ECE 580 Spring 2009

Funwork #1

Solutions

1.

- For the matrix,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 2 & -1 & 3 & 0 & 1 \\ 3 & 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 1 & 1 \end{bmatrix},$$

  find its rank by first transforming the matrix by means of the row elementary operations into an upper triangular form.

- Find the rank of the following matrix for different values of the parameter γ by first transforming the matrix by means of the row elementary operations into an upper triangular form,

$$\boldsymbol{A} = \begin{bmatrix} 1 & \gamma & -1 & 2 \\ 2 & -1 & \gamma & 5 \\ 1 & 10 & -6 & 1 \end{bmatrix}.$$

Solution:

- Performing a sequence of row elementary operations, we obtain

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 2 & -1 & 3 & 0 & 1 \\ 3 & 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & -5 & 5 & -6 & -3 \\ 0 & -5 & 5 & -6 & -3 \\ 0 & 0 & 4 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & -5 & 5 & -6 & -3 \\ 0 & 0 & 4 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \boldsymbol{B}$$

  Because elementary operations do not change the rank of a matrix, hence $\text{rank}(\boldsymbol{A}) = \text{rank}(\boldsymbol{B})$. Therefore $\text{rank}(\boldsymbol{A}) = 3$.

- Performing a sequence of row elementary operations, we obtain

$$\boldsymbol{A} = \begin{bmatrix} 1 & \gamma & -1 & 2 \\ 2 & -1 & \gamma & 5 \\ 1 & 10 & -6 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 10 & -6 & 1 \\ 1 & \gamma & -1 & 2 \\ 2 & -1 & \gamma & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 10 & -6 & 1 \\ 0 & \gamma - 10 & 5 & 1 \\ 0 & -21 & \gamma + 12 & 3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 10 & -6 \\ 0 & 1 & \gamma - 10 & 5 \\ 0 & 3 & -21 & \gamma + 12 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 10 & -6 \\ 0 & 1 & \gamma - 10 & 5 \\ 0 & 0 & -3(\gamma - 3) & \gamma - 3 \end{bmatrix} = \boldsymbol{B}$$

Because elementary operations do not change the rank of a matrix, hence rank$(\boldsymbol{A})$ = rank$(\boldsymbol{B})$. Therefore rank$(\boldsymbol{A}) = 3$ if $\gamma \neq 3$ and rank$(\boldsymbol{A}) = 2$ if $\gamma = 3$.

2. Consider the following system of equations,

$$\left.\begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 1 \\ x_1 - 2x_2 - x_4 &= -2 \end{aligned}\right\}$$

Use Theorem 2.1 to check if the system has a solution. Then, use the method of the proof of Theorem 2.2 to find a general solution to the system.

Solution: We represent the given system of equations in the form $\boldsymbol{Ax} = \boldsymbol{b}$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & -2 & 0 & -1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \text{ and } \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Using row elementary operations yields

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & -2 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -3 & -2 & -2 \end{bmatrix}, \text{and}$$

$$[\boldsymbol{A}, \boldsymbol{b}] = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & -2 & 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & -3 & -2 & -2 & -3 \end{bmatrix},$$

from which rank$(A) = 2$ and rank$\,[\boldsymbol{A}, \boldsymbol{b}] = 2$.

Therefore, by Theorem 2.1 the system has a solution.

We next represent the system of equations as

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 - 2x_3 - x_4 \\ -2 + x_4 \end{bmatrix}$$

Assigning arbitrary values to x_3 and x_4 ($x_3 = d_3$, $x_4 = d_4$), we get

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 - 2x_3 - x_4 \\ -2 + x_4 \end{bmatrix} \\ &= -\frac{1}{3} \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 - 2x_3 - x_4 \\ -2 + x_4 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{4}{3}d_3 - \frac{1}{3}d_4 \\ 1 - \frac{2}{3}d_3 - \frac{2}{3}d_4 \end{bmatrix}. \end{aligned}$$

Therefore, a general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3}d_3 - \frac{1}{3}d_4 \\ 1 - \frac{2}{3}d_3 - \frac{2}{3}d_4 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} d_3 + \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ 0 \\ 1 \end{bmatrix} d_4 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

where d_3 and d_4 are arbitrary values.

3. Find the nullspace of

$$\boldsymbol{A} = \begin{bmatrix} 4 & -2 & 0 \\ 2 & 1 & -1 \\ 2 & -3 & 1 \end{bmatrix}$$

Solution: The null space of $\boldsymbol{A}$ is $\mathcal{N}(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{A}\boldsymbol{x} = 0\}$. Using elementary row operations and back-substitution, we can solve the system of equations:

$$\begin{bmatrix} 4 & -2 & 0 \\ 2 & 1 & -1 \\ 2 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 4 & -2 & 0 \\ 0 & 2 & -1 \\ 0 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 4 & -2 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} 4x_1 - 2x_2 &= 0 \\ 2x_2 - x_3 &= 0 \end{aligned}$$

$$\Rightarrow \qquad x_2 = \frac{1}{2}x_3\ , \qquad x_1 = \frac{1}{2}x_2 = \frac{1}{4}x_3 \qquad \Rightarrow \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} x_3\ .$$

Therefore, $\mathcal{N}(\boldsymbol{A}) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} c : c \in \mathbb{R} \right\}$.

4. Find the transformation matrix $\boldsymbol{T}$ from $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ to $\{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3\}$, where

(a) $\boldsymbol{e}'_1 = \boldsymbol{e}_1 + 3\boldsymbol{e}_2 - 4\boldsymbol{e}_3$, $\boldsymbol{e}'_2 = 2\boldsymbol{e}_1 - \boldsymbol{e}_2 + 5\boldsymbol{e}_3$, $\boldsymbol{e}'_3 = 4\boldsymbol{e}_1 + 5\boldsymbol{e}_2 + 3\boldsymbol{e}_3$.

(b) $\boldsymbol{e}_1 = \boldsymbol{e}'_1 + \boldsymbol{e}'_2 + 3\boldsymbol{e}'_3$, $\boldsymbol{e}_2 = 2\boldsymbol{e}'_1 - \boldsymbol{e}'_2 + 4\boldsymbol{e}'_3$, $\boldsymbol{e}_3 = 3\boldsymbol{e}'_1 + 5\boldsymbol{e}'_3$.

Solution:

(a)

$$\begin{bmatrix} \boldsymbol{e}'_1 & \boldsymbol{e}'_2 & \boldsymbol{e}'_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & -1 & 5 \\ -4 & 5 & 3 \end{bmatrix} .$$

Therefore,

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{e}'_1 & \boldsymbol{e}'_2 & \boldsymbol{e}'_3 \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & -1 & 5 \\ -4 & 5 & 3 \end{bmatrix}^{-1} = \frac{1}{42} \begin{bmatrix} 28 & -14 & -14 \\ 29 & -19 & -7 \\ -11 & 13 & 7 \end{bmatrix} .$$

(b)

$$\begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}'_1 & \boldsymbol{e}'_2 & \boldsymbol{e}'_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 3 & 4 & 5 \end{bmatrix} .$$

Therefore,

$$
\boldsymbol{T} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 3 & 4 & 5 \end{bmatrix} .
$$

5. Given two bases, $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ and $\{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3, \boldsymbol{e}'_4\}$ of $\mathbb{R}^4$, where $\boldsymbol{e}'_1 = \boldsymbol{e}_1$, $\boldsymbol{e}'_2 = \boldsymbol{e}_1 + \boldsymbol{e}_2$, $\boldsymbol{e}'_3 = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$, $\boldsymbol{e}'_4 = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 + \boldsymbol{e}_4$, and the matrix representation of a linear transformation in $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ of the form

$$
\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 1 & 0 \\ -3 & 2 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & 0 & 3 \end{bmatrix} .
$$

Find the matrix representation of the linear transformation in the basis $\{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3, \boldsymbol{e}'_4\}$.

Solution:

$$
\begin{bmatrix} \boldsymbol{e}'_1 & \boldsymbol{e}'_2 & \boldsymbol{e}'_3 & \boldsymbol{e}'_4 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} .
$$

Therefore, the transformation matrix from $\{\boldsymbol{e}_1\ ,\ \boldsymbol{e}_2\ ,\ \boldsymbol{e}_3\ ,\ \boldsymbol{e}_4\}$ to $\{\boldsymbol{e}'_1\ ,\ \boldsymbol{e}'_2\ ,\ \boldsymbol{e}'_3\ ,\ \boldsymbol{e}'_4\}$ is

$$
\boldsymbol{T} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} .
$$

Now consider a linear transformation $L : \mathbb{R}^4 \to \mathbb{R}^4$, and let $\boldsymbol{A}$ be its representation with respect to $\{\boldsymbol{e}_1\ ,\ \boldsymbol{e}_2\ ,\ \boldsymbol{e}_3\ ,\ \boldsymbol{e}_4\}$, and $\boldsymbol{B}$ its representation with respect to $\{\boldsymbol{e}'_1\ ,\ \boldsymbol{e}'_2\ ,\ \boldsymbol{e}'_3\ ,\ \boldsymbol{e}'_4\}$. Let $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ and $\boldsymbol{y}' = \boldsymbol{B}\boldsymbol{x}'$. Then,

$$
\boldsymbol{y}' = \boldsymbol{T}\boldsymbol{y} = \boldsymbol{T}(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{T}\boldsymbol{A}(\boldsymbol{T}^{-1}\boldsymbol{x}') = (\boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1})\boldsymbol{x}' .
$$

Therefore,

$$\boldsymbol{B} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} = \begin{bmatrix} 5 & 3 & 4 & 3 \\ -3 & -2 & -1 & -2 \\ -1 & 0 & -1 & -2 \\ 1 & 1 & 1 & 4 \end{bmatrix}.$$

6. Given two bases, $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and $\{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3\}$ of $\mathbb{R}^3$, where $\boldsymbol{e}_1 = 2\boldsymbol{e}'_1 + \boldsymbol{e}'_2 - \boldsymbol{e}'_3$, $\boldsymbol{e}_2 = 2\boldsymbol{e}'_1 - \boldsymbol{e}'_2 + 2\boldsymbol{e}'_3$, $\boldsymbol{e}_3 = 3\boldsymbol{e}'_1 + \boldsymbol{e}'_3$, and the matrix representation of a linear transformation in $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of the form

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find the matrix representation of the linear transformation in the basis $\{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3\}$.

Solution:

$$\begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}'_1 & \boldsymbol{e}'_2 & \boldsymbol{e}'_3 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix}.$$

Therefore, the transformation matrix from $\{\boldsymbol{e}'_1\ ,\ \boldsymbol{e}'_2\ ,\ \boldsymbol{e}'_3\}$ to $\{\boldsymbol{e}_1\ ,\ \boldsymbol{e}_2\ ,\ \boldsymbol{e}_3\}$ is

$$\boldsymbol{T} = \begin{bmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{bmatrix},$$

and the representation of the linear transformation with respect to $\{\boldsymbol{e}'_1\ ,\ \boldsymbol{e}'_2\ ,\ \boldsymbol{e}'_3\}$ is

$$\boldsymbol{B} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} = \begin{bmatrix} 3 & -10 & -8 \\ -1 & 8 & 4 \\ 2 & -13 & -7 \end{bmatrix}.$$

7. Find the basis in which the matrix

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 5 & 2 & 1 \\ -1 & 1 & 0 & 3 \end{bmatrix}$$

is diagonal

Solution: Let $\{\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \boldsymbol{v}_3,\ \boldsymbol{v}_4\}$ be a set of linearly independent eigenvectors of $\boldsymbol{A}$ corresponding to the eigenvalues λ_1, λ_2, λ_3, and λ_4. Let $T = [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3\ \boldsymbol{v}_4]$. Then,

$$\begin{aligned} \boldsymbol{AT} &= \boldsymbol{A}[\,\boldsymbol{v}_1\ \ \boldsymbol{v}_2\ \ \boldsymbol{v}_3\ \ \boldsymbol{v}_4\,] = [\,\boldsymbol{Av}_1\ \ \boldsymbol{Av}_2\ \ \boldsymbol{Av}_3\ \ \boldsymbol{Av}_4\,] \\ &= [\,\lambda_1\boldsymbol{v}_1\ \ \lambda_2\boldsymbol{v}_2\ \ \lambda_3\boldsymbol{v}_3\ \ \lambda_4\boldsymbol{v}_4\,] = [\,\boldsymbol{v}_1\ \ \boldsymbol{v}_2\ \ \boldsymbol{v}_3\ \ \boldsymbol{v}_4\,]\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}. \end{aligned}$$

Hence, $\boldsymbol{AT} = \boldsymbol{T}\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$, or $\boldsymbol{T}^{-1}\boldsymbol{AT} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$.

Therefore, the matrix $\boldsymbol{A}$ has a diagonal form with respect to the basis formed by a linearly independent set of eigenvectors.

Because

$$\det(\boldsymbol{A}) = (\lambda - 2)(\lambda - 3)(\lambda - 1)(\lambda + 1),$$

the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 1$, and $\lambda_4 = -1$.

From $\boldsymbol{Av}_i = \lambda_i\boldsymbol{v}_i$, where $\boldsymbol{v}_i \neq 0$ $(i = 1, 2, 3, 4)$, the corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 9 \\ 1 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{v}_4 = \begin{bmatrix} 24 \\ -12 \\ 1 \\ 9 \end{bmatrix}.$$

Therefore, the basis is $\{\boldsymbol{v}_1\ ,\ \boldsymbol{v}_2\ ,\ \boldsymbol{v}_3\ ,\ \boldsymbol{v}_4\} = \left\{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 9 \\ 1 \end{bmatrix}, \begin{bmatrix} 24 \\ -12 \\ 1 \\ 9 \end{bmatrix}\right\}$.

8. Determine if the following quadratic forms are positive definite, negative definite, positive semidefinite, negative semidefinite, or indefinite:

(a) $f(x_1, x_2, x_3) = x_2^2$;

(b) $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - x_1x_3$;

(c) $f(x_1, x_2, x_3) = x_1^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$.

Solution:

(a)

$$f(x_1, x_2, x_3) = x_2^2 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Then, $\boldsymbol{Q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and the eigenvalues of $\boldsymbol{Q}$ are $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 0$. Therefore, the quadratic form is positive semidefinite.

(b)

$$f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - x_1x_3 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 2 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Then, $\boldsymbol{Q} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 2 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix}$ and the eigenvalues of $\boldsymbol{Q}$ are $\lambda_1 = 2$, $\lambda_2 = (1 - \sqrt{2})/2$, and $\lambda_3 = (1 + \sqrt{2})/2$. Therefore, the quadratic form is indefinite.

(c)

$$f(x_1, x_2, x_3) = x_1^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Then, $\boldsymbol{Q} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and the eigenvalues of $\boldsymbol{Q}$ are $\lambda_1 = 0$, $\lambda_2 = 1 - \sqrt{3}$, and $\lambda_3 = 1 + \sqrt{3}$. Therefore, the quadratic form is indefinite.

9. Find a transformation that brings the following quadratic form into the diagonal form,

$$f(x_1, x_2, x_3) = 4x_1^2 + x_2^2 + 9x_3^2 - 4x_1x_2 - 6x_2x_3 + 12x_1x_3.$$

Solution:

$$\begin{aligned} f(x_1, x_2, x_3) &= 4x_1^2 + x_2^2 + 9x_3^2 - 4x_1x_2 - 6x_2x_3 + 12x_1x_3 \\ &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & -2 & 6 \\ -2 & 1 & -3 \\ 6 & -3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \end{aligned}$$

Let $\boldsymbol{Q} = \begin{bmatrix} 4 & -2 & 6 \\ -2 & 1 & -3 \\ 6 & -3 & 9 \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3$, where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ are the natural basis for $\mathbb{R}^3$.

Let $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ be another basis for $\mathbb{R}^3$. Then, the vector $\boldsymbol{x}$ is represented in the new basis as $\tilde{\boldsymbol{x}}$, where $\boldsymbol{x} = [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3]\,\tilde{\boldsymbol{x}} = \boldsymbol{V}\tilde{\boldsymbol{x}}$.

Now, $f(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} = (\boldsymbol{V}\tilde{\boldsymbol{x}})^T\boldsymbol{Q}(\boldsymbol{V}\tilde{\boldsymbol{x}}) = \tilde{\boldsymbol{x}}^T(\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V})\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{x}}^T\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{x}}$, where

$$\tilde{\boldsymbol{Q}} = \begin{bmatrix} \tilde{q}_{11} & \tilde{q}_{12} & \tilde{q}_{13} \\ \tilde{q}_{21} & \tilde{q}_{22} & \tilde{q}_{23} \\ \tilde{q}_{31} & \tilde{q}_{32} & \tilde{q}_{33} \end{bmatrix},$$

and $\tilde{q}_{ij} = \boldsymbol{v}_i^T\boldsymbol{Q}\boldsymbol{v}_j$ for $i, j = 1, 2, 3$.

We are going to find a basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ such that $\tilde{q}_{ij} = 0$ for $i \neq j$, which is in the form

$$\begin{aligned} \boldsymbol{v}_1 &= \alpha_{11}\boldsymbol{e}_1 \\ \boldsymbol{v}_2 &= \alpha_{21}\boldsymbol{e}_1 + \alpha_{22}\boldsymbol{e}_2 \\ \boldsymbol{v}_3 &= \alpha_{31}\boldsymbol{e}_1 + \alpha_{32}\boldsymbol{e}_2 + \alpha_{33}\boldsymbol{e}_3 \end{aligned}$$

Since $\tilde{q}_{ij} = \boldsymbol{v}_i^T \boldsymbol{Q} \boldsymbol{v}_j = \boldsymbol{v}_i^T \boldsymbol{Q}(\alpha_{j1}\boldsymbol{e}_1 + \cdots + \alpha_{jj}\boldsymbol{e}_j) = \alpha_{j1}(\boldsymbol{v}_i^T \boldsymbol{Q}\boldsymbol{e}_1) + \cdots + \alpha_{jj}(\boldsymbol{v}_i^T \boldsymbol{Q}\boldsymbol{e}_j)$, if $\boldsymbol{v}_i^T \boldsymbol{Q}\boldsymbol{e}_j = 0$ for $j < i$ the $\boldsymbol{v}_i^T \boldsymbol{Q}\boldsymbol{v}_j = 0$.

In this case $\tilde{q}_{ii} = \boldsymbol{v}_i^T \boldsymbol{Q} \boldsymbol{v}_i = \boldsymbol{v}_i^T \boldsymbol{Q}(\alpha_{i1}\boldsymbol{e}_1 + \cdots + \alpha_{ii}\boldsymbol{e}_i) = \alpha_{i1}(\boldsymbol{v}_i^T \boldsymbol{Q}\boldsymbol{e}_1) + \cdots + \alpha_{ii}(\boldsymbol{v}_i^T \boldsymbol{Q}\boldsymbol{e}_i) = \alpha_{ii}(\boldsymbol{v}_i^T \boldsymbol{Q}\boldsymbol{e}_i)$.

Our task therefore is to find $\boldsymbol{v}_i$ $(i = 1, 2, 3)$ such that

$$\begin{aligned} \boldsymbol{v}_i^T \boldsymbol{Q}\boldsymbol{e}_j &= 0\,, \qquad j < i \\ \boldsymbol{v}_i^T \boldsymbol{Q}\boldsymbol{e}_i &= 1\,, \end{aligned}$$

and, in this case, we get

$$\tilde{\boldsymbol{Q}} = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{bmatrix} .$$

- Case of $i = 1$.

  From $\boldsymbol{v}_1^T \boldsymbol{Q}\boldsymbol{e}_1 = 1$,

$$(\alpha_{11}\boldsymbol{e}_1)^T \boldsymbol{Q}\boldsymbol{e}_1 = \alpha_{11}(\boldsymbol{e}_1^T \boldsymbol{Q}\boldsymbol{e}_1) = \alpha_{11} q_{11} = 1\,.$$

  Therefore,

$$\alpha_{11} = \frac{1}{q_{11}} = \frac{1}{\Delta_1} = \frac{1}{4} \qquad \Rightarrow \qquad \boldsymbol{v}_1 = \alpha_{11}\boldsymbol{e}_1 = \begin{bmatrix} \frac{1}{4} \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

- Case of $i = 2$.

  From $\boldsymbol{v}_2^T \boldsymbol{Q}\boldsymbol{e}_1 = 0$,

$$(\alpha_{21}\boldsymbol{e}_1 + \alpha_{22}\boldsymbol{e}_2)^T \boldsymbol{Q}\boldsymbol{e}_1 = \alpha_{21}(\boldsymbol{e}_1^T \boldsymbol{Q}\boldsymbol{e}_1) + \alpha_{22}(\boldsymbol{e}_2^T \boldsymbol{Q}\boldsymbol{e}_1) = \alpha_{21} q_{11} + \alpha_{22} q_{21} = 0\,.$$

  From $\boldsymbol{v}_2^T \boldsymbol{Q}\boldsymbol{e}_2 = 1$,

$$(\alpha_{21}\boldsymbol{e}_1 + \alpha_{22}\boldsymbol{e}_2)^T \boldsymbol{Q}\boldsymbol{e}_2 = \alpha_{21}(\boldsymbol{e}_1^T \boldsymbol{Q}\boldsymbol{e}_2) + \alpha_{22}(\boldsymbol{e}_2^T \boldsymbol{Q}\boldsymbol{e}_2) = \alpha_{21} q_{12} + \alpha_{22} q_{22} = 1\,.$$

Therefore,

$$\begin{bmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{bmatrix} \begin{bmatrix} \alpha_{21} \\ \alpha_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

But, since $\Delta_2 = 0$, this system of equations is inconsistent.

Hence, in this problem $\boldsymbol{v}_2^T \boldsymbol{Q} \boldsymbol{e}_2 = 0$ should be satisfied instead of $\boldsymbol{v}_2^T \boldsymbol{Q} \boldsymbol{e}_2 = 1$ so that the system can have a solution. In this case, the diagonal matrix becomes

$$\tilde{\boldsymbol{Q}} = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_{33} \end{bmatrix} ,$$

and the system of equations become

$$\begin{bmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{bmatrix} \begin{bmatrix} \alpha_{21} \\ \alpha_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha_{21} \\ \alpha_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \alpha_{22} ,$$

where α_{22} is an arbitrary real number. Thus,

$$\boldsymbol{v}_2 = \alpha_{21} \boldsymbol{e}_1 + \alpha_{22} \boldsymbol{e}_2 = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} a ,$$

where a is an arbitrary real number.

- Case of $i = 3$.

  Since in this case $\Delta_3 = \det(\boldsymbol{Q}) = 0$, we will have to apply the same reasoning of the previous case and use the condition $\boldsymbol{v}_3^T \boldsymbol{Q} \boldsymbol{e}_3 = 0$ instead of $\boldsymbol{v}_3^T \boldsymbol{Q} \boldsymbol{e}_3 = 1$. In this way the diagonal matrix becomes

$$\tilde{\boldsymbol{Q}} = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Thus, from $\boldsymbol{v}_3^T \boldsymbol{Q} \boldsymbol{e}_1 = 0$, $\boldsymbol{v}_3^T \boldsymbol{Q} \boldsymbol{e}_2 = 0$ and $\boldsymbol{v}_3^T \boldsymbol{Q} \boldsymbol{e}_3 = 0$,

$$\begin{bmatrix} q_{11} & q_{21} & q_{31} \\ q_{12} & q_{22} & q_{32} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} \alpha_{31} \\ \alpha_{32} \\ \alpha_{33} \end{bmatrix} = \boldsymbol{Q}^T \begin{bmatrix} \alpha_{31} \\ \alpha_{32} \\ \alpha_{33} \end{bmatrix} = \boldsymbol{Q} \begin{bmatrix} \alpha_{31} \\ \alpha_{32} \\ \alpha_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 & 6 \\ -2 & 1 & -3 \\ 6 & -3 & 9 \end{bmatrix} \begin{bmatrix} \alpha_{31} \\ \alpha_{32} \\ \alpha_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} \alpha_{31} \\ \alpha_{32} \\ \alpha_{33} \end{bmatrix} = \begin{bmatrix} \alpha_{31} \\ 2\alpha_{31} + 3\alpha_{33} \\ \alpha_{33} \end{bmatrix},$$

where α_{31} and α_{33} are arbitrary real numbers. Thus,

$$\boldsymbol{v}_3 = \alpha_{31}\boldsymbol{e}_1 + \alpha_{32}\boldsymbol{e}_2 + \alpha_{33}\boldsymbol{e}_3 = \begin{bmatrix} b \\ 2b + 3c \\ c \end{bmatrix},$$

where b and c are arbitrary real numbers.

Finally,

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{v}_1 \ \boldsymbol{v}_2 \ \boldsymbol{v}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{a}{2} & b \\ 0 & a & 2b + 3c \\ 0 & 0 & c \end{bmatrix},$$

where a, b, and c are arbitrary real numbers.