix}\right|_{\boldsymbol{x}=\boldsymbol{x}^{(0)}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Hence,

$$\begin{aligned} l(\boldsymbol{x}) &= 1 + \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \\ &= 2x_1 - x_2. \end{aligned}$$

(ii) The quadratic approximation of f at the point $\boldsymbol{x}^{(0)}$ has the form,

$$q(\boldsymbol{x}) = f\left(\boldsymbol{x}^{(0)}\right) + \nabla f\left(\boldsymbol{x}^{(0)}\right)^T \left(\boldsymbol{x} - \boldsymbol{x}^{(0)}\right) + \frac{1}{2}\left(\boldsymbol{x} - \boldsymbol{x}^{(0)}\right)^T \boldsymbol{F}\left(\boldsymbol{x}^{(0)}\right)\left(\boldsymbol{x} - \boldsymbol{x}^{(0)}\right),$$

where

$$\boldsymbol{F}\left(\boldsymbol{x}^{(0)}\right) = \left.\begin{bmatrix} 6x_1 - 2x_2^2 & 1 - 4x_1x_2 \\ 1 - 4x_1x_2 & -2x_1^2 \end{bmatrix}\right|_{\boldsymbol{x}=\boldsymbol{x}^{(0)}} = \begin{bmatrix} 4 & -3 \\ -3 & -2 \end{bmatrix}.$$

Hence,

$$\begin{aligned} q(\boldsymbol{x}) &= 1 + \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x_1 - 1 & x_2 - 1 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \\ &= 2x_1^2 - 3x_1x_2 - x_2^2 + x_1 + 4x_2 - 2 \end{aligned}$$

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4. (a) Find $Df(\boldsymbol{x})$ of

$$f(\boldsymbol{x}) = \boldsymbol{x}^T \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \boldsymbol{x} - \boldsymbol{x}^T \begin{bmatrix} -2 \\ 3 \end{bmatrix} + \pi$$

(b) Find the Hessian of

$$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T \begin{bmatrix} 2 & 3 \\ 7 & 1 \end{bmatrix} \boldsymbol{x} + \boldsymbol{x}^T \begin{bmatrix} 1 \\ -3 \end{bmatrix} + \log 3$$

Answer:

(a) Let

$$f(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{b} + c = \frac{1}{2}\boldsymbol{x}^T\left(\boldsymbol{Q} + \boldsymbol{Q}^T\right)\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{b} + c.$$

Then,

$$\begin{aligned} Df(\boldsymbol{x}) &= \boldsymbol{x}^T(\boldsymbol{Q} + \boldsymbol{Q}^T) - \boldsymbol{b}^T \\ &= \boldsymbol{x}^T \begin{bmatrix} 2 & 7 \\ 7 & 6 \end{bmatrix} - \begin{bmatrix} -2 & 3 \end{bmatrix}. \end{aligned}$$

(b) The Hessian of $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{b} + c = \frac{1}{4}\boldsymbol{x}^T\left(\boldsymbol{Q}+\boldsymbol{Q}^T\right)\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{b} + c$ is

$$\boldsymbol{F}(\boldsymbol{x}) = \frac{1}{2}\left(\boldsymbol{Q}+\boldsymbol{Q}^T\right) = \frac{1}{2}\begin{bmatrix} 4 & 10 \\ 10 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 1 \end{bmatrix}.$$

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5. For the function

$$f = f(x_1, x_2) = e^{x_1 x_2^2},$$

(a) find the gradient of f at $\boldsymbol{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$;

(b) find the rate of increase of f at the point $\boldsymbol{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ in the direction $\boldsymbol{d} = \begin{bmatrix} -3 & 4 \end{bmatrix}^T$;

(c) find the direction of maximum rate of increase at $\boldsymbol{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. What is the rate of increase in this direction?

Answer:

(a) The gradient of f is

$$\nabla f(\boldsymbol{x}) = \begin{bmatrix} x_2^2 e^{x_1 x_2^2} & 2x_1 x_2 e^{x_1 x_2^2} \end{bmatrix}^T.$$

The gradient of f at $\boldsymbol{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ is

$$\nabla f = \begin{bmatrix} e & 2e \end{bmatrix}^T.$$

(b) We first normalize the vector $\boldsymbol{d}$ to obtain

$$\frac{\boldsymbol{d}}{\|\boldsymbol{d}\|} = \frac{1}{\sqrt{25}}\begin{bmatrix} -3 & 4 \end{bmatrix}^T = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \end{bmatrix}^T.$$

Then the rate of increase of f at the point $\boldsymbol{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ in the direction $\boldsymbol{d} = \begin{bmatrix} -3 & 4 \end{bmatrix}^T$ is

$$\nabla f^T \frac{\boldsymbol{d}}{\|\boldsymbol{d}\|} = e$$

(c) The direction of maximum rate increase at $\boldsymbol{x} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ is the gradient of f at that point. The rate of increase is

$$\nabla f^T \frac{\nabla f}{\|\nabla f\|} = \|\nabla f\| = \left(e^2 + 4e^2\right)^{1/2} = \sqrt{5}e$$

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6. Find the range of values of the parameter ξ for which the function

$$f(x_1, x_2, x_3) = 2x_1x_3 - x_1^2 - x_2^2 - 5x_3^2 - 2\xi x_1x_2 - 4x_2x_3,$$

is negative semi-definite.

Answer: The function f is negative semi-definite if and only if its negative is positive semi-definite. We therefore find the range of the parameter ξ values for which the function $-f$ is positive semi-definite. The above quadratic form can be represented as

$$-f = \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} = \boldsymbol{x}^T \begin{bmatrix} 1 & \xi & -1 \\ \xi & 1 & 2 \\ -1 & 2 & 5 \end{bmatrix} \boldsymbol{x}.$$

The function $-f$ is positive semi-defnite if and only if all principal minors of $\boldsymbol{Q}$, not just the leading principal minors, are non-negative. It is easy to see that the three first-order principal minors (diagonal elements of $\boldsymbol{Q}$) are all positive. There are three second-order principal minors. Only one of them, the leading principal minor, is a function of the parameter ξ,

$$\det \boldsymbol{Q} \begin{pmatrix} 1\,2 \\ 1\,2 \end{pmatrix} = \det \begin{bmatrix} 1 & \xi \\ \xi & 1 \end{bmatrix} = 1 - \xi^2.$$

Thus the above second-order leading principal minor is non-negative if and only if

$$\xi \in \left[\ -1,\ \ 1\ \right]. \tag{1}$$

The other second-order principal minors are

$$\det \boldsymbol{Q} \begin{pmatrix} 1\,3 \\ 1\,3 \end{pmatrix} \qquad \text{and} \qquad \det \boldsymbol{Q} \begin{pmatrix} 2\,3 \\ 2\,3 \end{pmatrix}$$

and they are positive. There is only one third-order principal minor, $\det \boldsymbol{Q}$, where

$$\begin{aligned}\det \boldsymbol{Q} &= \det \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} - \xi \det \begin{bmatrix} \xi & -1 \\ 2 & 5 \end{bmatrix} - \det \begin{bmatrix} \xi & -1 \\ 1 & 2 \end{bmatrix} \\ &= 1 - \xi(5\xi + 2) - (2\xi + 1) \\ &= -5\xi^2 - 4\xi.\end{aligned}$$

The third-order principal minor is non-negative if and only if, $-\xi(5\xi+4) \geq 0$, that is, if and only if

$$\xi \in \left[\ -4/5, \quad 0\ \right]. \tag{2}$$

Combining (1) and (2), we conclude that the function f is negative semi-definite if and only if

$$\boxed{\xi \in \left[\ -4/5, \quad 0\ \right]}$$

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7. For the function

$$f = f(x_1, x_2) = \frac{1}{3}x_1^3 + \frac{1}{2}x_1^2 + 2x_1x_2 + \frac{1}{2}x_2^2 - x_2 + 5,$$

(a) find points that satisfy the first-order necessary conditions for the extremum;

(b) which point is a strict local minimizer? Justify your answer.

Answer:

(a) To find the points that satisfy the first-order necessary conditions for the extremum, we solve the equation

$$\nabla f(\boldsymbol{x}) = \begin{bmatrix} x_1^2 + x_1 + 2x_2 \\ 2x_1 + x_2 - 1 \end{bmatrix} = \mathbf{0}.$$

The second component of the above condition gives, $x_2 = 1 - 2x_1$. Substituting the above into the first component yields

$$x_1^2 - 3x_1 + 2 = (x_1 - 1)(x_1 - 2) = 0.$$

From the above, we obtain two points that satisfy the first-order necessary conditions for the extremum,

$$\boldsymbol{x}^{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}^{(2)} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

(b) We first compute the Hessian of f,

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{bmatrix} 2x_1 + 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

The Hessian evaluated at $\boldsymbol{x}^{(1)}$ is

$$\boldsymbol{F}\left(\boldsymbol{x}^{(1)}\right) = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix},$$

which is indefinite. So $\boldsymbol{x}^{(1)}$ is neither a minimizer nor a maximizer of f. The Hessian of f evaluated at $\boldsymbol{x}^{(2)}$ is

$$\boldsymbol{F}\left(\boldsymbol{x}^{(2)}\right) = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix},$$

which is positive definite. So $\boldsymbol{x}^{(2)}$ is a strict local minimizer.
